Chapter 1

Stochastic Geometric Partial Differential Equations

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Dedicated to David Elworthy, our teacher and friend

We describe some results about existence of manifold valued stochastic partial differential equations obtained in recent years.

1.1. Introduction

Over the past two decades the exploration of Stochastic Partial Differential Equations (briefly SPDEs) has become a rapidly expanding area in Mathematics and Physics. In addition to applications to some fundamental problems in Mathematical, Physical and Life Sciences, interest in such studies is motivated by a desire to understand and control the behaviour of complex systems that appear in many areas of natural and social sciences. Small random fluctuations such as thermal are present in all complex systems even if their fundamental theory is deterministic. For example, the 1D Nonlinear Schrödinger Equation (NLSE) arises in optical waveguide propagation and in optical communication, see e.g. Falkovich et al.[27]

It is generally accepted that differential equations serve as a mathematical and rigorous support of models in natural sciences. In general, they model ideal physical situations (propagation of waves or heat equations) neglecting external influences such as turbulence or random impacts. These effects are too complicated to be modeled in detail due to its immense quantity and unpredictable character. Therefore, stochastic perturbations are introduced to model the statistical errors caused by the external forces and random fluctuations, and computer simulations corroborate the accuracy of models of this type. During the last sixty years, it has become clear that sometimes better and more realistic descriptive results can be achieved by adding a stochastic perturbation to a particular differential equation.

This corresponds to unknown and unpredictable fluctuations that are omnipresent in the nature, known only from the statistical point of view, and without them, every model is a mere simplification of reality that neglects background noises. Equations with stochastic terms are not, of course, already perfect descriptions of reality but they take into account random influences that are not present in the corresponding deterministic equations.

For example thermal effects in micromagnetics have been studied since the work[1] by Brown. They can be incorporated into the model by modifying the effective energy in the LLEs to include a random term. One of many important applications of these equations is in magneto-electronics, where submicron-sized ferromagnetic elements are the main building blocks of information storage devices. The smaller these elements are the effects of thermal noise become more crucial, e.g. the ability of the noise to change the magnetization what leads to reducing the data-retention time of the memory element[61] and thus the **noise-induced** magnetization reversal has received a lot of attention in the magnetics community, from experimental, analytical, and numerical points of view, see e.g. Braun (1994) and E (2003). Recently, these equations have been investigated by Grinstein and Koch (2005) from the numerical point of view, and by Kohn et al. in[44] from more mathematical viewpoint.

In the physical literature equations similar to geometric evolution equations appear especially concerning the kinetic theory of phase transitions, e.g.[40] and the theory of stochastic quantization, e.g.[59] In these theories, the solution $f \in C(S^1, M)$ of the SPDE expresses a continuum like spin field distributed over the unit circle S^1 with the space of (spin-)variables constrained to the manifold M. In this field the first rigorous steps towards stochastic models have been undertaken in the direction of parabolic equations, see e.g. Funaki[30] and Carroll,[15] who investigated the existence, uniqueness, regularity and approximations of global solutions of stochastic heat equations in loop manifolds. On the other hand, there has not been any work published on stochastic geometric wave equations so far.

The present paper consists of essentially two independent parts. However there is a unifying background for both of them, i.e. they are about stochastic geometric equations. The second part of this article (Sections 1.4 - 1.10) is devoted to description of some recent results obtained by the 1st and the 3rd named authours on stochastic geometric wave equations while in the the first part (sections 1.2 and 1.3) we present new results on geometric heat flow where the target manifold is a general compact riemannian manifold. In Part I we planned to present the results from the PhD thesis by A Carroll and still unfinished paper by the 1st named authour and Carroll but while working on this project we realised that it would be much more natural to write down an account on how the detailed approach from the first paper by the 1st and the 3rd named authours,[11 12] can work in the case of stochastic heat flow equation in the case when the domain is one dimensional. The one dimensionality of the domain makes it possible to work with the energy space, i.e. the Hilbert space $H^{1,2}(\mathbb{S}^1, \mathbb{R}^d)$ as the state space since only in this case the embedding the energy space into the Banach space $C(\mathbb{S}^1, \mathbb{R}^d)$ of continuous functions holds. During researching on this new project we observed that some techniques that had been developed in,[9] see Lemma 1.4, are essential. Let us point out a difference between our proof of the global existence and the one in the deterministic case by Eells-Sampson[24] and Hamilton.[37] While in the latter papers the crucial step is to prove that the energy density solves certain scalar parabolic equation, in our case the crucial step is to prove an inequality for the L^2-norm of the gradient of the solution, see (1.43) which is based on certain geometric property (1.41) of the manifold M.

It is our pleasure to acknowledge that our interest in the field of geometric stochastic PDEs grew out of studying the fundamental works of David Elworthy on infinite dimensional stochastic analysis, see for instance his monograph.[25] In particular, our works[6] and[4] were strongly motivated by some aspects of his research. Moreover, David's ideas directed us towards it's PDEs generalisation.[4]

1.2. Stochastic Geometric heat flow on $\mathbb{S}^1 \times \mathbb{R}_+$

In this section we assume that M is a compact riemannian manifold that is isometrically embedded into an Euclidean space $\mathbb{R}^d$. We consider the following one-dimensional stochastic geometric heat flow equation

$$\partial_t u = \mathbf{D}_x \partial_x u + Y_u \circ \dot{W}, \tag{1.1}$$

with initial data

$$u(0, \cdot) = \xi(\cdot), \tag{1.2}$$

where $\mathbb{S}^1$ is the unit circle (usually identified with the interval $[0, 2\pi)$, Y is a C^1-class section of a certain vector bundle $\mathbb{M}$ over M, see Theorem 1.1, and $\xi : \mathbb{S}^1 \to M$ is a continuous map. We assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration such that $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets. Let us denote by $\mathfrak{S} = (e^{-tA})_{t \geq 0}$ the C_0 analytic semigroup of bounded linear operators on the space $L^2(\mathbb{S}^1, \mathbb{R}^d)$, generated by an operator $A := -\Delta$ whose domain $\mathrm{Dom}(A)$ is equal to the Sobolev space $H^{2,2}(\mathbb{S}^1, \mathbb{R}^d)$.

Let S denote the second fundamental tensor (form) of the manifold M with respect to the above mentioned isometric embedding $M \subset \mathbb{R}^d$. In particular, for each $p \in M$, $S_p : T_pM \times T_pM \to N_pM$, where $N_p := \mathbb{R}^d \ominus T_pM$ is the normal space to p with respect to the standard scalar product in $\mathbb{R}^d$. The operator $\mathbf{D}_x \partial_x$ that appears in equation (1.1) acts on smooth curves $\gamma : \mathbb{S}^1 \to M$ and is defined by the formula (see for instance [12, section 2] and references therein)

$$\mathbf{D}_x \partial_x \gamma(x) = \partial_{xx}\gamma(x) - S_{\gamma(x)}(\partial_x \gamma(x), \partial_x \gamma(x)), \quad x \in \mathbb{S}^1. \tag{1.3}$$

We note that the following fundamental property of the operator $\mathbf{D}_x \partial_x$, see [12, formula (2.6)],

$$\langle \partial_{xx}\gamma(x) - S_{\gamma(x)}(\partial_x \gamma(x), \partial_x \gamma(x)), \partial_{xx}\gamma(x) \rangle = |\partial_{xx}\gamma(x) - S_{\gamma(x)}(\partial_x \gamma(x)|^2, \ x \in I. \tag{1.4}$$

As far as the noise is concerned, we make the following standing assumption.

Assumption 1.2.1. $W = (W(t))_{t\geq 0}$ is an E-valued $\mathbb{F}$-Wiener process, where E is a separable Banach space such that for some fixed natural number n, $E \subset H^1(\mathbb{S}^1, \mathbb{R}^n)$ continuously.

Remark 1.1. It follows from Assumption 1.2.1 that the Reproducing Kernel Hilbert Space K of the law of $W(1)$ is contained in E (and so in $H^1(\mathbb{S}^1, \mathbb{R}^n)$) and the natural embedding $i : K \hookrightarrow E$ is γ-radonifying.
Let us recall that if $\Lambda : E \to H$ is a bounded linear map, where H is a separable Hilbert space, then $\Lambda \circ i : \mathrm{K} \to H$ is γ-radonifying, i.e. Hilbert-Schmidt.
If $\Lambda : E \times E \to X$ is a bounded bilinear map where X is a separable Banach space then

$$\mathrm{tr}_{\mathrm{K}}(\Lambda) := \sum_j \Lambda(e_j, e_j) \in X, \tag{1.5}$$

where $(e_j)_j$ is an ONB basis of K is well defined. In other words the series on the RHS of equality (1.5) is absolutely convergent, its sum is independent of choice of the ONB $(e_j)_j$ and the map $\mathbb{L}(E, E; X) \ni \Lambda \mapsto \mathrm{tr}_{\mathrm{K}}(\Lambda)$ is linear and bounded. In particular, if $G : X \to \mathcal{L}(E, X)$ is of C^1-class, then for every $a \in X$, $G'(a)G(a) \in \mathcal{L}(E, \mathcal{L}(E, H)) \cong \mathbb{L}(E, E; X)$ and so $\mathrm{tr}_{\mathrm{K}}[G'(a)G(a)] \in X$ is well defined.

For the deterministic version of our problem one can consult the fundamental works by Eells-Sampson[24] and Hamilton.[37]

Contrary to the case of a wave equation the solutions to the stochastic heat flow equation can only be defined using an external formulation. However, we hope to be able to find an appropriate definition of an *intrinsic solution*.

What concerns the initial data ξ we make the following assumption.

Assumption 1.2.2. The initial data ξ is an $\mathcal{F}_0$-measurable random variable with values in $H^1(\mathbb{S}^1, M)$.

Remark 1.2. As is,[10] see the end of the proof of Theorem 1.1 on page 133. it is sufficient to assume that ξ is such that for some $p > 2$,

$$\mathbb{E}|\xi|^p_{H^1(\mathbb{S}^1,\mathbb{R}^d)} < \infty. \tag{1.6}$$

Definition 1.1. A process $u : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to M$ is called an **extrinsic solution** to equation (1.1) if and only if the following five conditions are satisfied

(i) $u(t, x, \cdot)$ is $(\mathcal{F}_t)$-progressively measurable for every $x \in \mathbb{S}^1$,
(ii) $u(\cdot, \cdot, \omega)$ belongs to $C(\mathbb{R}_+ \times \mathbb{S}^1; M)$ for every $\omega \in \Omega$,
(iii) $\mathbb{R}^+ \ni t \mapsto u(t, \cdot, \omega) \in H^1(\mathbb{S}^1, M)$ is continuous for every $\omega \in \Omega$,
(vii) for all $t \geq 0$ the following equality holds in $H^{-1}(\mathbb{S}^1, \mathbb{R}^d)$, $\mathbb{P}$ almost surely,

$$\begin{aligned} u(t) = u(0) + \int_0^t \left[\partial_{xx} u(s) - S_{u(s)}(\partial_x u(s), \partial_x u(s))\right]\, ds \\ + \int_0^t Y_{u(s)} \circ dW(s). \end{aligned} \tag{1.7}$$

Moreover, if Assumption 1.2.2 is satisfied, then process $u : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to M$ is called an **extrinsic solution** of problem (1.1)-(1.2) if and only if u is an extrinsic solution to equation (1.1) and

(v) $u(0,x,\omega) = \xi(x,\omega)$ for every $x \in \mathbb{S}^1$, $\mathbb{P}$-a.s.

Finally, u is called a **regular extrinsic solution** if in addition the following two conditions are satisfied

(viii) $\mathbb{E} \int_0^T |u(t)|^2_{H^2(\mathbb{S}^1,\mathbb{R}^d)}\, dt < \infty$ for each $T > 0$,
(ix) and for all $t \geq 0$ the equality (1.7) holds in $L^2(\mathbb{S}^1, \mathbb{R}^d)$, $\mathbb{P}$ almost surely.

