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GENERALIZED SEMIFLOWS AND CHAOS IN MULTIVALUED DYNAMICAL SYSTEMS

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This contribution addresses a possible description of the chaotic behavior in multivalued dynamical systems. For the multivalued system formulated via differential inclusion the practical conditions on the right-hand side are derived to guarantee existence of a solution, which leads to the chaotic behavior. Our approach uses the notion of the generalized semiflow but it does not require construction of a selector on the set of solutions. Several applications are provided by concrete examples of multivalued dynamical systems including the one having a clear physical motivation.

Keywords: Multivalued dynamical systems; chaos; differential inclusions.

1. Introduction

Satisfactory solution of numerous particular problems in mechanics, engineering sciences and other related fields are often heavily influenced by non-smooth phenomena. As a practical examples, let us mention, e.g., the noise of a squeaking chalk on a black-board or, sometimes more pleasantly, the sounds of stringed instruments like a violin. More relevant applications include noise generation in railway wheels, the drilling machines, etc. Physically speaking, these effects often are due to the fact that there are rigid bodies, which are in contact (they "stick"), whereas these contact phases are interrupted by "slip" phase during which one of the bodies moves relative to another. In addition to such behavior mainly induced by friction, there may also be impacts between different parts of the system.

From mathematical point of view, the resulting models are dynamical systems whose right-hand sides are non-smooth or even discontinuous. In many cases the solution has to satisfy additional restrictions that frequently appear in the form of inequality constraints. Since many concepts from classical dynamical systems theory do rely on the smoothness of the underlying system or (semi-) flow, it is necessary to cover non-smooth dynamical systems as well. Such a generalization is not a trivial task.

Recently, attempts have been made to extend the theory of chaos to differential equations with discontinuous right-hand sides. For example, planar discontinuous differential equations are investigated in Refs. 1–2, piecewise linear three-dimensional discontinuous differential equations are investigated in Refs. 3–4 and weakly discontinuous systems are studied in Refs. 5–7. Melnikov type analysis is also presented for discontinuous differential equations in Refs. 8–13. An overview of some aspects of chaotic dynamics in hybrid systems is given in Ref. 14. A survey of controlling chaotic differential equations is presented in Ref. 15. The switchability of flows of general discontinuous differential equations is discussed in Refs. 16–18. Planar discontinuous differential equations are investigated in Refs. 19–20 using analytic and numeric approaches. Periodic and almost periodic solutions of discontinuous differential equations are considered in Refs. 21–25. In Ref. 26 bifurcations of bounded solutions from homoclinic orbits are investigated for time perturbed discontinuous differential equations in any finite dimensional space. It was anticipated that under the conditions of Ref. 26 not only the existence of bounded solutions on R, but also chaotic solutions could occur.

Very deep results concerning analysis of chaos in the case of multivalued systems were published also in Ref. 27.

The goal of this paper is to generalize the results dealing with chaotic behavior of single valued flows to the case of the so-called *generalized semi-flows*, see precise definition later on. Briefly, the main result of this paper claims that there exists *at least one* trajectory of the generalized semi-flow such that for arbitrary covering of the solution set, possessing certain surjection-like property defined later on, such a trajectory connects mutually all subsets of that covering in a finite time. As a consequence, the trajectory of the generalized dynamical semi-flow can be described (in fact, coded) via the index set of the covering of the solution set and, subsequently, the methods of symbolic dynamics can be used to analyze its dynamics. To compare with Ref. 27, our results presented in the current paper will show the existence of chaotic behavior as well. Moreover, no *a priori* hypothesis concerning the existence of homo/heteroclinical trajectory will be needed here.

Recent theoretical result and techniques in Ref. 28 are now used in this paper. We will develop more specific conditions than those on Ref. 28 in order to analyse particular dynamical systems in a much more comfortable way.

