Economic and Financial Problems via Multiobjective Stochastic Optimization

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Abstract. Multiobjective optimization problems depending on a probability measure correspond to many economic and financial activities. Evidently if the probability measure is completely known, then we can try to influence economic process employing methods of a multiobjective deterministic optimization theory. Since this assumption is fulfilled very seldom we have mostly to analyze the mathematical model and consequently also economic process on the data base. The aim of the talk will be to investigate a relationship between "characteristics" obtained on the base of complete knowledge of the probability measure and them obtained on the above mentioned data base. To this end, the results of the deterministic multiobjective optimization theory and the results obtained for stochastic one objective problems will be employed.

Keywords: Stochastic multiobjective optimization problems, efficient solution, Wasserstein metric, $\mathcal{L}_1$ norm, stability, empirical estimates, Lipschitz property.

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1 Introduction

To introduce a "rather general" multiobjective stochastic programming problem, let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F(z),\, z \in R^s)$, P_F and Z_F denote the distribution function, the probability measure and the support corresponding to ξ. Let, moreover, $g_i := g_i(x,\, z)$, $i = 1, \ldots, l$, $l \geq 1$ be real–valued (say, continuous) functions defined on $R^n \times R^s$; $X_F \subset X \subset R^n$ be a nonempty set generally depending on F, and $X \subset R^n$ be a nonempty deterministic set. If the symbol E_F denotes the operator of mathematical expectation corresponding to F and if for every $x \in X$ there exist finite $\mathsf{E}_F g_i(x, \xi)$, $i = 1, \ldots, l$, then a rather general "multiobjective" one–stage stochastic programming problem can be introduced in the form:

$$\text{Find} \quad \min \mathsf{E}_F g_i(x, \xi),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in X_F. \tag{1}$$

The multiobjective problem (1) corresponds evidently to economic situation in which a "result" of an economic process is simultaneously influenced by a random factor ξ and a decision parameter x, it is reasonable to evaluate this process by a few (say l, $l \geq 1$) objective functions. The decision vector has to be determined without knowledge of the random element realization and it seems to be reasonable to determine "the decision" with respect to the mathematical expectation of the objectives.

It is possible only very seldom to find out simultaneously the solution with respect to all criteria in (1) and moreover, these problems depend on a probability measure P_F that usually has to be estimated on the data base. Consequently, in applications very often the "underlying" probability measure P_F has to be replaced by empirical one. Evidently, then the "solution" and an analysis of the problem have to be done with respect to an empirical problem:

$$\text{Find} \quad \min \mathsf{E}_{F^N} g_i(x, \xi),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in X_{F^N}, \tag{2}$$

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where F^N denotes an empirical distribution function determined by a random sample $\{\xi^i\}_{i=1}^N$ (not necessary independent) corresponding to the distribution function F.

To analyze the problem (1), first, the results of the multiobjective deterministic problems have to be recalled. Since, it follows from multiobjective theory that the results of uni–objective optimization theory can be useful (under rather general conditions) to investigate the relationship between the results obtained under complete knowledge of P_F and them obtained on the data base, we recall also the results obtained for uni–objective stochastic programming problems. Our aim will be to focus to "underlying" distributions with heavy tails, that correspond just to many economic and financial processes (for more details see e.g. [10] or [12]).

According to the above mentioned facts, the paper is organized as follows. First, we try to recall auxiliary assertions concerning deterministic multiobjective theory (subsection 2.1). Stability and empirical estimates obtained for uni–objective stochastic programming problems are recalled in section 2.2. Section 3 is devoted to the stability analysis of the problem (1). Essential results devoted to the multiobjective problems can be found in section 4.

2 Some Definition and Auxiliary Assertion

2.1 Deterministic Multiobjective Problems

To recall some results of the multiobjective deterministic optimization theory we consider a multiobjective deterministic optimization problem in the following form:

$$\text{Find} \quad \min f_i(x),\ i = 1,\,\ldots,\,l' \quad \text{subject to} \quad x \in \mathcal{K}, \tag{3}$$

where $f_i(x)$, $i = 1,\,\ldots,\,l'$ are real–valued functions defined on R^n, $\mathcal{K} \subset R^n$ is a nonempty set.