Remark 1.3. Since a function $\xi : \Omega \to C(\mathbb{S}^1; M)$ is $\mathcal{F}$-measurable if and only if for every $x \in \mathbb{S}^1$ the function $i_x \circ \xi \to M$ is $\mathcal{F}$-measurable, in view of the Kuratowski Theorem, Assumption 1.2.2 is equivalent to the following one.
The initial data ξ is a function taking values in $H^1(\mathbb{S}^1, M)$ such that for every $x \in \mathbb{S}^1$ the function $i_x \circ \xi \to M$ is a $\mathcal{F}_0$-measurable.
In a similar vein, condition (i) in Definition 1.1 can be replaced by the following one

(i$'$) $\Omega \ni \omega \mapsto u(t,\cdot,\omega) \in H^1(\mathbb{S}^1, M)$ is $\mathcal{F}_t$-measurable for every every $t \geq 0$.

Remark 1.4. Let us observe that the following is an informal version of equation (1.7)

$$\partial_t u = \partial_{xx} u - S_u(u_x, u_x) + Y_u \circ \dot{W}. \tag{1.8}$$

Both equations can also be formulated in the following mild form.

$$u(t) = e^{-tA}\xi - \int_0^t e^{-(t-s)A} S_u(u_x,u_x)\, ds + \int_0^t e^{-(t-s)A} Y_u \circ dW(s), \quad t \geq 0. \tag{1.9}$$

Next we formulate the main result of this part of the paper.

Theorem 1.1. *Let us denote by $\mathbb{M}$ a vector bundle over M whose fiber at $m \in M$ is equal to $\mathcal{L}(\mathbb{R}^n; T_m M)$, where m is a fixed natural number. Assume that Y is a C^1 class section of the vector bundle $\mathbb{M}$. Then there exists an $\mathbb{F}$-adapted process $u = \big(u(t)\big)_{t\geq 0}$ such that u is a* **regular extrinsic solution** *to problem (1.1-1.2).*
Moreover, suppose that $u = \big(u(t)\big)_{t\geq 0}$ and $\bar{u} = \big(\bar{u}(t)\big)_{t\geq 0}$ are two $\mathbb{F}$-adapted processes such that for some $T > 0$, they are **extrinsic solutions** *to problem (1.1-1.2). Then $\bar{u}(t,x,\omega) = u(t,x,\omega)$ for all $x \in \mathbb{S}^1$ and $t \in [0,T)$, $\mathbb{P}$-almost surely.*

In the following generalization of the Itô Lemma, see [11, Lemma 6.5] we denote by $\mathcal{T}_2(\mathrm{K}, H)$ the Hilbert space of Hilbert-Schmidt operators acting between separable Hilbert spaces K and H.

Lemma 1.1. *Let K and H be separable Hilbert spaces, and let f and g be progressively measurable processes with values in H and $\mathcal{T}_2(\mathrm{K}, H)$ respectively, such that*

$$\int_0^T \left\{ |f(s)|_H + \|g(s)\|^2_{\mathcal{T}_2(\mathrm{K},H)} \right\} ds < \infty \ \text{ almost surely.}$$

For some H-valued $\mathcal{F}_0$-measurable random variable ξ define a process u by

$$u(t) = e^{-tA}\xi + \int_0^t e^{-(t-s)A} f(s)\,ds + \int_0^t e^{-(t-s)A} g(s)\,dW(s), \qquad t \in [0,T],$$

where W is a cylindrical Wiener process on K, and $(e^{-tA})_{t\geq 0}$ is a C_0-semigroup on H with an infinitesimal generator $-A$. Let V be another separable Hilbert space and let $(e^{-tB})_{t\geq 0}$ be a C_0-semigroup on V with an infinitesimal generator $-B$. Suppose that $Q : H \to V$ is a C^2-smooth function such that $Q[D(A)] \subseteq D(B)$ and there exists a continuous function $F : H \to V$ such that

$$-Q'(u)Au = -BQ(u) + F(u), \qquad u \in D(A). \tag{1.10}$$

Then, for all $t \geq 0$,

$$\begin{aligned} Q(u(t)) = e^{-tB}Q(\xi) &+ \int_0^t e^{-(t-s)B} Q'(u(s))g(s)\,dW(s) \\ &+ \int_0^t e^{-(t-s)B}\Big[Q'(u(s))f(s) + F(u(s)) \,+\, \frac{1}{2}\,\mathrm{tr}_K Q''(u(s)) \circ (g(s), g(s))\Big]\,ds. \end{aligned}$$

1.3. Proof of Theorem 1.1

The basic idea of the proof of the main result comes from[37] and.[4] The nonlinearities S and Y in equation (1.8) are extended from their domains (products of tangent bundles) to the ambient space, and thus we obtain a classical SPDE in Euclidean space for which the existence of global solutions is known. However our proof of the existence of the manifold valued solutions requires, that from the many extensions that can be constructed, we choose those which satisfy certain "symmetry" properties.

1.3.1. *Differential Geometry preliminaries*

Let us denote by TM and NM the tangent and the normal bundle respectively, and denote by $\mathcal{E}$ the exponential function $T\mathbb{R}^d \ni (p,\xi) \mapsto p + \xi \in \mathbb{R}^d$ relative to the Riemannian manifold $\mathbb{R}^d$ equipped with the standard Euclidean metric. The following result about tubular neighbourhood of M can be found in,[57] see Proposition 7.26, p. 200.

Proposition 1.1. *There exists an $\mathbb{R}^d$-open neighbourhood O of M and an NM-open neighbourhood V around the set $\{(p,0) \in NM : p \in M\}$ such that the restriction of the exponential map $\mathcal{E}|_V : V \to O$ is a diffeomorphism. Moreover, V can be chosen in such a way that $(p, t\xi) \in V$ whenever $t \in [-1,1]$ and $(p,\xi) \in V$.*

Remark 1.5. In what follows, we will denote the diffeomorphism $\mathcal{E}|_V : V \to O$ by $\mathcal{E}$, unless there is a danger of ambiguity.

Denote by $i : NM \to NM$ the diffeomorphism $(p,\xi) \mapsto (p,-\xi)$ and define

$$h = \mathcal{E} \circ i \circ \mathcal{E}^{-1} : O \to O. \tag{1.11}$$

The function h defined above is an involution on the normal neighbourhood O of M and corresponds to multiplication by -1 in the fibers, having precisely M for its fixed point set. The identification of the manifold M as a fixed point set of a smooth function enables to prove that solutions of heat equations with initial values on the manifold remain thereon, see[37] for deterministic heat equations in manifolds and[4] for stochastic heat equations in manifolds. Employing a partition of unity argument we may assume that $h : \mathbb{R}^d \to \mathbb{R}^d$ is such that properties (1)-(5) of Corollary 1.1 are valid on O. Therefore, without loss of generality we may assume that the function h is defined on the whole $\mathbb{R}^d$.

Corollary 1.1. *The function h has the following properties: (i) $h : O \to O$ is a diffeomorphism, (ii) $h(h(q)) = q$ for every $q \in O$, (iii) if $q \in O$, then $h(q) = q$ if and only if $q \in M$, (iv) if $p \in M$, then $h'(p)\xi = \xi$, provided $\xi \in T_pM$ and $h'(p)\xi = -\xi$, provided $\xi \in N_pM$.*

Next we define, for $q \in \mathbb{R}^d$ and $a, b \in \mathbb{R}^d$,

$$B_q(a,b) = d_q^2 h(a,b), \quad \mathcal{S}_q(a,b) = \frac{1}{2} B_{h(q)}(h'(q)a, h'(q)b). \tag{1.12}$$

Let us recall that the second fundamental tensor S was introduced before the formula (1.4). We will be studying problem (1.7) with S replaced by $\mathcal{S}$. The following result which is essential for our paper is taken from [12, Proposition 4.2].

Proposition 1.2. *If $p \in M$ and $q \in O$, then*

$$\mathcal{S}_p(\xi,\eta) = \frac{1}{2} B_p(a,b) = S_p(\xi,\eta), \;\; \xi,\eta \in T_pM, \tag{1.13}$$

$$\mathcal{S}_{h(q)}(h'(q)a, h'(q)b) = h'(q)\mathcal{S}_q(a,b) + B_q(a,b), \;\; a,b \in \mathbb{R}^d. \tag{1.14}$$

Let us formulate and prove the following result which shows importance of Proposition (1.2).

Corollary 1.2. *Let us put*

$$\underline{\mathbf{\Delta}}(u) = u_{xx} - \mathcal{S}_u(u_x,u_x), \;\; u \in H^2(\mathbb{S}^1, \mathbb{R}^d). \tag{1.15}$$

Then,

$$\underline{\mathbf{\Delta}}(h \circ u) = h'(u)\underline{\mathbf{\Delta}}(u), \;\; u \in H^2(\mathbb{S}^1, \mathbb{R}^d). \tag{1.16}$$

Proof. Assume that $u \in C^2(\mathbb{S}^1, \mathbb{R}^d)$ and put $v = h \circ u$. Then,

$$\begin{aligned}\underline{\mathbf{\Delta}}(v) &= [h \circ u]_{xx} - \mathcal{S}_{h \circ u}((h \circ u)_x, h \circ u)_x) \\ &= h'(u)u_{xx} + h''(u)(u_x,u_x) - \mathcal{S}_{h(u)}(h'(u)u_x, h'(u)u_x) \\ &= h'(u)u_{xx} - h'(u)\mathcal{S}_u(u_x,u_x) = h'(u)\underline{\mathbf{\Delta}}(u),\end{aligned}$$

where the second line above follows from (1.14). □

To this end, let π_p, $p \in M$ be the orthogonal projection of $\mathbb{R}^d$ to T_pM and let us define $v_{ij}(p) = S_p(\pi_p e_i, \pi_p e_j)$ for $i, j \in \{1, \dots, n\}$ and extend the functions $v_{ij} = v_{ji}$ smoothly to the whole $\mathbb{R}^d$.

Now we will shortly recall the construction of extensions of vector fields on M to vector fields on O from,[37] cf.[15] To this end, let us define a new Riemannian metric g on O by

$$g_q(a,b) = \langle a, b\rangle_{\mathbb{R}^d} + \langle h'(q)a, h'(q)b\rangle_{\mathbb{R}^d}, \qquad q \in O, \quad a, b \in \mathbb{R}^d. \tag{1.17}$$

Remark 1.6. $h : (O, g) \to (O, g)$ is an isometric diffeomorphism.

If $q \in O$ then, by Proposition 1.1, there exists a unique $(p, \xi) \in V$ such that $q = \mathcal{E}(p, \xi)$. We will write $p(q) = p$ for this dependence. Moreover, also by Proposition 1.1, $\mathcal{E}(p, t\xi) \in O$ for $t \in [0, 1]$. Hence we can define the curve $\gamma_q : [0, 1] \ni t \mapsto \mathcal{E}(p, t\xi) \in O$. If $a \in \mathbb{R}^d$ and $(X(t))_{t\in[0,1]}$, $X(0) = a$ is the parallel translation of a along γ_q in (O, g) then we denote by $P_q a$ the endpoint vector $X(1)$.

Proposition 1.3. *[11, Proposition 3.9] $P : O \to \mathcal{L}^{\mathrm{isom}}(\mathbb{R}^d, \mathbb{R}^d)$ is a smooth function. Moreover, $P_q = I$ for $q \in M$ and*

$$h'(q)P_q = P_{h(q)} h'(p(q)), \qquad q \in O.$$

Due to this setting, it is possible to extend conveniently various mappings defined on the manifold M to its neighbourhood O, c.f.[15] For example, if X is a vector field on M, i.e. a section of the tangent bundle, then we can define a map $\mathbf{X} : O \to \mathbb{R}^d$ by the following formula

$$\mathbf{X}_q = P_q X_{p(q)}, \;\; q \in O. \tag{1.18}$$

1.3.2. *Existence of solutions to approximating equations*

Note that the tangent bundle $T\mathbb{R}^d$ is isomorphic to $\mathbb{R}^d \times \mathbb{R}^d$. Using formula (1.18) and Proposition 1.3 we can find a C^1-class map $\mathbf{Y} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ that such

$$\mathbf{Y}_m = Y_m : \mathbb{R}^n \to T_m M, \;\; m \in M, \tag{1.19}$$

where we identify $T_m M$ with the corresponding subspace of $\mathbb{R}^d$, and

$$\mathbf{Y}_{h(q)} = h'(q) \circ \mathbf{Y}_q, \; q \in O. \tag{1.20}$$

Note that both sides of (1.20) belong to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$.