The main results of this paper are collected in Theorems 1 - 4 in the third section. Here, Theorem 1 gives conditions on the right-hand side of a differential inclusion to guarantee the existence of the Filippov's solution that means the existence of an absolutely continuous solutions, compactness of the set of solutions and upper semicontinuity of the solution. Theorem 2 provides conditions on the right-hand side of the differential inclusion to guarantee that the solution set is globally compact and connected. Finally, Theorem 3 and Theorem 4 show conditions on the right-hand side of the differential inclusion to simplify verification of conditions of the previous theorems in the case when one deals with maximal monotone multivalued flow.

Several examples will illustrate these theoretical results.

The rest of the paper is organized as follows. The next section summarizes some preliminaries and terminology to be used later on. Main results are collected in Section 3, while illustrative examples are given in Section 4. The last short section draws conclusions and provides some outlooks for the future research.

2. Preliminaries

Some basic notions are repeated in this section. Interested reader is referred to Ref. 28 for further details.

Definition 1. A *generalized semiflow* $\mathcal{G}$ on (not necessarily complete) metric space X is a family of maps $\varphi : [0,\infty) \longrightarrow X$, called *solutions*, satisfying the hypotheses:

(H1) (*Existence*). For each $z \in X$ there exists at least one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$.

(H2) (*Translates of solutions are solutions*). If $\varphi \in \mathcal{G}$ and $\tau \geq 0$, then $\varphi^{\tau} \in \mathcal{G}$, where $\varphi^{\tau}(t) \triangleq \varphi(t+\tau), t \in [0,\infty)$

(H3) (*Concatenation*). If $\varphi, \psi \in \mathcal{G}, t \geq 0$, with $\psi(0) = \varphi(\tau)$ then $\theta \in \mathcal{G}$, where

$$\theta(t) \triangleq \begin{cases} \varphi(t) \text{ for } 0 \leq t \leq \tau, \\ \psi(t-\tau) \text{ for } \tau < t \end{cases}$$

(H4) (*Upper-semicontinuity with respect to initial data*). Consider the sequence of flows $\{\varphi_j\}_{j=1}^{\infty} \in \mathcal{G}$, with $\varphi_j(0) \to z$ as $j \to \infty$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

Remark. Let $\mathcal{G}$ be a generalized semiflow and let $E \subset X$. Define for $t \geq 0$

$$T(t)E \triangleq \{\varphi(t) | \varphi \in \mathcal{G} \; with \; \varphi(0) \in E\},$$

so that $T(t) : 2^X \longrightarrow 2^X$, where 2^X is the space of all subsets of X. It follows from (H2), (H3) that $\{T(t)\}_{t\geq 0}$ is a semigroup action on the powerset 2^X. Note that (H4) implies that $T(t)\{z\}$ is compact for each $z \in X, t \geq 0$.

Notation. The expression $\varphi(\cdot) \in \mathcal{G}(x)$ stands for the solution $\varphi(\cdot)$ that starts at $x \in X$.

If for each $z \in X$ there is exactly one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ then $\mathcal{G}$ is called a *semiflow*.

Definition 2. The generalized semiflow $\mathcal{G}$ is said to be *upper-semicompact* from X to $\mathcal{C}([0,\infty); X)$ ($\mathcal{C}$ means a space of continuous mappings from $[0,\infty)$ into X) if for any solution $\varphi_n \in X$ converging to $x \in X$ and for any generalized semiflow $\varphi_n(\cdot) \in \mathcal{G}$ starting at φ_n, there exists a subsequence of $\varphi_n(\cdot)$ converging to a generalized semiflow $\varphi(\cdot) \in \mathcal{G}$ uniformly on compact intervals.

Definition 3. Let D be a closed set and let us consider a sequence of nonempty closed subsets $S_n \subset D$, $n \in \mathbb{N} \cup 0$, $\mathcal{S} = \{S_n\}$, such that $S_n \cap S_{n+1} \neq \emptyset$. Let $\varphi(\cdot) \in \mathcal{G}(x)$ be a solution. We say that $\mathcal{S}$ *forms a* $\varphi(\cdot)$*-chain* when there exists a nondecreasing sequence of times $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots$ such that for all $n \geq 0$, for any $t \in [t_n, t_{n+1}], \varphi(t) \in S_n$ and $\varphi(t_{n+1}) \in S_{n+1}$.