Definition 1. The vector x^* is an efficient solution of the problem (3) if and only if there exists no $x \in \mathcal{K}$ such that $f_i(x) \le f_i(x^*)$ for $i = 1,\,\ldots,\,l'$ and such that for at least one i_0 one has $f_{i_0}(x) < f_{i_0}(x^*)$.

Definition 2. The vector x^* is properly efficient solution of the multiobjective optimization problem (3) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in \mathcal{K}$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \le M. \tag{4}$$

Proposition 1. ([4]) *Let $\mathcal{K} \subset R^n$ be a nonempty convex set and let $f_i(x)$, $i = 1,\,\ldots,\,r$ be convex functions on $\mathcal{K}$. Then x^0 is a properly efficient solution of the problem (3) if and only if x^0 is optimal in*

$$\min_{x\in\mathcal{K}} \sum_{i=1}^{r} \lambda_i f_i(x) \quad \text{for some} \quad \lambda_1,\,\ldots,\,\lambda_r > 0; \quad \sum_{i=1}^{r} \lambda_i = 1.$$

Definition 3. Let $\bar{h}(x)$ be a real–valued function defined on a nonempty convex se $\mathcal{K} \subset R^n$. $\bar{h}(x)$ is a strongly convex function with a parameter $\rho > 0$ if

$$\bar{h}(\lambda x^1 + (1-\lambda)x^2) \le \lambda \bar{h}(x^1) + (1-\lambda)\bar{h}(x^2) - \lambda(1-\lambda)\rho\|x^1 - x^2\|^2 \quad \text{for every} \quad x^1,\, x^2 \in \mathcal{K},\ \lambda \in \langle 0,\, 1\rangle.$$

Proposition 2. ([6]) *Let $\mathcal{K} \subset R^n$ be a nonempty, compact, convex set. Let, moreover, $\bar{h}(x)$ be a strongly convex with a parameter $\rho > 0$, continuous, real–valued function defined on $\mathcal{K}$. If x^0 is defined by the relation $x^0 = \arg\min_{x\in\mathcal{K}} \bar{h}(x)$, then*

$$\|x - x^0\|^2 \le \frac{2}{\rho}|\bar{h}(x) - \bar{h}(x^0|| \quad \text{for every} \quad x \in \mathcal{K}.$$

2.2 Uni–Objective Stochastic Programming Problems

To recall suitable for us assertions of one criteria stochastic optimization theory we start with the problem:

$$\text{Find} \quad \varphi(F,\, X_F) = \inf \mathsf{E}_F g_0(x,\, \xi) \quad \text{subject to} \quad x \in X_F, \tag{5}$$

where $g_0(x, z)$ is a real–valued function defined on $R^n \times R^s$.

First, if F and G are two s–dimensional distribution functions for which the Problem (5) is well defined, then we can obtain by the triangular inequality that

$$|\varphi(F, X_F) - \varphi(G, X_G)| \le |\varphi(F, X_F) - \varphi(G, X_F)| + \varphi(G, X_F) - \varphi(G, X_G)|. \tag{6}$$

According to the relation (6) we can study separately stability of the problem (5) with respect to perturbation in the objective function and in constraints set. In this paper we restrict our consideration to the case $X_F = X$ independently of F. To this end we introduce the following assumptions:

A.1
- X is a convex set and there exists $\varepsilon > 0$ such that $g_0(x, z)$ is a convex bounded function on $X(\varepsilon)$ ($X(\varepsilon)$ denotes the ε–neighborhood of X),
- $g_0(x, z)$ is a Lipschitz function of $z \in R^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x.

To introduce the first assertion dealing with the stability of the problem (5) (with $X_F = X$) we denote by $F_i, i = 1, \ldots, s$ one–dimensional marginal distribution functions corresponding to F; $\mathcal{P}(R^s)$ the set of Borel measures on R^s, $\mathcal{M}_1(R^s) = \{P \in \mathcal{P}(R^s) : \int\limits_{R^s} \|z\|_s^1 P(dz) < \infty\}$, $\| \cdot \|_s^1$ denote $\mathcal{L}_1$ norm in R^s.