Let us fix $T > 0$. In what follows we put $H^1 = H^1(\mathbb{S}^1, \mathbb{R}^d)$, $L^1 = L^1(\mathbb{S}^1, \mathbb{R}^d)$ etc. We will also denote by $H^1(\mathbb{S}^1, M)$ the Hilbert manifold consisting of those $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^d)$ which satisfy $\gamma(x) \in M$ for all $x \in \mathbb{S}^1$.

Let us recall that E is a Banach space from Assumption 1.2.1.

We define maps $\mathbf{F} : H^1 \to L^1$, $\mathbf{G} : H^1 \to \mathcal{L}(E, H^1)$ and $\mathbf{Q} : H^1 \to H^1$ by the following formulae,

$$
\begin{aligned}
[\mathbf{F}(u)](x) &= -\mathcal{S}_{u(x)}(u_x(x), u_x(x)),\ u \in H^1,\ x \in \mathbb{S}^1,\\
\big(\mathbf{G}(u)\xi\big)(x) &= \mathbf{Y}_{u(x)}(\xi(x)),\ u \in H^1,\ \xi \in E,\ x \in \mathbb{S}^1,\\
\mathbf{Q}(u) &= h \circ u,\ u \in H^1.
\end{aligned}
$$

We begin with the following result which follows directly from Corollary 1.1 parts (3) and (4) and the definitions of the function $\mathbf{Q}$.

Lemma 1.2. *If $u \in H^1(\mathbb{S}^1, M)$, then $\mathbf{Q}(u) = u$. Conversely, if $u \in H^1(\mathbb{S}^1, \mathbb{R}^d)$ is such that for all $x \in \mathbb{S}^1$ $u(x) \in O$ and $\mathbf{Q}(u) = u$, then $u \in H^1(\mathbb{S}^1, M)$.*

Let π_n be the projection from H^1 onto the ball $B(0,n) \subset H^1$ defined by

$$
\pi_n(v) = \begin{cases} v, & \text{if } |v|_{H^1} \le n,\\ \frac{n}{|v|_{H^1}} v, & \text{if } |v|_{H^1} > n. \end{cases} \tag{1.21}
$$

It is well known, see for instance [5, Lemma 2.3], that the map $\pi_n : H^1(0,1) \to H^1(0,1)$ is globally Lipschitz with Lipschitz constant 1. Moreover,

$$
\begin{aligned}
|(D_x(\pi_n u))^2 - (D_x(\pi_n v))^2|_{L^1} &\le 2n|u-v|_{H^1}, \ \text{ for all } u, v \in H^1, \\
|(\pi_n u)^2|_{L^1} &\le \big[n \wedge |u|_{H^1}\big]|u|_{H^1}, \ \text{ for all } u \in H^1.
\end{aligned} \tag{1.22}
$$

Next we define maps $\mathbf{F}_n : H^1 \to L^2$ and $\mathbf{G}_n : H^1 \to \mathcal{L}(E, H^1)$ by analogous formulae

$$
\begin{aligned}
\mathbf{F}_n(u) &= -\mathcal{S}_u((\pi_n \circ u)_x, (\pi_n \circ u)_x),\ u \in H^1,\\
\mathbf{G}_n(u) &= G(\pi_n u),\ u \in H^1.
\end{aligned}
$$

Note that $\mathbf{F}_n = \mathbf{F}$ and $\mathbf{G}_n = \mathbf{G}$ on on the ball $B(0,n)$ in H^1. Moreover, since the function G is Lipschitz continuous on the ball $B(0,n)$ in H^1, it follows, see for instance the proof of [3, Corollary 3] that $\mathbf{G}_n : H^1 \to H^1$ is globally Lipschitz. Finally, the Lipschitz continuity and boundedness of the map $\mathbf{F}_k : H^1 \to L^1$ can be derived as in.[5] Thus have the following result.

Lemma 1.3. *The functions $\mathbf{F} : H^1 \to L^1$ and $\mathbf{G} : H^1 \to \mathcal{L}(E, H^1)$ are Lipschitz continuous on balls, the function $\mathbf{G}$ is of C^1-class and the the function*

$$
\mathrm{tr}_K(G' \circledast G) : H^1 \ni u \mapsto \mathrm{tr}_K(G'(u)G(u)) \in H^1
$$

is Lipschitz continuous on balls. For each $k \in \mathbb{N}$, the functions $\mathbf{F}_k : H^1 \to L^1$, $\mathbf{G}_k : H^1 \to \mathcal{L}(E, H^1)$ and $\mathrm{tr}_K(\mathbf{G}_k' \circledast \mathbf{G}_k) : H^1 \to H^1$ are globally Lipschitz continuous, i.e. there exists a constant C_k such that for all $u, v \in H^1$

$$
\begin{aligned}
&|\mathbf{F}_k(u) - \mathbf{F}_k(v)|_{L^1} + |\mathbf{G}_k(u) - \mathbf{G}_k(v)|_{\mathcal{L}(E,H^1)}\\
&\qquad + |\mathrm{tr}_K(\mathbf{G}_k'(u)\mathbf{G}_k(u)) - \mathrm{tr}_K(\mathbf{G}_k'(v)\mathbf{G}_k(v))|_{H^1} \le C_k|u-v|_{H^1}.
\end{aligned} \tag{1.23}
$$

We will also need the following result which is related to the proofs of Lemmae 2.11 and 2.12 in.[5]

Lemma 1.4. *Let* $\left(e^{-tA}\right)_{t\geq 0}$ *be the heat semigroup on the scale of Banach spaces* $L^p(\mathbb{S}^1, \mathbb{R}^d)$, $p \in [1, \infty)$. *Then for each* $\alpha \geq 0$, *there exists a constant* $C = C_\alpha > 0$ *such that*

$$\|e^{-tA}\|_{\mathcal{L}(L^1, H^{\alpha,2})} \leq Ct^{-\frac{1}{4}-\frac{\alpha}{2}}, \; t > 0. \tag{1.24}$$

In particular, for each for each $T > 0$ *and* $\alpha \in [0, \frac{3}{2})$ *and for any bounded and strongly-measurable function* $v : (0,t) \to L^1(0,1)$ *the following inequality holds*

$$\sup_{t\in[0,T]} |\int_0^t |e^{-(t-s)A}v(s)\, ds|_{H^{\alpha,2}} \leq C_\alpha T^{\frac{3}{4}-\frac{\alpha}{2}} \sup_{t\in[0,T]} |v(t)|_{L^1}. \tag{1.25}$$

The following result can be proved by using Lemma 1.3 and employing similar methods as used in the proof of Theorem 2.14 in.[5] One should point out here that this result is different from more standard existence results as those for instance in Theorem 4.3 in.[2]

Proposition 1.4. *Let us fix* $p > 2$. *Let the initial data* ξ *from Assumptions 1.2.2 satisfy in addition condition* (1.6).

Then there exists a unique H^1*-valued continuous process* u_k *satisfying*

$$\mathbb{E} \sup_{t\in[0,T]} |u_k(s)|^p_{H^1} < \infty \tag{1.26}$$

and such that for all $t \in [0,T]$, $\mathbb{P}$*-a.s.*

$$\begin{aligned} u_k(t) = e^{-tA}\xi + \int_0^t e^{-(t-s)A}\mathbf{F}_k(u_k(s))\, d\, s + \int_0^t e^{-(t-s)A}\mathbf{G}_k(u_k(s))\, dW(s) \\ + \int_0^t e^{-(t-s)A}\mathrm{tr}_K[\mathbf{G}_k' \circledast \mathbf{G}_k](u_k(s))\, d\, s. \end{aligned} \tag{1.27}$$

Moreover,

$$\mathbb{E}\int_0^T |u_k(s)|^2_{H^2}\, ds < \infty. \tag{1.28}$$

Proof. [Proof of Proposition 1.4] As mentioned above the proof of the first part follows the ideas from the proof of Theorem 2.14 in.[5] The proof of the second part uses ideas from.[9] Since $\xi \in L^p(\Omega, H^1) \subset L^2(\Omega, H^1)$ and $H^1 = \mathrm{Dom}(A^{1/2}) = (D(A), L^2)_{1/2,2}$, by invoking[48] we infer that the 1st term on the RHS of (1.27) satisfies the condition (1.28). Because u_k satisfies the condition (1.26) in view of the Lipschitz condition (1.23) satisfied by $\mathbf{G}_k$ and $\mathrm{tr}_{\mathrm{K}}(\mathbf{G}_k' \circledast \mathbf{G}_k)$, both the 3rd and the 4th terms on the RHS of (1.27) satisfy the condition (1.28). The only difficulty lies with the 2nd term because so far we only know that for each $t > 0$, $\mathbf{F}_k(u_k(t))$ belongs to L^1. By[48] again, it is enough to show that $\mathbb{E}\int_0^T |\mathbf{F}_k(u_k(s))|^2_{L^2}\, ds < \infty$. In view of the definition of $\mathbf{F}_k$, it is enough to show that

$$\mathbb{E}\int_0^T |(u_k)_x(s)|^4_{L^4}\, ds < \infty. \tag{1.29}$$

Obviously, it is enough to show that each term on the RHS of (1.27) satisfies the above condition (1.29). To this end it is sufficient to verify the following two claims.

Claim 1. If $v \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$ then $v \in L^4(0,T;L^4)$ and there exists a constant $C > 0$ such that

$$\int_0^T |v(s)|_{L^4}^4 \, ds \le CT^{1/2} \sup_{t\in[0,T]} |v(t)|_{L^2}^2 \Big(\int_0^T |v(s)|_{L^2}^2 \Big)^{1/2}.$$

Claim 2. If $\alpha > \frac{5}{4}$, then there exists a constant $C > 0$ such that for all v

$$\int_0^T |v_x(s)|_{L^4}^4 \, ds \le C \sup_{t\in[0,T]} |v(t)|_{H^\alpha}^4.$$

Claim 1 follows from a special case of the Gagliardo-Nirenberg inequality $|v|_{L^4}^4 \le C|v|_{L^2}^3 |v_x|_{L^2}$. Claim 2 follows from a special case of the Sobolev embedding (valid for $\beta > \frac{1}{4}$) that

$$|v|_{L^4} \le C|v|_{H^\beta}.$$

Therefore, the proof of Proposition 1.4 is completed by applying Claim 2 to the 2^{nd} term on the RHS of (1.27) and Claim 1 to all three remaining terms. □

We will apply Lemma 1.1 to the process u_k and the function $\mathbf{Q}$. We have the following result.