Definition 4. Let D be a closed set and let us consider a sequence of nonempty closed subsets $S_n \subset D$ and we assume that there exists $T < +\infty$ such that for each nonnegative n and for each $z \in S_{n+1}$ there exists $x \in S_n$ with solution $\varphi_n(\cdot) \in S_n$ and exists $\tau \in [0, T)$ with $\varphi(\tau) = z$, then the system $\mathcal{S} = \{S_n\}$ is called to be $T-$ *surjective under* $\mathcal{G}$. When $T \longrightarrow +\infty$, then the system $\mathcal{S} = \{S_n\}$ is called to be *surjective under* $\mathcal{G}$

Definition 5. Let $D \subset X$ be a constrained set. A solution $\varphi(\cdot)$ is *locally positively D-invariant* when there exists $T > 0$ such that for each $t \in [0, T]$ we have $\varphi(t) \in D$. When $T = +\infty$ we call $\varphi(\cdot)$ *positively D-invariant*. When all $\varphi(\cdot) \in \mathcal{G}$ are (locally) positively D-invariant, we say that generalized semiflow $\mathcal{G}$ is (locally) positively D-invariant.

Definition 6. The generalized semiflow $\mathcal{G}$ possesses the *chaotic behaviour* on the compact set D, if for any its at most countable closed covering $\mathcal{S} = \{S_m\}_{m \in \mathcal{I}}$, $D \subseteq \bigcup_{m \in \mathcal{I}} S_m$, $\mathcal{I}$ being a suitable index set, and any sequence $\{m_0, m_1, \ldots, m_n, \ldots\} \subset \mathcal{I}$ there exists *at least one* solution $\varphi(\cdot) \in \mathcal{G}$ and a nondecreasing sequence $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots$ such that system $\overline{\mathcal{S}} := \{\mathcal{S}_m\}_{m=m_0,m_1,m_2\ldots}$ is surjective under $\varphi(\cdot) \in \mathcal{G}$ with $\varphi(t_i) \in \mathcal{S}_{m_i}, i = 0, 1, \ldots$.

Theorem A Ref.(28). *Let D be a compact subset. Assume:*

(1) generalized semiflow $\mathcal{G}$ is positively D-invariant and upper semicompact,
(2) let exists a covering $\mathcal{S}$ that is T-surjective under $\mathcal{G}$ for some $T < +\infty$.

Then the generalized semiflow $\mathcal{G}$ possesses the chaotic behaviour.

3. Main Results

In this section, we formulate the main results of our contribution. To start with, let us give the following definitions, cf. Ref.(29).

Definition 7. Let $J = [a, b] \subset R$. Then we denote by $L^1(J)$ the Banach space of Lebesgue integrable $\psi : J \to \overline{R} = R \cup \{-\infty, \infty\}$ with norm $|\psi|_1 = \int_J |\psi|\, dt$.

Definition 8. Given $X = R^n, J = [0, a] \subset R$, a closed $D \subset X$, a multivalued mapping $F : J \times D \to 2^X \backslash \{\emptyset\}$, we define a norm $\|F(t, x)\| = sup\, \{|y| : y \in F(t, x)\}$.

Definition 9. Tangent cone $T_D(x)$, $x \in D$, where $\rho(x, D) = inf_{d \in D}\, |x - d|$ is defined as follows
$T_D(x) = \left\{y \in X : \underline{lim}_{\lambda \to 0+} \lambda^{-1} \rho\,(x + \lambda y, D) = 0\right\}$.

Definition 10. We say that a map F from X into the subset of Y is $\sigma-$ *selectionable* if there exists a sequence of compact valued maps $\{F_n\}$ with closed graph satisfying

i $\forall n \geq 0,\ F_n$ has a continuous selection,
ii $\forall x \in X,\ F(x) = \bigcap_n F_n(x)$.