Proposition 3. ([7]) *Let $P_F, P_G \in \mathcal{M}_1(R^s)$, X be a nonempty set. If A.1 is fulfilled, then*

$$|\mathsf{E}_F g_0(x, \xi) - \mathsf{E}_G g_0(x, \xi)| \le L \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i \quad \text{for every} \quad x \in X.$$

Proposition 3 reduces (from the mathematical point of view) s–dimensional case to one–dimensional. Of course, stochastic dependence between components of the random vector ξ is there neglected. Replacing G by an empirical estimate F^N of F we can employ Proposition 3 to investigate empirical estimates of Problem (5) (with $X_F = X$) and according to Proposition 1 also to analyze relationship between Problems (1) and (2). Evidently, according to Proposition 3 it is reasonable to investigate the behaviour of $\int\limits_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i$, $i = 1, \ldots, s$. To this end, we recall the following assumptions:

A.2 $\{\xi^i\}_{i=1}^{\infty}$ is independent random sequence corresponding to F, F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$,

A.3 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w. r. t. the Lebesgue measure on R^1.

Proposition 4. ([15]) *Let $s = 1$ and $P_F \in \mathcal{M}_1(R^1)$. Let, moreover A.2 be fulfilled . Then*

$$P\{\omega : \int\limits_{-\infty}^{\infty} |F(z) - F^N(z)| dz \longrightarrow_{N \longrightarrow \infty} 0\} = 1.$$

Proposition 5. [8] *Let $s = 1$, $t > 0$ and Assumptions A.2, A.3 be fulfilled. If there exists $\beta > 0$, $R := R(N) > 0$ defined on $\mathcal{N}$ such that $R(N) \xrightarrow[N \to \infty]{} \infty$ and, moreover,*

$$\begin{aligned} & N^{\beta} \int\limits_{-\infty}^{-R(N)} F(z) dz \xrightarrow[N \to \infty]{} 0, \qquad && N^{\beta} \int\limits_{R(N)}^{\infty} [1 - F(z)] dz \xrightarrow[N \to \infty]{} 0, \\ & 2NF(-R(N)) \xrightarrow[N \to \infty]{} 0, && 2N[1 - F(R(N))] \xrightarrow[N \to \infty]{} 0, \\ & (\tfrac{12N^{\beta}R(N)}{t} + 1) \exp\{-2N(\tfrac{t}{12R(N)N^{\beta}})^2\} \xrightarrow[N \longrightarrow \infty]{} 0, \end{aligned} \tag{7}$$

then

$$P\{\omega : N^{\beta} \int\limits_{-\infty}^{\infty} |F(z) - F^N(z)| dz > t\} \xrightarrow[N \to \infty]{} 0. \tag{8}$$

($\mathcal{N}$ denotes the set of natural numbers.)

Evidently, it follows from the relations (7), (8) and from the classical result of [2] that the validity of the relation (8) depends on the tails behaviour (for more details see e.g. [9]).

Proposition 6 ([5])**.** *Let $s = 1$, $t > 0$, $r > 0$, the assumptions A.2, A.3 be fulfilled. Let, moreover, ξ be a random variable such that $\mathsf{E}_F|\xi|^r < \infty$. If constants $\beta, \gamma > 0$ fulfil the inequalities $0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1-r)\gamma < 0$, then*

$$P\{\omega : N^{\beta} \int\limits_{-\infty}^{\infty} |F(z) - F^N(z)| dz > t\} \longrightarrow_{N \longrightarrow \infty} 0.$$

Analyzing Proposition 6 we can obtain $\beta := \beta(r)$ fulfilling this assertion and simultaneously

$$\beta(r) \longrightarrow_{r \longrightarrow \infty} 1/2, \quad \beta(r) \longrightarrow_{r \longrightarrow 2^+} 0.$$

Proposition 6 covers also some cases of heavy tails distributions. Unfortunately, we cannot obtain by this Proposition any results for the case when there exist only $\mathsf{E}_F|\xi|^r$ for $r < 2$. But just this case corresponds to stable distributions with the tail (shape) parameter $\nu < 2$ (for more details see e.g. [11] or [13]). The shape parameter expresses how "heavy" tails of distribution are. The case $\nu = 2$ corresponds to normal distribution, when the second moment exists. To obtain at least weaker result for the case when the finite moment exists only for $r < 2\,(\nu < 2)$, we recall the results of [1].