Lemma 1.5. *The map $\mathbf{Q} : H^1 \to H^1$ is of C^2-class and, with $u, v, z \in H^1$, it satisfies*

$$\mathbf{Q}'(u)v = h'(u)v, \quad \mathbf{Q}''(u)(v,z) = h''(u)(v,z),$$

$$\mathbf{Q}'(u)[-Au] = -A\mathbf{Q}(u) + \mathbf{L}(u), \quad u \in H^2, \tag{1.30}$$

$$\mathbf{Q}'(u)[\mathbf{G}(u)] = [\mathbf{G} \circ \mathbf{Q}](u), \quad u \in H^1(\mathbb{S}^1, O), \tag{1.31}$$

where, with the map B being defined in (1.12)*, the function $\mathbf{L} : H^2 \to L^2$ is defined by*

$$\mathbf{L}(u) = B_u(u_x, u_x).$$

Proof. Identity (1.30) can be proved in the same way as identity (1.16). Identity (1.31) is a consequence of the invariance property (1.20). Indeed, if $\xi \in E$, then

$$\begin{aligned} \mathbf{Q}'(u)[\mathbf{G}(u)\xi] &= h'(u)Y_u\xi = Y_{h(u)}\xi \\ &= \mathbf{G}(h(u))(\xi) = (\mathbf{G}(\mathbf{Q}(u))(\xi). \end{aligned}$$

□

Define now the following two auxiliary functions

$$\widetilde{\mathbf{F}}_k : H^1 \ni u \mapsto \mathbf{Q}'(u)\big(\mathbf{F}_k(u)\big) - \mathbf{L}(u) = h'(u)(\mathbf{F}_k(u)) - B_u(u_x, u_x) \in L^1 \tag{1.32}$$

$$\widetilde{\mathbf{G}}_k : H^1 \ni u \mapsto \mathbf{Q}'(u) \circ \mathbf{G}_k(u) = h'(u)(\mathbf{G}_k(u)) \in \mathcal{L}(E, H^1). \tag{1.33}$$

This leads to the following result that rigorously expresses the fact that $\widetilde{\mathbf{F}}_k$, resp. $\widetilde{\mathbf{G}}_k$ is the push forward by $\mathbf{Q}$ of $\mathbf{F}_k$, resp. $\mathbf{G}_k$.

Proposition 1.5. *The following identities hold.*

$$\widetilde{\mathbf{F}}_k(u) = \mathbf{F}_k(\mathbf{Q}(u)), \quad \widetilde{\mathbf{G}}_k(u) = \mathbf{G}_k(\mathbf{Q}(u)), \; u \in H^1, \tag{1.34}$$

$$\widetilde{\mathbf{G}}'_k(u)\widetilde{\mathbf{G}}_k(u) = \mathbf{G}'_k(\mathbf{Q}(u))\mathbf{G}_k(\mathbf{Q}(u)), \;\; u \in H^1(\mathbb{S}^1, O). \tag{1.35}$$

Proof. [Proof of Proposition 1.5] It is enough to prove the identities for the original operators, i.e. without the subscript k.

We begin with the second identity in (1.34). By the invariance property (1.20) of the function Y and the identities (1.31) and (1.33) we have,

$$\begin{aligned}[\mathbf{G}(\mathbf{Q}(u))](\xi) &= Y_{\mathbf{Q}(u)(\cdot)}\xi(\cdot) = Y_{h(u(\cdot))}\xi(\cdot) \\ &= h'(u(\cdot))Y_{u(\cdot)}\xi(\cdot) = [h'(u)\mathbf{G}(u)](\xi) = [\widetilde{\mathbf{G}}(u)](\xi).\end{aligned}$$

To prove the first part of (1.34) we can argue as in the proof of identity (1.16).

Identity (1.35) is a consequence of identity (1.31). □

Let us also observe that it follows from Lemma 1.5 that the assumptions of Lemma 1.1 are satisfied with the linear operator B being equal to A. Thus we have the following fundamental result.

Corollary 1.3. *Let u_k be the solution to* (1.27) *as in Proposition 1.4 and a process $\widetilde{u}_k$ be defined by the following formula*

$$\widetilde{u}_k = \mathbf{Q} \circ u_k. \tag{1.36}$$

Then for all $t \in [0, T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\widetilde{u}_k(t) = e^{-tA}\mathbf{Q}(\xi) + \int_0^t e^{-(t-s)A}\widetilde{\mathbf{F}}_k(u_k(s))\,ds + \int_0^t e^{-(t-s)A}\widetilde{\mathbf{G}}_k(u_k(s))\,dW(s) \\ + \int_0^t e^{-(t-s)A}\mathrm{tr}_K[\widetilde{\mathbf{G}}'_k(u_k(s))\widetilde{\mathbf{G}}_k(u_k(s))]\,d\,s.\end{aligned} \tag{1.37}$$

1.3.3. *Construction of a maximal local solution*

In the first part of this subsection we will show that the approximate solutions stay on the manifold M. This will follow from Corollary 1.3. As usual, we begin with some notation.

Let for each $k \in \mathbb{N}$, et u_k be the solution to problem (1.27). Let us define the following four $[0, \infty]$-valued functions on Ω.

$$\begin{aligned}\tau_k^1 &= \inf\{t \in [0, T] : |u_k(t)|_{H^1} > k\}, \\ \tau_k^2 &= \inf\{t \in [0, T] : |\widetilde{u}_k(t)|_{H^1} > k\}, \\ \tau_k^3 &= \inf\{t \in [0, T] : \exists x \in \mathbb{S}^1 : \; u_k(t, x) \notin O\}, \\ \tau_k &= \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3.\end{aligned}$$

The following result is borrowed from,[12] see Lemma 5.4.

Lemma 1.6. *Each function τ_k^j, $j = 1, 2, 3$, $k \in \mathbb{N}$, is a stopping time.*

Proposition 1.6. *The process u_k and $\widetilde{u}_k$ coincide on $[0,\tau_k)$ almost surely. In particular, $u_k(t,x) \in M$ for $x \in \mathbb{S}^1$ and $t \le \tau_k$ almost surely. Consequently,*

$$\tau_k = \tau_k^1 = \tau_k^2 \le \tau_k^3.$$

Proof.

By Proposition 1.5 we infer that for all $s \in [0,T]$, $x \in \mathbb{S}^1$, $\mathbb{P}$-a.s.

$$\begin{aligned}
1_{[0,\tau_k)}(s)[\widetilde{\mathbf{F}}_k(u_k(s))](x) &= 1_{[0,\tau_k)}(s)[\mathbf{F}_k(\widetilde{u}_k(s))](x), \\
1_{[0,\tau_k)}(s)[\widetilde{\mathbf{G}}_k(u_k(s))e](x) &= 1_{[0,\tau_k)}(s)[\mathbf{G}_k(\widetilde{u}_k(s))e](x), \quad e \in \mathrm{K}, \\
1_{[0,\tau_k)}(s)\mathrm{tr}_{\mathrm{K}}[\widetilde{\mathbf{G}}_k' \circledast \widetilde{\mathbf{G}}_k](u_k(s)) &= 1_{[0,\tau_k)}(s)\mathrm{tr}_{\mathrm{K}}[\mathbf{G}_k' \circledast \mathbf{G}_k](\widetilde{u}_k(s)).
\end{aligned}$$

Let us denote

$$p(t) = |u_k(t) - \widetilde{u}_k(t)|_{L^2}^2, \ \ t \in [0,T]. \tag{1.38}$$

Then the process p stopped at τ_k is continuous and uniformly bounded. Note that since $\xi = \mathbf{Q}(\xi)$, we infer that $p(0) = 0$. Moreover, by the Itô Lemma from[58] and Lemma 1.3, we can find a continuous martingale I with $I(0) = 0$ such that for all $k \in \mathbb{N}$,

$$\begin{aligned}
p(t \wedge \tau_k) \le & \int_0^t 1_{[0,\tau_k)(s)} |\mathbf{F}_k(u_k(s)) - \mathbf{F}_k(\widetilde{u}_k(s))|_{H^{-1}}^2 \, ds \\
& + \int_0^t 1_{[0,\tau_k)(s)} |\mathbf{G}_k'(u_k(s))\mathbf{G}_k(u_k(s)) - \mathbf{G}_k'(u_k(s))\mathbf{G}_k(\widetilde{u}_k(s))|_{L^2}^2 \, ds \\
& + \int_0^t 1_{[0,\tau_k)(s)} |\mathbf{G}_k(u_k(s)) - \mathbf{G}_k(\widetilde{u}_k(s))|_{\mathcal{T}_2(\mathrm{K},L^2)}^2 \, ds + I(t \wedge \tau_k) \\
& \le 3C \int_0^t 1_{[0,\tau_k)}(s) |u_k(s) - \widetilde{u}_k(s)|_{H^1}^2 \, ds + I(t \wedge \tau_k) \\
& \le 3C \int_0^t p(s \wedge \tau_k) \, ds + I(t \wedge \tau_k), \ t \in [0,T].
\end{aligned}$$

Therefore, by taking the expectation and then applying the the Gronwall lemma, we infer that $p = 0$ on $[0,\tau_k]$ almost surely. In other words, $\mathbb{P}$ almost surely, $u_k = \widetilde{u}_k$ on $[0,\tau_k]$. Consequently, $\mathbb{P}$-a.s. $u_k(t,x) \in O$ and $u_k(t,x) = h(u_k(t,x))$ for $x \in \mathbb{S}^1$ and $t \le \tau_k$. Hence, by Corollary 1.1 (or Lemma 1.2), $\mathbb{P}$-a.s. $u_k(t,x) \in M$ for $x \in \mathbb{S}^1$ and $t \in [0,\tau_k]$. Therefore, $\tau_k \le \tau_k^3$ and so $\tau_k = \tau_k^1 \wedge \tau_k^2$. Finally, since $p = 0$ on $[0,\tau_k]$ we infer that $\tau_k^1 = \tau_k^2$. □

Remark 1.7. Although the process $u_k - \widetilde{u}_k$ is H^1-valued, there is no error in considering the L^2 norm of it (and not the H^1 norm) in order to prove that this process is equal to a 0 process. Moreover, we had to use the framework of Pardoux for the Gelfand triple $H^1 \subset L^2 \subset H^{-1}$ (and not the $H^2 \subset H^1 \subset L^2$ one) because we had to use the Lipschitz property of $\mathbf{F}_k$. We have implicitly used an embedding $L^1 \subset H^{-1}$.

The same comments apply to the proof of Proposition 1.7 below.

In the second part of this subsection we will show that the approximate solutions extend each other. To be precise we will prove the following result.

Proposition 1.7. *Let* $k \in \mathbb{N}$. *Then* $u_{k+1}(t,x,\omega) = u_k(t,x,\omega)$ *on* $x \in \mathbb{S}^1$, $t \leq \tau_k(\omega)$, *and* $\tau_k(\omega) \leq \tau_{k+1}(\omega)$ *almost surely.*

Proof. Define a process p as before by formula (1.38). As in the proof of Proposition 1.6, we apply the Itô Lemma from.[58] Since $p(0) = 0$ we can find continuous martingale I satisfying $I(0) = 0$ such that for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned} p(t\wedge\sigma_k) \leq & \int_0^t 1_{[0,\sigma_k)}(s)|\mathbf{F}_{k+1}(u_{k+1}(s)) - \mathbf{F}_k(u_k(s))|^2_{L^2}\,ds \\ & + \int_0^t 1_{[0,\sigma_k)}(s)|1_{[0,\tau_{k+1})}\mathrm{tr}_{\mathrm{K}}[\mathbf{G}'_{k+1} \circledast \mathbf{G}_{k+1}(u_{k+1}(s))] \\ & - \mathrm{tr}_{\mathrm{K}}[\mathbf{G}'_k \circledast \mathbf{G}_k(u_k(s))]|^2_{L^2}\,ds \\ & + \int_0^t 1_{[0,\sigma_k)}(s)|\mathbf{G}_{k+1}(u_{k+1}(s)) - \mathbf{G}_k(u_k(s))|^2_{\mathcal{T}_2(\mathrm{K},L^2)}\,ds + I(t\wedge\sigma_k), \end{aligned}$$

where $\sigma_k := \tau_k \wedge \tau_{k+1}$. Since for $s \in [0,\sigma_k)$, $\mathbf{F}_k(u_k(s)) = \mathbf{F}(u_k(s)) = \mathbf{F}_{k+1}(u_{k+1}(s))$ and similarly, $\mathbf{G}_k(u_k(s)) = \mathbf{G}(u_k(s)) = \mathbf{G}_{k+1}(u_{k+1}(s))$, by the Lipschitz continuity of the functions $\mathbf{F}_{k+1}$, $\mathbf{F}_{k+1}$ and $\mathbf{G}'_{k+1} \circledast \mathbf{G}_{k+1}$ we infer that for some constant $C > 0$,

$$p(t\wedge\sigma_k) \leq C\int_0^t 1_{[0,\sigma_k)}(s)p(s)\,ds + I(t\wedge\sigma_k) = C\int_0^{t\wedge\sigma_k} p(s\wedge\sigma_k)\,ds.$$