Let $\overline{conv}(K)$ be a closed convex hull of K, i.e., the smallest closed convex set containing K. Then we make a usage of the following theorem , Ref.(29):

Approximation Theorem. *Let X be a metric space, Y be a Banach space and F be a strict (i.e. a map, which domain $Dom(F) = \{x \in X | F(x) \neq \emptyset\} = X$) upper semicontinuous (u.s.c.) map from X to the closed convex subset of Y. It is $\sigma-$ selectionable. Actually, there exists a sequence of u.s.c. maps F_n from X to $\overline{conv}(F(x))$, which approximate F in the sense that for all $x \in X$*

i $\forall\, n \geq 0,\ \ F(x) \subset \ldots \subset F_{n+1}(x) \subset F_n(x) \subset \ldots \subset F_0(x)$,
ii $\forall\, \epsilon > 0\ \ \exists\ N(\epsilon, x)$ *such that* $\forall\, n \geq N(\epsilon, x), \quad F_n(x) \subset F(x) + \epsilon\, B$.

The maps F_n can be written in the following form:

$$\forall\, x \in X, \quad F_n(x) = \sum_{i \in J_n} \Psi_i^n(x) C_i^n$$

where the subsets C_i^n are closed and convex and where the functions Ψ_i^n form a locally Lipschitzian locally finite partition of unity.

To present the main results of the current paper, we restrict ourself to the case of finite-dimensional $X = R^n$. The main object of our investigation is initial value problem

$$\dot{u} = \frac{du}{dt} \in F(t,u) \quad a.e.\ on\ J = [0,a]$$
$$u(0) = x \in D \subset X \tag{1}$$

with $F : J \times D \to 2^X \setminus \{\emptyset\}$. Solution of (1) means the Filippov solution, that means following Ref.(30): $u : [0,\tau[\to R^n$ is an absolutely continuous function so that, for almost all $t \in [0,\tau)$ we have $\frac{du}{dt} \in \tilde{F}(t,u)$, where $\tilde{F}(t,u)$ is a convex regularization of $F(t,u)$, see Ref.(30).

The purpose of the following theorem is to give technical conditions to guarantee the valididty of the Assumption 1. of the Theorem A.

Theorem 1. *Let $Y = \{u \in C_X(J) : u(t) \in D\ on\ J\}$ with $|u|_0 = max_J |u|$ and be $Sol(x) \subset Y$ the solution set of (1). Let $F(t,\cdot)$ is u.s.c., $F(\cdot,x)$ is measurable, let $\|F(t,x)\| \leq c(t)(1+|x|)$ on $J \times D$ and $c \in L^1(J)$. If $F(t,x) \cap T_D(x) \neq \emptyset$ on $[0,a) \times D$, then (1) has an a.c. solution for every $u_0 \in D$ and $Sol(x)$ is compact and $Sol(\cdot) : D \to 2^Y \setminus \{\emptyset\}$ is u.s.c.*

Proof. We assume that D is compact. Using the Gronwall's Lemma, it is straightforward that $\|F(t,x)\| \leq c(t)(1+|x|)$ on $J \times D$ implies that $\|F(t,x)\| \leq 1$. As $F(t,\cdot)$ is u.s.c. and $F(\cdot,x)$ is measurable, the conditions of the Approximaation Theorem are satisfied. As a result, there exists $\sigma-$ selectionable map F. Moreover, there is a sequence F_n of u.s.c. compact valued maps with closed graphs. Conditions (i), (ii) of the Approximation theorem imply that $F(x) = \cap_{n\geq 0} F_n(x)$. As a result, the solution set $Sol(.)$ is upper semicontinuous and compact.
We finish the proof with proof that solution of the (1) is a.c. We assume that D is compact and F is (due to previously stated Gronwall's Lemma) locally uniformly bounded. That means that the approximate solutions F_n are Lipschitzean. Now, due to (ii) of the Approximation Theorem and due to the form of the maps F_n, it is straightforward that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_i dist(F_n(x_i), F_n(x_{i+1}) \leq \epsilon$ for every finite sequence of points $\{x_i\}$ with the property that $\sum_i dist(x_i, x_{i+1}) \leq \delta$ (definition of absolute continuity of F_n). As $F(x) = \cap_{n\geq 0} F_n(x)$, we can stated that solution of (1) is a.c., too □

In order to guarantee that also the Assumptions 2 of the Theorem A is satisfied, and also the existence of at least one covering in Definition 6, one needs the global compactness and the connectedness of the solution set Sol. These prerequisites are provided by the following theorem.