Proposition 7 ([1])**.** *Let $s = 1, \quad \{\xi^i\}_{i=1}^N$, $N = 1, 2, \ldots$ be a sequence of independent random values corresponding to a heavy tailed distribution F with the shape parameter $\nu \in (1, 2)$ and let*

$$\sup_{t > 0} t^{\nu} P\{\omega : |\xi| > t\} < \infty, \tag{9}$$

then

$$\lim_{\bar{M} \longrightarrow \infty} \sup_{N} P\{\omega : \frac{N}{N^{1/\nu}} \int\limits_{-\infty}^{\infty} |F(z) - F^N(z)| > \bar{M}\} = 0. \tag{10}$$

3 Problem Analysis

To analyze the stability of the multiobjective stochastic problem (1) we define the sets $\mathcal{G}(F, X_F)$, $\bar{\mathcal{X}}(F, X_F)$, $\bar{\mathcal{G}}(F, X_F)$ and the function $\bar{g}(x, z, \lambda)$ by the relations

$$\begin{array}{rcl} \mathcal{G}(F, X_F) & = & \{y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi),\, j = 1, \ldots, l \quad \text{for some} \quad x \in X_F;\, y = (y_1, \ldots, y_l)\}, \\ \bar{\mathcal{X}}(F, X_F) & = & \{x \in X_F : x \quad \text{is a properly efficient point of the problem (1)}\}, \\ \bar{\mathcal{G}}(F, X_F) & = & \{y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi),\, j = 1, \ldots, l \quad \text{for some} \quad x \in \bar{\mathcal{X}}(F, X_F)\} \\ \bar{g}(x, z, \lambda) & = & \sum\limits_{i=1}^{l} \lambda_i g_i(x, z), \quad x \in R^n,\, z \in R^s,\, \lambda = (\lambda_1, \ldots, \lambda_l),\, \lambda_i > 0,\, \sum\limits_{i=1}^{l} \lambda_i = 1. \end{array} \tag{11}$$

Evidently, if the following assumptions are fulfilled

B.1
- X is a convex set and, moreover, there exists $\varepsilon > 0$ such that $g_i(x, z)$, $i = 1, \ldots, s$ are for every $z \in R^s$ a convex functions on $X(\varepsilon)$,
- $g_i(x, z)$, $\imath = 1, \ldots, l$ are Lipschitz functions of $z \in R^s$ with the Lipschitz constant L (corresponding to $\mathcal{L}_1$ norm) not depending on x,

then $\bar{g}(x, z, \lambda)$ is a convex function on $X(\varepsilon)$ and, moreover, it is a Lipschitz function of z with the Lipschitz constant L not depending on x, λ. Consequently, according to to Proposition 3 we can obtain.

Proposition 8. *Let $P_F, P_G \in \mathcal{M}_1(R^s)$, X be a nonempty set. If B.1 is fulfilled, then*

$$|\mathsf{E}_F \bar{g}(x, \xi, \lambda) - \mathsf{E}_G \bar{g}(x, \xi, \lambda)| \le L \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i, \quad x \in X,\, \lambda_i > 0,\, i = 1, \ldots, s,\, \sum_{i=1}^{s} \lambda_i = 1.$$

Proposition 9. *Let P_F, $P_G \in \mathcal{M}_1(R^s)$, X be a compact set. If B.1 is fulfilled, then*

$$\Delta_n[\mathcal{G}(F, X), \mathcal{G}(G, X)] \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i,$$

where the symbol $\Delta n[\cdot, \cdot]$ is reserved for the Hausdorff distance of the subsets of R^n (for the definition of the Hausdorff distance see e.g. [14]).

Proposition 10. *Let X be a convex set. If $g_i(x, z)$, $i = 1, \ldots, l$ are strongly convex (with a parameter $\rho > 0$) function on X, then $\mathsf{E}_F \bar{g}(x, \xi, \lambda)$ is a strongly convex function on X with the parameter ρ.*

Proof. The assertion of Proposition 10 follows from Definition 3 and Relation (11). □

Employing the assertion of Proposition 1 we can investigate the relationship between the Problems (1), and (2).