Hence by the Gronwall Lemma we infer that $p = 0$ on $[0,\sigma_k]$. This implies that τ_k, $\tau_k \leq \tau_{k+1}$. Indeed, if $|\xi|_{H^1} > k+1$ then $\tau_{k+1} = \tau_k = 0$ and if if $k < |\xi|_{H^1} \leq k+1$ then $\tau_{k+1} > 0$ and $\tau_k = 0$. Thus, one can assume that $|\xi|_{H^1} \leq k$. If τ_{k+1} were smaller than τ_k then by the just proved property we would have $u_k(t) = u_{k+1}(t)$ for $t \in [0,\tau_{k+1}]$. Hence $|u_k(0)|_{H^1} \leq k$ and $|u_k(\tau_{k+1})|_{H^1} \geq k+1$ and therefore we can find $\bar{t} \in [0,\tau_{k+1})$ such that $|u_k(\bar{t})|_{H^1} = k + \frac{1}{2}$. This implies that $\tau_k \leq \bar{t}$ and this contradicts the assumption that $\tau_{k+1} < \tau_k$. The proof is complete. □

By Proposition 1.7 the sequence $(\tau_k)_{k=1}^\infty$ of stopping times is non-decreasing and so the limit of (τ_k) exists. We denote it by τ, i.e. $\tau = \lim_{k\to\infty}\tau_k$. Moreover, we can define a process $\tilde{u}(t,x)$, $t \in [0,\tau)$, $x \in \mathbb{S}^1$ by $\tilde{u}(t,x,\omega) = u_k(t,x,\omega)$ provided k is so large that $t \in [0,\tau_k(\omega))$. Note that $\tilde{u}(t,\cdot) \in H^1$.

In the following subsection we will show that $\tau = T$ $\mathbb{P}$ almost surely.

1.3.4. *No explosion for approximate solutions*

In this final subsection we will show that the maximal local solution constructed in the previous subsection is a global solution. We begin with proving that the local maximal solution is a global one.

Proposition 1.8. $\tau = T$ *almost surely.*

Proof. We first notice that we have, for $t \in [0,T]$,

$$u_k(t) = \xi - \int_0^t Au_k(s)\,ds + \int_0^t \mathbf{F}_k(u_k(s))\,ds + \int_0^t [\mathbf{G}_k' \circledast \mathbf{G}_k(u_k(s))]\,ds + \int_0^t \mathbf{G}_k(u_k(s))\,dW(s). \tag{1.39}$$

By applying the Itô Lemma from[58] and Lemma 1.3 we can find a continuous local martingale J_0 such that for $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\frac{1}{2}|\nabla u_k(t)|^2 + \int_0^t 1_{[0,\tau_k)}(s)\langle Au_k(s), Au_k(s)\rangle\,ds &= \frac{1}{2}|\nabla\xi|^2 + J_0(t)\\ &+ \int_0^t 1_{[0,\tau_k)}(s)\langle Au_k(s), \mathbf{F}_k u_k(s)\rangle\,ds\\ &+ \int_0^t 1_{[0,\tau_k)}(s)\langle \nabla u_k(s), \nabla\mathbf{G}_k'(u_k(s))\mathbf{G}_k(u_k(s))u_k(s)\rangle\,ds,\end{aligned} \tag{1.40}$$

where the norms and the scalar product are those from the $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^d)$ space. Note the following fundamental property. If $u \in D(A)$ then, see (1.4),

$$\langle -\Delta u(x) + F(u(x)), F(u(x))\rangle = 0, \quad \text{for a.a. } x \in \mathbb{S}^1. \tag{1.41}$$

Since for $s \in [0,\tau_k)$, $\mathbf{F}_k(u_k(s)) = \mathbf{F}(u_k(s))$, in view of identity (1.41), equality (1.40) can be rewritten as

$$\begin{aligned}\frac{1}{2}|\nabla u_k(t\wedge\tau_k)|^2 + \int_0^t 1_{[0,\tau_k)}(s)|Au_k(s) - \mathbf{F}_k(u_k(s))|^2\,ds - \frac{1}{2}|\nabla\xi|^2 - J_0(t)\\ = \int_0^t 1_{[0,\tau_k)}(s)\langle \nabla u_k(s), \mathrm{tr}_{\mathrm{K}}[\nabla\mathbf{G}_k'(u_k(s))\mathbf{G}_k(u_k(s))u_k(s)]\rangle\,ds,\\ \le C\int_0^t 1_{[0,\tau_k)}(s)\big[1 + |\nabla u_k(s)|^2\big]\,ds\\ = C\int_0^{t\wedge\tau_k}\big[1 + |\nabla u_k(s\wedge\tau_k)|^2\big]\,ds.\end{aligned} \tag{1.42}$$

Hence, for each $j \in \mathbb{N}$ there exists a constant K_j such that with $B_j = \big\{\omega \in \Omega : |\nabla\xi(\omega)|^2_{L^2} \le j\big\}$, one has, by the Gronwall Lemma,

$$\mathbb{E}\,1_{B_j}[1 + |\nabla u_k(t\wedge\tau_k)] \le K_j, \qquad t \in [0,T], \quad j \in \mathbb{N}. \tag{1.43}$$

Let us now fix $t \in [0,T)$. Then, since $1_{\{\tau_k \le t\}}|u_k(\tau_k)|_{\mathcal{H}_{r-\tau_k}} \ge k1_{\{\tau_k\le t\}}$, we infer that

$$\log(1+k^2)\mathbb{P}\left(\{\tau_k \le t\} \cap B_j\right) \le \mathbb{E}\,1_{B_j}q(t\wedge\tau_k) \le C_{r,j}. \tag{1.44}$$

Since $\tau_k \nearrow \tau$ as $k \to \infty$, from (1.44) we infer that for all $t \in [0,T)$, $j \in \mathbb{N}$, $\mathbb{P}\left(\{\tau \le t\} \cap B_j\right) = 0$ what in turn implies that $\tau = T$ almost surely. This completes the proof. $\square$

1.4. Stochastic geometric wave equations

Keeping the notation from Section 1.2, we let M be a compact Riemannian manifold isometrically embedded in some ambient Euclidean space $\mathbb{R}^d$, we let $S_p : T_pM \times T_pM \to N_pM$ be the associated second fundamental form and $\mathbf{D}$ is the generic symbol denoting induced connections on pull-back bundles relative to given mappings. We also use H^k_{loc} to denote the local Sobolev spaces $W^{k,2}_{loc}(\mathbb{R}^m;\mathbb{R}^d)$ and we assume that every filtration $(\mathscr{F}_t)$ on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ is such that $\mathscr{F}_0$ contains all $\mathbb{P}$-negligible sets in $\mathscr{F}$. Finally, we use the short notation u_y to denote the partial derivatives $\partial_y u = \frac{\partial u}{\partial y}$.

1.4.1. *Deterministic theory*

The geometric wave equation has *an intrinsic form* (i.e. independent of the ambient space)

$$\mathbf{D}_t u_t = \sum_{j=1}^{m} \mathbf{D}_{x_j} u_{x_j} \tag{1.45}$$

or, equivalently, in local coordinates

$$u^i_{tt} - \Delta u^i = \sum_{l=1}^{\dim M}\sum_{k=1}^{\dim M} \Gamma^i_{lk}(u)(-u^l_t u^k_t + \sum_{j=1}^{m} u^l_{x_j} u^k_{x_j}), \qquad i \in \{1,\dots,\dim M\} \tag{1.46}$$

where $\{\Gamma^i_{lk}\}$ are the Christoffel symbols, or, equivalently, *an extrinsic form* (i.e. dependent on the ambient space)

$$u_{tt} = \Delta u + S_u(u_t,u_t) - \sum_{j=1}^{m} S_u(u_{x_j},u_{x_j}) \tag{1.47}$$

where $u : \mathbb{R}\times\mathbb{R}^m \to M$ is a differentiable mapping. To get a better insight, we mention that the equations (1.45) - (1.47) are equivalent to

$$\mathbf{P}_u(u_{tt} - \Delta u) = 0$$

where $\mathbf{P}_p : \mathbb{R}^d \to T_pM$ is the orthogonal projection of $\mathbb{R}^d$ onto the tangent space of M at a point $p \in M$ - that is, the geometric wave equation is just the "projected" classical wave equation. Finally, we may also interpret geometric wave equations as the Euler-Lagrange equations for the Lagrangian

$$\int_{\mathbb{R}\times\mathbb{R}^m} \left\{ -|u_t|^2_{T_uM} + \sum_{j=1}^{m} |u_{x_j}|^2_{T_uM} \right\} dt\,dx.$$

Example 1.1. If M is a unit sphere in a Euclidean space of dimension greater or equal 2 with the Riemannian structure inherited from the ambient space then the second fundamental form satisfies $S_p(\xi,\nu) = -\langle\xi,\nu\rangle p$ for $\xi,\nu \in T_pM$, $p \in M$ and the equation (1.47) has the form

$$u_{tt} = \Delta u + (|\nabla u|^2 - |u_t|^2)u, \qquad |u| = 1.$$

Geometric wave equations arise in many fields of the modern theoretical physics - for instance, in the analysis of the more difficult hyperbolic Yang-Mills equations either as a special case or as an equation for certain families of gauge transformations, and also in general relativity for spacetimes with two Killing vector fields.

1.4.2. *Historical remarks*

The reader is kindly referred to nice surveys in[63] and[65] for various aspects of these equations, from which we select just a few results concerning existence and uniqueness. Namely, it is known that (1.45) has a unique global (strong) solution in $H^2_{loc} \times H^1_{loc}$ on the Minkowski space $\mathbb{R}^{1+1}$, i.e. if the space dimension $m = 1$ by,[34],[3646] and[62] for every compact Riemannian manofld M. This result was further extended by[67] and[63] to cover initial conditions from $H^1_{loc} \times L^2_{loc}$ on the Minkowski space $\mathbb{R}^{1+1}$, thus obtaining existence of unique weak solutions in $H^1_{loc} \times L^2_{loc}$ if $m = 1$. Climbing the ladder further, the tence of weak solutions in $H^1_{loc} \times L^2_{loc}$ was proved in the Minkowski space $\mathbb{R}^{1+2}$ for general target manifolds in,[51] leaving the problem of uniqueness open. In higher spatial dimensions $m \geq 3$ solutions may blow up or be non-unique even for smooth initial data and for a large class of target manifolds including convex manifolds or manifolds with negative sectional curvature, as shown e.g. in[16] and.[62] On the other hand, there is still a plenty of positive existence results in higher dimensions when additional assumptions are imposed, see e.g.[18,20,45,64] and especially the work[28] which yields the existence of global weak $H^1_{loc} \times L^2_{loc}$-valued solutions on any Minkowski space $\mathbb{R}^{1+m}$ provided the target manifold is a compact Riemannian homogeneous space (for instance the unit sphere).

1.4.3. *Objectives*

The aim of this part of the paper is to survey the recent achievements in the field of stochastic geometric wave equations and explain their principles. It should be noted that this area is ints infancy and few results are available so far.

1.5. Spatially homogeneous Wiener process

Random perturbations of wave equations in flat spaces have been predominantly modelled by spatially homogeneous Wiener processes for various physically motivated reasons (see e.g.[60]) that, in our opinion, remain valid for stochastic geometric wave equations, and that is why we consider them here as well. These perturbations correspond to centered Gaussian random fields $(\mathcal{W}(t,x) : t \geq 0,\, x \in \mathbb{R}^d)$ satisfying

$$\mathbb{E}\, \mathcal{W}(t,x)\mathcal{W}(s,y) = (t \wedge s)\Gamma(x-y), \qquad t,s \geq 0, \quad x,y \in \mathbb{R}^m \tag{1.48}$$

for some function or even a distribution Γ called *the spatial correlation* of $\mathcal{W}$.