Theorem 2. *Let $X = R^n, J = [0,a] \subset R$, $D \subset X$ be closed convex, $F : J \times D \to 2^X \setminus \{\emptyset\}$ have closed convex values and be such that $F(\cdot,x)$ has a measurable selection, $F(t,\cdot)$ is u.s.c., $F(t,x) \subset T_D(x)$ on $[0,a) \times D$ and $\|F(t,x)\| \leq c(t)(1+|x|)$*

on $J \times D$ with $c \in L^1(J)$. Let $x \in D$. The the set $Sol = \bigcup_{x \in M} Sol(x)$ is compact and connected.

Proof. This theorem is a special case of the previous theorem when $Y = 2^X$. So only connectedness has to be proven.
Sequence (u_n) with $u_n \in S_n$ for $n \geq 1$ has a uniformly convergent subsequence with limit in $S(x)$ and we have $dist(S_n,\ S(x)) \to 0$ as $n \to \infty$ because $S(x) \subset S_n$ is compact. As the sets S_n are connected then $S(x)$ cannot be the union of two nonempty disjoint compact subsets. □

In the case of maximal monotone multivalued maps in Hilbert space, which will be useful in our example, e.g. Ref.(29), is the situation much more simple.

Definition 9. A multivalued map A from Hilbert space X to Hilbert space Y is called *monotone* if and only if $\forall x_1, x_2 \in Dom(A),\ \forall v_i \in A(X_i),\ i = 1,2 \implies \langle v_1 - v_2, x_1 - x_2 \rangle \geq 0$.

Definition 10. A monotone multivalued map is *maximal* if there is no other monotone multivalued map $\tilde{A}$ whose graph contains strictly the graph of A.

We point out some remarks.

A multivalued map is monotone (maximal monotone) if and only if its inverse A^{-1} is monotone (maximal monotone).

The graph of any monotone multivalued map is contained in the graph of maximal monotone multivalued map by means of Zorn's Lemma because of the union of an increasing family of graphs of monotone multivalued maps is obviously the graph of a monotone multivalued map.

As a direct consequence of the definition, we have the following useful criterion to check if u belongs to $A(x)$.

Theorem 3. *A multivalued map A is maximal monotone if and only if the following statements are equivalent:*

$$\forall (y, v) \in Graph(A), \langle u - v, x - y \rangle \geq 0 \;\; u \in A(x).$$

The next theorem guaranties the fulfillment of the conditions of the Theorem 1 and Theorem 2 in the case of maximal monotone multivalued maps:

Theorem 4. *Let A be maximal monotone multivalued map. Then:*

a *Its images are closed and convex*

b *Its graph is strongly-weakly closed in the sense that if x_n converges to x and if u_n converges weakly to u, then $u \in A(x)$.*

Proof. To prove the statement a), one can see that $A(x)$ is the intersection of the closed half-spaces $\{u \in : \langle u - v, x - y \rangle \geq 0\}$ in the case when $(y, v) \in Graph(A)$. So, $A(x)$ is closed and convex.
To prove the statement b), we suppose that x_n converge to x and simultaneously $u_n \in A(x_n)$ converge weakly to u. Let $(y, v) \in Graph(A)$. Then $\langle u_n - v, x_n - y \rangle \geq 0$ implies, by the limiting process, that $\langle u - v, x - y \rangle \geq 0$. Thus, $u \in A(x)$ using the Theorem 3. □

4. Examples

In this section, we will apply previously obtained theoretical results to examples.