Following[60] (which we recommend as a useful survey of properties and examples of spatially homogeneous Wiener processes) let μ be a finite symmetric measure on $\mathbb{R}^m$, that we will call *a spectral measure*, and let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis. A spatially homogeneous Wiener process with spectral measure μ may be introduced in two equivalent ways. The first one is to think of a centered Gaussian process $(\mathcal{W}(t,x) : t \geq 0, x \in \mathbb{R}^m)$ such that $(\mathcal{W}(t,x) : t \geq 0)$ is an $(\mathscr{F}_t)$-Wiener process for every $x \in \mathbb{R}^m$ and (1.48) holds for $\Gamma : \mathbb{R}^m \to \mathbb{R}$ which is the Fourier transform of $(2\pi)^{-\frac{1}{2}}\mu$. Another way is to consider an $(\mathscr{F}_t)$-Wiener process in the space of tempered distributions on $\mathbb{R}^m$ satisfying

$$\mathbb{E}\left\{\langle W(s), \varphi_0\rangle\langle W(t), \varphi_1\rangle\right\} = \min\{s,t\}\langle\widehat{\varphi}_0, \widehat{\varphi}_1\rangle_{L^2(\mu)}, \qquad t,s \geq 0, \quad \varphi_0, \varphi_1 \in \mathscr{S}_{\mathbb{R}},$$

where $\mathscr{S}_{\mathbb{R}}$ is the real Schwartz space of smooth rapidly decreasing real functions on $\mathbb{R}^m$. The equivalence between $\mathcal{W}$ and W is given by the formula (see e.g. page 190 in[60])

$$\langle W(t), \varphi\rangle = \int_{\mathbb{R}} \mathcal{W}(t,x)\varphi(x)\,dx, \qquad t \geq 0, \quad \varphi \in \mathscr{S}_{\mathbb{R}}.$$

The following proposition describes the reproducing kernel Hilbert space (RKHS) of a spatially homogeneous Wiener process and some of its properties (see Proposition 1.2 in[60] and Lemma 1 in[53]).

Proposition 1.9. *Let W be a spatially homogeneous Wiener process with a finite spectral measure μ. Then the reproducing kernel Hilbert space of W (denoted by H_μ) is described as*

$$H_\mu = \{\widehat{\psi\mu} : \psi \in L^2_{\mathbb{C}}(\mathbb{R}^m, \mu), \overline{\psi(x)} = \psi(-x)\}, \qquad \langle\widehat{\psi_0\mu}, \widehat{\psi_1\mu}\rangle_{H_\mu} = \langle\psi_0, \psi_1\rangle_{L^2(\mu)},$$

H_μ is continuously embedded in the space of real continuous bounded functions on $\mathbb{R}^m$ and

$$\|\xi \mapsto h\xi\|_{\mathcal{T}_2(H_\mu, L^2(\mathbb{R}^m))} = \mathbf{c}\|h\|_{L^2(\mathbb{R}^m)}, \qquad h \in L^2_{\mathbb{R}}(\mathbb{R}^m) \tag{1.49}$$

holds for $\mathbf{c} = c[\mu(\mathbb{R}^m)]^{\frac{1}{2}}$ where $c \in \mathbb{R}_+$ is a constant.

Corollary 1.4. *A spatially homogeneous $(\mathscr{F}_t)$-Wiener process W with a finite spectral measure μ is a cylindrical $(\mathscr{F}_t)$-Wiener process on H_μ. Thus, if K is a Hilbert space and ψ an $(\mathscr{F}_t)$-progressively measurable process with paths in $L^2_{loc}(\mathbb{R}_+; \mathcal{T}_2(H_\mu, K))$ a.s. then the stochastic integral $\int_0^{\cdot} \psi\,dW$ is understood in the classical sense as a K-valued continuous local $(\mathscr{F}_t)$-martingale (see e.g.[22]).*

1.6. Intrinsic and extrinsic solutions

1.6.1. *Posing the problem*

Let us observe that equation (1.45) is assumed to hold in $T_{u(t,x)}M$ for every $(t,x) \in \mathbb{R}^{1+m}$. Thus, any first order stochastic Itô perturbation of (1.45) by a spatially homogeneous Wiener process W acting by multiplication of some diffusion Y must have the form

$$\mathbf{D}_t u_t = \sum_{j=1}^{m} \mathbf{D}_{x_j} u_{x_j} + Y_u(u_t, \nabla u)\,dW \tag{1.50}$$

where $Y_p : (T_pM)^{m+1} \to T_pM$ is given. Contrary to the deterministic equation (1.45), where one can express its terms in local coordinates, the stochastic equation (1.50) is purely formal and two natural rigorous definitions of its solution were proposed in.[11] In general, minimal requirements upon a reasonable solution lead us to look for

(**Sol**) a progressively measurable process $u : \mathbb{R}_+ \times \mathbb{R}^m \times \Omega \to M$ with $(H^1_{loc}, \tau_{\text{weak}})$-continuous paths continuously differentiable in $(L^2_{loc}, \tau_{\text{weak}})$.

At this point, there are two possibilities to give a meaning to equation (1.50):

1.6.2. *Extrinsic form*

First possibility is to consider *an extrinsic equation* by which we mean the stochastic partial differential equation

$$u_{tt} = \Delta u + S_u(u_t, u_t) - \sum_{j=1}^{m} S_u(u_{x_j}, u_{x_j}) + Y_u(u_t, \nabla u)\, dW \tag{1.51}$$

in the ambient space $\mathbb{R}^d$ in the classical sense, i.e. when the equation

$$\begin{aligned}\langle u_t(t), \varphi\rangle = \langle u_t(0), \varphi\rangle + \int_0^t \langle u(s), \Delta\varphi\rangle\, ds + \int_0^t \langle Y_u(u_t, \nabla u)\, dW, \varphi\rangle \\ + \int_0^t \langle S_{u(s)}(u_t(s), u_t(s)) - \sum_{j=1}^{m} S_{u(s)}(u_{x_j}(s), u_{x_j}(s)), \varphi\rangle\, ds\end{aligned} \tag{1.52}$$

holds almost surely for every $t \in \mathbb{R}_+$ and every smooth compactly supported $\varphi : \mathbb{R}^m \to \mathbb{R}^d$, assuming implicitly convergence of the integrals in (1.52).

Remark 1.8. The adjective *"extrinsic"* is related to the presence of the second fundamental form S in the equation (1.52) which is a geometrically extrinsic object.

1.6.3. *Intrinsic form*

The second possibility is to consider *an intrinsic equation* when the equation (1.50) is formally multiplied by $X(u)$ for every smooth vector field X on M, developing (1.50) further to the stochastic partial differential equation

$$\begin{aligned}d\,\langle u_t, X(u)\rangle_{T_uM} = \sum_{j=1}^{m} \Big[\partial_{x_j}\langle u_{x_j}, X(u)\rangle_{T_uM} - \langle u_{x_j}, \nabla_{u_{x_j}} X\rangle_{t_uM}\Big]\, dt \\ + \langle u_t, \nabla_{u_t} X\rangle_{T_uM}\, dt + \langle Y_u(u_t, \nabla u)\, dW, X(u)\rangle_{T_uM},\end{aligned} \tag{1.53}$$

i.e.

$$\begin{aligned}\langle\langle u_t(t), X(u(t))\rangle_{T_{u(t)}M}, \varphi\rangle &= \langle\langle u_t(0), X(u(0))\rangle_{T_{u(0)}M}, \varphi\rangle \\ &+ \int_0^t \langle\langle u_t(s), \nabla_{u_t(s)}X\rangle_{T_{u(s)}M}, \varphi\rangle\, dt \\ &- \int_0^t \sum_{j=1}^m \langle\langle u_{x_j}(s), X(u(s))\rangle_{T_{u(s)}M}, \varphi_{x_j}\rangle \\ &- \int_0^t \sum_{j=1}^m \langle\langle u_{x_j}(s), \nabla_{u_{x_j}(s)}X\rangle_{t_{u(s)}M}, \varphi\rangle\, ds \\ &+ \int_0^t \langle\langle Y_{u(s)}(u_t(s), \nabla u(s))\, dW, X(u(s))\rangle_{T_{u(s)}M}, \varphi\rangle. \quad (1.54)\end{aligned}$$

holding almost surely for every $t \in \mathbb{R}_+$, every smooth vector field X on M and for every smooth compactly supported $\varphi : \mathbb{R}^m \to \mathbb{R}^d$, assuming again implicitly convergence of the integrals in (1.54).

1.6.4. *Extrinsic vs. Intrinsic*

The proof of the following theorem can be found in[11] for strong solutions and in[14] for weak solutions.

Theorem 1.2. *Let a process u satisfy the assumption* $(\mathbf{Sol})$. *Then u is an intrinsic solution if and only if u is an extrinsic solution.*

Remark 1.9. The reader will not be certainly surprised by a revelation that the implication from extrinsic to intrinsic is based on the Itô formula. The proof of the converse statement is however somewhat unexpectedly based on purely algebraic arguments.

1.7. Strong solutions in $H^2_{loc} \times H^1_{loc}$ on $\mathbb{R}^{1+1}$

The existence and uniqueness of strong solutions on the Minkowski space $\mathbb{R}^{1+1}$ was established already early in the 80s in[36] and[46] by classical PDE methods based on the fact that solutions of all wave equations, in particular those of the type (1.46), propagate at finite speed. This local character of the equation in unbounded Euclidean domains enabled to work with the equation (1.46) in each patch of the manifold separately and glue the particular solutions to a unique global solution eventually. Simpler proofs were given later on in[34] and,[62] mostly profitting from the extrinsic form of the geometric wave equation (1.47) avoiding local coordinates. This approach was also used in[11] and[12] to prove the following result (after it had been observed that working in local coordinates with equation (1.46) in the stochastic case was nearly impossible).

Theorem 1.3. *Let $m = 1$, let $Y \in C^1$ be such that*

$$|Y(p,\xi,\eta)| + |\nabla_p Y(p,\xi,\eta)| \le C(1+|\xi|+|\eta|), \qquad |\nabla_\xi Y(p,\xi,\eta)| + |\nabla_\eta Y(p,\xi,\eta)| \le C$$

holds for every $\xi, \eta \in T_pM$, $p \in M$ *and let the spectral measure satisfy*

$$\int_{\mathbb{R}^m} (1 + |x|^2)\, \mu(dx) < \infty. \tag{1.55}$$

Then existence and uniqueness holds for the equation (1.50) *within the class of adapted processes* u *such that paths of* u *belong to* $C(\mathbb{R}_+; H^2_{loc}) \cap C^1(\mathbb{R}_+; H^1_{loc})$.

Remark 1.10. Let us observe that the fact that we solve equation (1.50) in the high energy space $H^2_{loc} \times H^1_{loc}$ (which is the state space for the pair (u, u_t)) requires a stronger assumption on the spectral measure (1.55). In particular, only finer spatially homogeneous noises are admissible and the constraint (1.55) seems to be inevitable.

Proof. There are mainly two ideas in the proof (presented in detail in[11] or[12]). We are solving the extrinsic equation (1.51) and we first observe that H^2_{loc} and H^1_{loc} are algebras (i.e. the multiplication is a continuous operation) embedded in the space of continuous functions by the Sobolev embedding theorem on one-dimensional domains (that is why we assumed $m = 1$). That means that the quadratic non-linearities in (1.51) are "locally Lipschitz" on the state space $H^2_{loc} \times H^1_{loc}$ and, consequently, we obtain unique local solution of (1.51). The second idea is to extend the nonlinearities in (1.51) to the ambient space in such a way that this local solution stays on the manifold. This is done by the trick with involutions on tubular neighbourhoods of M explained in Section 1.3.1. The final step of the proof - that the local solution does not explode - is based on the Lyapunov method and on particular properties of another suitable extension of the second fundamental form which appears in the terms of the Lyapunov formulas in such a fortunate form that it can be estimated by a linear function (though it is, prima facie, quadratic). Then non-explosion follows from the Gronwall lemma. □

1.8. Weak solutions in $H^1_{loc} \times L^2_{loc}$ on $\mathbb{R}^{1+1}$

Considering the existence and uniqueness of strong solutions of (1.45) in $H^2_{loc} \times H^1_{loc}$ for $m = 1$ satisfactory, the main motivation for studying weak solutions on $\mathbb{R}^{1+1}$ in the deterministic theory is to have solutions of the Cauchy problem (1.45) for less regular initial data, especially from the more natural basic energy space $H^1_{loc} \times L^2_{loc}$. In the stochastic case, there is also an additional interest in having solutions of the equation (1.50) for a larger class of rougher spatially homogeneous Wiener processes W with finite spectral measures μ that do not satisfy (1.55).