Example 1. We consider a generalized Lorenz system with discontinuous right hand side $(\dot{x}, \dot{y}, \dot{z})^\top = f(x, y, z)$ of the form:

$$\dot{x} = -\sigma x + \sigma y$$

$$\dot{y} = rx - y - Sign(y)\,|x|\,z$$
$$\dot{z} = -bz + |xy| \tag{2}$$

The differential inclusion associated with this discontinuous system is $(\dot{x}, \dot{y}, \dot{z})^T \in F(x, y, z)$ where $F(x, y, z)$ is the convex regularization of $f(x, y, z)$, which has the following form:

$$F(x, y, z) = \begin{bmatrix} -\sigma x + \sigma y \\ rx - y - Sign(y)\,|x|\,z \\ -bz + |xy| \end{bmatrix}, \ y \neq 0,$$
$$F(x, y, z) = \begin{bmatrix} -\sigma x \\ \left[rx - |xz|\,, rx + |xz|\, \right] \\ -bz \end{bmatrix}, \ y = 0.$$

We try to estimate the domain of existence of the chaotic attractor. In order to do the estimate, we will construct a Lyapunov function to the system. We start with a general form of the Lyapunov function:

$$V(x, y, z) = \alpha(\delta x + \xi)^2 + \beta(\epsilon y + \rho)^2 + \gamma(\mu z + \tau)^2$$

where $\alpha > 0, \beta > 0, \gamma > 0, \ \delta, \ \epsilon, \ \mu, \ \xi, \ \rho, \ \tau$ are parameters to determine. We evaluate the gradient of V for both cases $y \neq 0, \ y = 0$ and we estimate a maximum of that gradient from above. For both cases one has the following:

$$\max \tfrac{1}{2}\dot{V}(x, y, z) \leq -\alpha\sigma\delta^2 x^2 - \beta\epsilon^2 y^2 - \gamma b\mu^2 z^2 + \alpha\sigma\delta^2 xy + (\beta\epsilon^2 r + \gamma\mu\tau)\,|xy| + \beta\epsilon\rho\,|xz| + (\gamma\mu^2 - \beta\epsilon^2)\,|xy|\,z - (\alpha\sigma\delta\xi - \beta r\epsilon\rho)x - (\beta\epsilon\rho - \alpha\sigma\delta\xi)y - \gamma\mu b\tau z.$$

Now, if we choose $\tau = -\beta\epsilon^2 r/\gamma\mu$ and $\gamma\mu^2 = \beta\epsilon^2$, the right hand side $c(x, y, z)$ of the above inequality takes the form

$-\frac{1}{2}c(x,y,z) = \alpha\sigma\delta^2x^2+\beta\epsilon^2y^2+\gamma\mu^2bz^2-\alpha\sigma\delta^2xy-\beta\epsilon\rho\,|xz|+(\alpha\sigma\delta\xi-\beta r\epsilon\rho)x+(\beta\epsilon\rho-\alpha\sigma\delta\xi)y+\gamma\mu b\tau z.$

One can easily see that the function $c(x,y,z)$ has the form of general quadric, so we can use a set of transformations in order to simplify the form of the function $c(x,y,z)$. After a lengthy calculations and final choice of parameters $\alpha=\sigma,\ \beta=\gamma=\epsilon=\mu=1,\ \xi=\rho=0,\ \tau=-r, \delta=1/\sigma$ the function $c(x,y,z)$ takes the form

$$-\frac{1}{2}c(x,y,z)=x^2+y^2+bz^2-xy-rbz$$

The last evaluation that is needed is to find out the

$$\sup_{\{(x,y,z):c(x,y,z)<0\}} V(x,y,z).$$

To do that, we utilize the very well known Lagrange multipliers method. After a rather long calculations, we get the result

$$\sup_{\{(x,y,z):c(x,y,z)<0\}} V(x,y,z)=r^2.$$

Consequently, we have estimated the domain of existence of the chaotic attractor by the manifold

$$\frac{1}{\sigma}x^2+y^2+(z-r)^2\le r^2.$$

Due to compactness of that domain and due to the convex regularization of the equation (3), we can conclude that all the conditions of Theorem A are satisfied thus we have proved the existence of a chaotic solution in the above domain. In fact, if we choose $\sigma=10,\ r=28.5,\ b=2.5$, we obtain the traditional chaotic solution of the Lorenz system. We can conclude that the traditional chaotic solution of the Lorenz system is one of the more possible chaotic solutions of the general differential inclusion (3).