The deterministic equation (1.45) is known to have uniqe global weak solutions by[63] or[67] and here is its partial stochastic counterpart.

Definition 1.2. A system of mappings $\lambda = \{\lambda_p\}_{p \in M}$ is a continuous vector bundles homomorphism from TM to TM if $\lambda_p : T_pM \to T_pM$ is linear for every $p \in M$ and $M \to TM : p \mapsto \lambda_p X_p$ is continuous for every continuous vector field X on M.

Theorem 1.4. *Let* $m = 1$, *let* μ *be a finite spectral measure on* $\mathbb{R}$, *let* Z *be a continuous vector field on* M, *let* $\lambda^{(t)}, \lambda^{(x)}$ *be continuous vector bundles homomorphisms from* TM

to TM and let Θ be a Borel probability measure on $H^1_{loc} \times L^2_{loc}$. Then there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, a spatially homogeneous $(\mathscr{F}_t)$-Wiener process W with spectral measure μ and a $(\mathbf{Sol})$*-satisfying solution u of the equation*

$$\mathbf{D}_t u_t = \mathbf{D}_x u_x + (Z_u + \lambda_u^{(t)} u_t + \lambda_u^{(x)} u_x)\, dW \tag{1.56}$$

such that Θ is the law of $(u(0), u_t(0))$.

Three remarks about Theorem 1.4 are in place.

Remark 1.11. Firstly, Theorem 1.4 is an existence result where uniqueness is not addressed being left open (contrary to the deterministic equation (1.45) for which uniqueness is known to hold). Moreover, the existence in Theorem 1.4 is understood in the weak probabilistic sense, i.e. the stochastic basis and the Wiener process are parts of the solution, and paths of the solution are continuous in the weak topology only.

Remark 1.12. Secondly, the diffusion Y in (1.56) has to be, comparing to the assumptions in Theorem 1.3, of a particular multilinear form to be $(L^2_{loc}, \tau_{\text{weak}})$-continuous in the variables corresponding to the first order terms. Yet, it need not be C^1-smooth and mere continuity is sufficient.

Remark 1.13. Thirdly, the constraint (1.55) upon the spectral measure of the Wiener process in Theorem 1.3 is not present in Theorem 1.4 and therefore rougher noises are admissible.

Proof. Following,[14] the straightforward idea is to approximate the initial condition, the multilinear diffusion Y in (1.56) by suitable smooth nonlinearities Y^k and the Wiener process W by spatially homogeneous Wiener processes W^k with spectral measures μ^k satisfying (1.55). If properly and carefully done, we may indeed find strong solutions u^k of equations

$$\mathbf{D}_t u_t^k = \mathbf{D}_x u_x^k + Y_{u^k}^k(u_t^k, u_x^k)\, dW^k$$

by Theorem 1.3 such that laws of (u^k, u_t^k) are tight in the space of weakly continuous functions $C_w(\mathbb{R}_+; H^1_{loc} \times L^2_{loc})$ and, therefore, converge in law to a process (u, u_t) in the locally uniform weak topology of $H^1_{loc} \times L^2_{loc}$ by the Skorokhod-Jakubowski construction when convergence of measures is modelled by a convergence of random variables in probability, see.[41]

Unfortunately, neither nonlinear term in the extrinsic equation (1.51) nor in the intrinsic equation (1.53) allows us to pass in the limit to their corresponding counterparts; this is due to the fact that weak convergence of (u^k, u_t^k) to (u, u_t) does not imply convergence of the terms $S_{u^k(s)}(u_t^k(s), u_t^k(s))$, $S_{u^k(s)}(\partial_x u^k(s), \partial_x u^k(s))$ in (1.51) and $\langle u_t^k(s), \nabla_{u_t^k(s)} X|_{u^k(s)}\rangle$ in (1.53) to $S_{u(s)}(u_t(s), u_t(s))$ and $S_{u(s)}(\partial_x u(s), \partial_x u(s))$ and $\langle u_t(s), \nabla_{u_t(s)} X|_{u(s)}\rangle$ respectively in any topological sense (where X is a smooth test vector field).

This problem is resolved by a forced strengthening of weak convergence to strong convergence which is done by mollification by a smooth compactly supported density b, i.e.

$b * u^k$ converges to $b * u$ locally uniformly in the inductive limit topology of H^1_{loc}. This approach is suitably applicable only for the intrinsic equation (1.53) that must be modified though to *a pseudo-intrinsic equation*

$$\begin{aligned} d\,\langle u^k_t, X(b*u^k)\rangle = {} & \big[\partial_x\langle u^k_x, X(b*u^k)\rangle - \langle u^k_x, d_{b*u^k_x}X(b*u^k)\rangle\big]\ dt \\ & + \langle u^k_t, d_{b*u^k_t}X(b*u^k)\rangle\, dt + \langle S_{u^k}(u^k_x,u^k_x) - S_{u^k}(u^k_t,u^k_t), X(b*u^k)\rangle\, dt \\ & + \langle Y^k_{u^k}(u^k_t,u^k_x)\, dW^k, X(b*u^k)\rangle \end{aligned} \tag{1.57}$$

for every smooth test vector field X on M where $d_\xi X(p)$ is the derivative of X in the direction ξ at the point p. The additional term $\langle S_{u^k}(u^k_x,u^k_x) - S_{u^k}(u^k_t,u^k_t), X(b*u^k)\rangle$ in (1.57) is also tight in $C(\mathbb{R}_+)$ and converges in law (by the Skorokhod-Jakubowski construction) to a limit Q^X_b which tends to zero as $b \to \delta_0$ for every test X.

In the final step, u is identified with a weak solution (in the probabilistic sense) of the equation (1.56), i.e. a stochastic basis and a Wiener process are found for which u becomes a solution of (1.56). □

Remark 1.14. Since the proof takes place in the non-separable and non-metrizable space $C_w(\mathbb{R}_+; H^1_{loc}\times L^2_{loc})$ the Jakubowski version[41] of the Skorokhod theorem has to be applied.

1.9. Weak solutions in compact homogeneous spaces on $\mathbb{R}^{1+m}$

Existence results for geometric wave equations on higher-dimensional Minkowski spaces $\mathbb{R}^{1+m}$, $m \geq 2$ are available either for $m = 2$ and any target M, or for any dimension $m \geq 2$ within specific targets M (see Section 1.4.2 for references).

Whereas there is no available result for stochastic geometric wave equations on $\mathbb{R}^{1+2}$ for general targets, we present here the only existence theorem for (1.50) in any dimension when the target M is *a compact Riemannian homogeneous space*.

Definition 1.3. The target M is a compact Riemannian homogeneous space provided that M is a compact Riemannian manifold with a compact Lie group G which acts on M transitively by isometries.

Remark 1.15. In other words, Definition 1.3 means that there exists a smooth mapping

$$G \times M \to M : (g,p) \mapsto gp$$

such that $ep = p$ and $(g_0g_1)p = g_0(g_1p)$, where e is the unit element in G, holds for every $g_0, g_1 \in G, p \in M$, transitivity means that there exists $p_0 \in M$ such that $\{gp_0 : g \in G\} = M$ and "by isometries" means that each $M \to M : p \mapsto gp$ is an isometry for every $g \in G$.

Example 1.2. The unit sphere is the simplest example of a compact Riemannian homogeneous space.

Theorem 1.5. *Let $m \in \mathbb{N}$, let μ be a finite spectral measure on $\mathbb{R}^m$, let M be a compact Riemannian homogeneous space, let Z be a continuous vector field on M, let $\lambda^{(t)}$ be a continuous function on M, let $\lambda^{(x_1)}, \dots, \lambda^{(x_m)}$ be continuous vector bundles homomorphisms* *from TM to TM and let Θ be a Borel probability measure on $H^1_{loc} \times L^2_{loc}$. Then there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, a spatially homogeneous $(\mathscr{F}_t)$-Wiener process W with spectral measure μ and a $(\mathbf{Sol})$-satisfying solution u of the equation*

$$\mathbf{D}_t u_t = \sum_{j=1}^{m} \mathbf{D}_{x_j} u_{x_j} + \left[Z_u + \lambda_u^{(t)} u_t + \sum_{j=1}^{m} \lambda_u^{(x_j)} u_{x_j} \right] dW \tag{1.58}$$

such that Θ is the law of $(u(0), u_t(0))$.

Remark 1.16. Equation (1.58) is a multidimensional version of equation (1.56) and Remarks 1.11 - 1.13 remain valid here as well, except for the diffusion term $\lambda^{(t)}$ which is a general continuous vector bundles homomorphisms from TM to TM in (1.56) whereas it may be just a real function in (1.58).

Proof. Following,[13] the solution is found by a combination of a penalization method and of a compactness method. Since M is a homogeneous space, we may construct a suitable identifier of M in $\mathbb{R}^d$ as a "level set", i.e. a smooth function $F : \mathbb{R}^d \to \mathbb{R}_+$ such that $M = [F = 0]$ and F has further useful properties of technical nature that enable the whole proof to go through. The idea is to consider weak solutions of penalized SPDEs

$$\partial_{tt} u^k = \Delta u^k - k \nabla F(u^k) + f^k(u^k, \nabla_{(t,x)} u^k) + g^k(u^k, \nabla_{(t,x)} u^k)\, dW^k, \qquad k \in \mathbb{N}$$

where f^k and g^k approximate the drift and the diffusion in (1.58). We remark that the solutions u^k need not be M-valued. By the properties of F, it is shown that the sequence $\{u^k\}$ is tight in the space of weakly continuous functions $C_w(\mathbb{R}_+; H^1_{loc})$, however, the sequence $\{u^k_t\}$ is tight in a space $(L^\infty_{loc}(\mathbb{R}_+; L^2_{loc}), weak^*)$ where the paths are not determined uniquely (but just almost everywhere) and this is a problem since the values of the potential limit at fixed times $u_t(t)$ would not be well defined. This inconvenience is overcome by a trick based on the properties of homogeneous spaces, namely there exist matrices A^i such that the processes $\langle u^k_t, A^i u^k \rangle$ are tight in $C_w(\mathbb{R}_+; L^r_{loc})$ for some $r \in (1, 2)$, thus converge to a weakly continuous limit by which we can express u_t showing that, in fact, u_t is weakly continuous in L^2_{loc}.

There is again a repeated Skorokhod-Jakubowski construction of random variables converging in probability in the background, (see[41]).

Finally, u is identified as an M-valued weak solution (in the probabilistic sense) of the equation (1.58) by some geometrical properties of homogeneous spaces, i.e. a stochastic basis and a Wiener process are found and complemented to u to become a solution of (1.58). $\square$

Remark 1.17. The technical-geometrical properties used in the proof were developed from the techniques in.[28]

1.10. Energy estimates

Results on the existence and uniqueness of solutions of (1.50) presented in Sections 1.7 - 1.9 are always accompanied by a qualitative property of the solution describing the growth of the energy of the solution on backward cones. To keep the exposition at a reasonable length, we present only the energy estimate for solutions in homogeneous spaces (Theorem 1.5) remarking that analogous results for Theorem 1.3 and Theorem 1.4 are very similar and can be found in[11] and[14] while a general exposition on local energy estimates for wave equations was presented in.[54]

Towards this end, let $\mathbf{s}^2 = \max\{|Z_p|^2 : p \in M\}$ and define the energy on spatial cuts of a backward cone $K(x,T) = \{(s,y) : |y-x| \le T-s\}$

$$\mathbf{e}_{x,T}(t,u,v) = \int_{B(x,T-t)} \left\{ \frac{1}{2}|u(y)|^2 + \frac{1}{2}|\nabla u(y)|^2 + \frac{1}{2}|v(y)|^2 + \mathbf{s}^2 \right\} dy$$

for $x \in \mathbb{R}^m$, $0 \le t \le T$ and $(u,v) \in H^1_{loc} \times L^2_{loc}$.