Example 2. Now, we will apply previous theoretical results to a practical problem. We have chosen a problem of modeling of the pendulum with friction, Ref.(31). Let us consider a forced pendulum with a viscous damping and Coulomb friction. This pendulum corresponds to the model:

$$\ddot{x}+a\dot{x}+\lambda sin(x)+\alpha Sign(\dot{x})-f(t)\ni 0$$

where $\dot{x}:=\frac{dx}{dt},\ \ddot{x}:=\frac{d^2x}{dt^2},\ \lambda\in R,\ \alpha\in R^+$ and $Sign(\cdot)$ denotes the graph of the function $Sign(u)=-1$ if $u<0, Sign(u)=+1$ if $u>0, Sign(u)=[-1,1]$ if $u=0$. This model has to be understood as a differential inclusion. $\alpha Sign(\dot{x})$ is the expression of a Coulomb friction applied to the pendulum. Hereafter, we choose a particular expression of the external forcing $f(t)=f\sin(\omega t),\ f\in R,\ \omega\in R.$

The previous model can be written in an obvious way in the form of a first order differential inclusion:

$$\dot{Y} + \begin{bmatrix} y_2 \\ ay_2 + \lambda \sin(y_1) - f(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha Sign(y_2) \end{bmatrix} \ni \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This relation gives that

$$\dot{Y} + F(Y,t) + H(Y) \ni 0, \tag{3}$$

where $Y = (y_1, y_2)^\top = (x, \dot{x})^\top$. The initial conditions are $Y(t_0) = Y_0 = (x_0, \dot{x_0})^\top$ and $x_0 \in [-\pi, \pi] \ni \dot{x_0}$.

Moreover, F is clearly a Lipschitz-continuous map: for usual Euclidean scalar product $(\cdot,\cdot)$ and its associated norm $\|\cdot\|$ of R^2 it holds that: $\forall t \in R,\ \forall (Y,Z) \in R^2 \times R^2,\ Y = (y_1, y_2)^T,\ Z = (z_1, z_2)^T$ we have $\|F(Y,t) - F(Z,t)\| \leq (1 + |a| + |\lambda|)\,\|Y - Z\|$.

Further, let us show that H is a monotone operator. Let $\forall Y = (y_1, y_2)^T \in R^2,\ \forall Z = (z_1, z2)^T \in R^2,\ \forall U = (u_1, u_2) \in R^2,\ \forall V = (v_1, v_2) \in R^2$. Then for $U \in H(Y),\ V \in H(Z)$ we have $(V - U, Z - Y) = \alpha(v_2 - u_2, z_2 - y_2) \geq 0$ because the function $Sign$ is monotone.

It is easy to show that H is maximal because $(I + \mu H)$ is invertible for every real $\mu \geq 0$. Due to Ref.(32), there exists unique solution of differential inclusion (4).

As a result, the operator $A = F + H$ meets the implication of the Theorem 4. So, the conditions of the Theorem 2 are fulfilled. Accordingly to the Theorem A, one can stated that the chaotic solution exists for the differential inclusion (4). This result coincides with the results of Ref.(31) where chaotic behavior has been observed by numerical experiments in the case of parameters $a = 0.052,\ \lambda = 0.87,\ f = 0.586,\ \omega = 0.666,\ \alpha = 0.144$.

5. Conclusion and Outlooks

The analysis of chaos in the case of multivalued nonlinear dynamical systems, modeled by differential inclusions, has been presented here. Chaotic phenomena in the case of multivalued dynamical systems are still at the beginning of their analysis, even the notion of the chaos itself is still open and discussed. Similarly to previously existing research, the results presented in the current paper shows the existence of chaotic behavior. Moreover, no *a priori* hypothesis concerning the existence of homo/heteroclical trajectory is needed here. Conditions are illustrated on several examples. On the other side, design of an efficient numerical method to compute a particular chaotic solution remains an open problem yet to be investigated.

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