Theorem 1.6. *Under the assumptions in Theorem 1.5, there exists a weak solution u of the equation* (1.58) *such that*

$$\mathbb{E}\left\{ \mathbf{1}_A(u(0),u_t(0)) \sup_{s\in[0,t]} L(\mathbf{e}_{x,T}(s,u(s),u_t(s))) \right\} \le 4e^{Ct} \int_A L(\mathbf{e}_{x,T}(0,\cdot))\, d\Theta$$

holds for every $x \in \mathbb{R}^m$, $0 \le t \le T$, $A \in \mathscr{B}(H^1_{loc} \times L^2_{loc})$ and every nondecreasing function $L \in C(\mathbb{R}_+) \cap C^2(0,\infty)$ satisfying

$$tL'(t) + \max\{0, t^2 L''(t)\} \le \kappa L(t), \qquad t > 0$$

where the constant C depends only on the constant $\mathbf{c}$ *from Proposition 1.9, on the constant κ and on the $L^\infty(M)$-norms of the nonlinearities $\lambda^{(t)}, \lambda^{(x_1)}, \dots, \lambda^{(x_m)}$.*

Remark 1.18. Typically, $L(t) = t^q$ for $q \in (0,\infty)$ or $L(t) = \log(t+1)$.

Remark 1.19. Besides the sub-exponential growth of the energy of the solution on backward cones, Theorem 1.6 also yields a pathwise estimation on the growth of the conditional expectation

$$\mathbb{E}\left[L(\sup_{s\in[0,t]} \mathbf{e}_{x,T}(s,u(s),u_t(s)))\,|(u(0),u_t(0)) = z \right] \le 4e^{Ct} L(\mathbf{e}_{x,T}(0,z)) \qquad \Theta-\text{a. e.}$$

1.11. Stochastic Landau-Lifschitz-Gilbert Equation

The Landau-Lifschitz-Gilbert equation is a fundamental equation of Micromagnetism proposed by Landau and Lifschitz in[47] and modified by Gilbert in.[33] Given a domain $D \subset \mathbb{R}^d$, $d \le 3$, filled in with a ferromagnetic material, the equation describes evolution of the magnetisation vector $u : D \to \mathbb{S}^2$. We will consider here the case when $D = [0,1]$, see[8] for the case $D \subset \mathbb{R}^3$. In its simplest version the equation takes form

$$\begin{cases} \partial_t u = \alpha u \times \Delta u + \beta\left(\Delta u + |\nabla u|^2 u\right) + (u \times h) \circ \dot{W} & t > 0,\ x \in (0,1), \\ \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0 & t > 0, \\ u(0,x) = u_0(x) & x \in [0,1], \end{cases} \tag{1.59}$$

where $h : [0,1] \to \mathbb{R}^3$ is a bounded measurable function and W is a one-dimensional Wiener process. Let us note that for $\alpha = 0$ equation (1.59) is a special case of equation (1.1) for the geometric heat flow of maps with values in $\mathbb{S}^2$ but the initial manifold is now an interval instead of a sphere $\mathbb{S}^1$ and the Neumann boundary conditions are dictated by the theory of micromagnetism. All the results presented below remain true for $\mathbb{S}^1$. If $\beta = 0$ then we obtain the so-called Heisenberg equation which is equivalent to the two-dimensional nonlinear Schrödinger equation. Precise values of the constants $\alpha \in \mathbb{R}$ and $\beta > 0$ are of physical importance but in this presentation we assume for simplicity $\alpha = \beta = 1$. The term $u \times \Delta u$ being non-dissipative the equation requires some modifications of the arguments used in previous sections to study the geometric heat flow. We start with a definition of weak martingale solutions. We will use the notation $\mathbb{L}^2$ for the space $L^2\left(0,,1;\mathbb{R}^3\right)$, $\mathbb{H}^1$ for the Sobolev space $H^{1,2}\left(0,1;\mathbb{R}^3\right)$ and so on. We will denote by A the selfadjoint extension of the operator $-\frac{\partial^2}{\partial x^2}$ with the domain

$$D(A) = \left\{ \phi \in \mathbb{H}^2 : \ \frac{\partial \phi}{\partial x}(x) = 0, \ x = 0,1 \right\}.$$

For $u \in D(A)$ and $\phi \in \mathbb{H}^1$ we have $u \times \Delta u \in \mathbb{L}^2$ and

$$\langle \phi, u \times Au \rangle = \langle Au, \phi \times u \rangle = \left\langle \frac{\partial u}{\partial x}, \frac{\partial (\phi \times u)}{\partial x} \right\rangle.$$

It is easy to see that if $u \in \mathbb{H}^1$ then $u \times Au \in \mathbb{H}^{-1}$.

Definition 1.4. A weak martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, u)$ to equation (1.59) consists of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the filtration satisfying the usual conditions, a one dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\geq 0}$, and a progressively measurable process $u : [0,\infty) \times \Omega \to \mathbb{L}^2$ such that:
(a) for all $T > 0$,

$$u(\cdot) \in C\left([0,T]; \mathbb{L}^2\right) \quad \mathbb{P} - a.s.,$$

(b) for every $T > 0$

$$\mathbb{E} \sup_{t \leq T} \left| \frac{\partial u}{\partial x} u(t) \right|^2_{\mathbb{L}^2} < \infty,$$

(c) for every $t \geq 0$ we have $|u(t,x)|_{\mathbb{R}^3} = 1$ $\ Leb \otimes \mathbb{P}$- a.e.,
(d) $u \times Au \in L^2\left(0,T; \mathbb{L}^2\right)$,
(e) For every $t \geq 0$

$$\begin{aligned} u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} u(s) \times Au(s) ds + \int_0^t e^{-(t-s)A} \left| \frac{\partial u}{\partial x} u(s) \right|^2 u(s) ds \\ + \int_0^t e^{-(t-s)A} (u(s) \times h) \circ dW(s). \end{aligned} \tag{1.60}$$

Theorem 1.7. *Assume that $u_0 \in \mathbb{H}^1$ and $h \in \mathbb{H}^{1,3}$. Then there exists a unique weak martingale solution to equation* (1.59)*. Moreover, for every $T > 0$ and $p \in [1, \infty)$*

$$\mathbb{E} \sup_{t \leq T} \left| \frac{\partial u}{\partial x} \right|^p_{\mathbb{L}^2} < \infty.$$

Proof. To show the existence we use the Galerkin approximations and the compactness method in the same way as in.[8] The uniqueness is proved in.[9] □

Theorem 1.8. *Assume that $u_0 \in \mathbb{H}^{1,2}$ and $h \in \mathbb{H}^{1,3}$. Then the weak martingale solution u has the property that $u(t) \in D(A)$ for almost every $t > 0$ and for every $T > 0$*

$$\mathbb{E} \int_0^T |Au(t)|^2_{\mathbb{L}^2} dt < \infty.$$

Proof. We will only sketch the arguments that are presented in detail in.[9] We write (1.60) in the form

$$u(t) = \sum_{i=1}^4 u_i(t),$$

and consider each term separately. Clearly, we have

$$\int_0^T |Ae^{-tA} u_0|^2_{\mathbb{L}^2} dt < \infty, \tag{1.61}$$

and by the maximal regularity

$$\mathbb{E} \int_0^T |Au_2(t)|^2_{\mathbb{L}^2} dt \leq \mathbb{E} \int_0^T |u(t) \times \Delta u(t)|^2_{\mathbb{L}^2} dt < \infty. \tag{1.62}$$

We will consider u_3. Let us recall first that the semigroup $\left(e^{-tA}\right)$ is ultracontractive. More precisely, if $1 \leq p \leq q \leq \infty$ then

$$\left|e^{-tA} f\right|_{\mathbb{L}^q} \leq \frac{1}{t^{\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}} |f|_{\mathbb{L}^p}, \tag{1.63}$$

and for a certain $C > 0$

$$\left| \frac{\partial}{\partial x} e^{-tA} f \right|_{\mathbb{L}^q} \leq \frac{C}{t^{\frac{1}{2} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}} |f|_{\mathbb{L}^p}. \tag{1.64}$$

Let $f(t) = \left| \frac{\partial u(t)}{\partial x} \right|^2 u(t)$. Then for every $t \geq 0$ we have $f(t) \in \mathbb{L}^1$ and therefore using (1.64) with $p = 1$ and $q = 4$ we find that for any $s \leq t \leq T$

$$\left| \frac{\partial}{\partial x} e^{-(t-s)A} f(s) \right|_{\mathbb{L}^4} \leq \frac{C}{(t-s)^{\frac{7}{8}}} \left| \frac{\partial u}{\partial x} u(s) \right|^2_{\mathbb{L}^2}. \tag{1.65}$$

Therefore, invoking (1.65) we obtain

$$\begin{aligned}\mathbb{E}\int_0^T \left|\frac{\partial}{\partial x}u_2(t)\right|_{\mathbb{L}^4}^4 dt &= \mathbb{E}\int_0^T \left(\int_0^t \left|\frac{\partial}{\partial x}e^{-(t-s)A}f(s)\right|_{\mathbb{L}^4} ds\right)^4 dt \\ &\le C\mathbb{E}\sup_{t\le T}\left|\frac{\partial}{\partial x}u(t)\right|_{\mathbb{L}^2}^8 \int_0^T\left(\int_0^t \frac{ds}{(t-s)^{7/8}}ds\right)^4 dt \\ &\le C_1\mathbb{E}\sup_{t\le T}\left|\frac{\partial}{\partial x}u(t)\right|_{\mathbb{L}^2}^8 < \infty.\end{aligned} \tag{1.66}$$

This estimate yields $f \in L^2\left(0,T;\mathbb{L}^2\right)$ hence

$$\mathbb{E}\int_0^T |Au_3(t)|_{\mathbb{L}^2}^2\, dt < \infty. \tag{1.67}$$

Since

$$\left|\frac{\partial}{\partial x}(u\times h)\right|^2 \le a\left|\frac{\partial}{\partial x}u\right|^2 + b, \tag{1.68}$$

we find easily that

$$\mathbb{E}\int_0^T |Au_4(t)|_{\mathbb{L}^2}^2\, dt = \mathbb{E}\int_0^T \left|\frac{\partial}{\partial x}(u(t)\times h)\right|_{\mathbb{L}^2}^2 dt < \infty. \tag{1.69}$$

Finally, combining (1.61), (1.62), (1.67), and (1.69) we conclude the proof. □

Remark 1.20. Let us consider equation (1.1) for the heat flow of $\mathbb{S}^2$-valued maps defined on $[0,1]$. In this case the argument leading to (1.66) can be easily modified to show that for every $p>1$

$$\int_0^T \left|\frac{\partial}{\partial x}u(t)\right|_{\mathbb{L}^p}^p dt < \infty,$$

and therefore

$$\int_0^T |Au(t)|_{\mathbb{L}^p}^p dt < \infty,$$

for all $p>1$. The last estimate yields $u(t)\in C^{1+\alpha}\left([0,1];\mathbb{R}^3\right)$.

The next results is an immediate consequence of Theorem 1.8.

Corollary 1.5. *Let u be a weak martingale solution to equation* (1.59)*. Then for every $t\ge 0$*

$$u(t) = u_0 + \int_0^t Au(s)ds + \int_0^t \left|\frac{\partial u(s)}{\partial x}\right|^2 ds + \int_0^t (u(s)\times h)\circ dW(s),$$

where the first two integrals are the Bochner integrals in $\mathbb{L}^2$ and the last one is the Stratonovitch integral in $\mathbb{L}^2$.

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