4 Empirical Estimates

Theorem 11. *Let Assumptions B.1, A.2, and A.3 be fulfilled, $P_F \in \mathcal{M}_1(R^s)$, X be a compact set. Then*

$$P\{\omega : \Delta_n[\mathcal{G}(F, X), \mathcal{G}(F^N, X)] \longrightarrow_{N \longrightarrow \infty}\} = 1.$$

Proof. The assertion of Theorem 11 follows from Propositions 4, 9 and the relation (11). □

Theorem 12. *Let $t > 0$, $r > 0$, Assumptions B.1, A.2, A.3 be fulfilled. Let, moreover, ξ be a random vector with the components ξ_i, $i = 1, \ldots, s$ such that $\mathsf{E}_F|\xi_i|^r < \infty$. If constants $\beta, \gamma > 0$ fulfil the inequalities $0 < \beta + \gamma < 1/2$, $\gamma > 1/r$, $\beta + (1 - r)\gamma < 0$, then*

$$P\{\omega : N^{\beta} \Delta_n[\mathcal{G}(F, X), \mathcal{G}(F^N, X)] > t\} \longrightarrow_{N \longrightarrow \infty}\} = 0.$$

If, moreover, $g_i(x, z)$, $i = 1, \ldots, l$ are strongly convex with a parameter $\rho > 0$ function on X, then also

$$P\{\omega : N^{\beta} \Delta_n[\bar{\mathcal{X}}(F, X)), \bar{\mathcal{X}}(F^N, X)]^2 > t\} \longrightarrow_{N \longrightarrow \infty} 0.$$

Proof. First assertion of Theorem 12 follows from Propositions 6 and 9. The second assertion follows from the first one and from Proposition 2, 10 and the relation 10. □

Theorem 13. *Let Assumptions B.1, A.2 and A.3 be fulfilled, $P_F \in \mathcal{M}_1(R^s)$, $\bar{M} > 0$, X be a compact set. If one–dimensional components ξ_i, $i = 1, \ldots, s$ of the random vector ξ have distribution functions F_i with tails parameter $\nu_i \in (1, 2)$ fulfilling the relations*

$$\sup_{t > 0} t^{\nu_i} P_F\{\omega : |\xi_i| > t\} < \infty, \quad i = 1, \ldots, s,$$

then

$$\lim_{\bar{M} \longrightarrow \infty} \sup_{N} P\{\omega : \frac{N}{N^{1/\nu}} \Delta_n[\mathcal{G}(F, X)), \mathcal{G}(F^N, X))], > \bar{M}\} = 0 \quad \text{with} \quad \nu = \min(\nu_1, \ldots, \nu_s\}.$$

If, moreover, $g_i(x, z)$, $i = 1, \ldots, l$ are strongly convex with a parameter $\rho > 0$ function on X, then also

$$\lim_{\bar{M} \longrightarrow \infty} \sup_{N} \{\omega : \frac{N}{N^{1/\nu}} \Delta_n[\bar{\mathcal{X}}(F, X)), \bar{\mathcal{X}}(F^N, X)]^2 > \bar{M}\} = 0 \quad \text{with} \quad \nu = \min(\nu_1, \ldots, \nu_s\}.$$

Proof. The assertion of Theorem 13 follows from the assertion of Propositions 2, 7 and 10. □

Remark 1. Let us assume that Assumptions of Theorem 13 are fulfilled and $\beta(\nu) := 1 - 1/\nu$. Then $\beta(\nu)$ is an increasing function of ν and holds up to

$$\lim_{\nu \longrightarrow 1^+} \beta(\nu) = 0, \quad \lim_{\nu \longrightarrow 2^-} \beta(\nu) = \frac{1}{2},$$

The assertions of Theorems 11, 12 and 13 are introduced (under the approach of properly efficient points and their functions mapping), however, since the set of properly efficient points is dense in the set of efficient points our results are not much restricted (for more details see e. g. [3]).

5 Conclusion

The paper deals with multiobjective stochastic programming problems, especially with a relationship between characteristics of these problems corresponding to complete knowledge of the probability measure and them determined on the data base. We have restricted ourselves to investigate the characteristics $\mathcal{G}(F, X)$, $\bar{\mathcal{X}}(F, X))$ and $X_F = X$, generally. Evidently the presented results can be generalized to the characteristic $\bar{\mathcal{G}}(F, X_F)$ and (employing Relation (6)) some type of constraints set depending on the probability measure (see the corresponding results achieved for one objective case [9]). However more detailed investigation in this direction is beyond the scope of this paper.

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References

[1] Barrio, E., Giné, E., and Matrán, C.: Central limit theorems for the Wassersten distance between the empirical and the true distributions. *Annals of Probability* **27** (1999), 2, 1009–1071.

[2] Dvoretski, A., Kiefer, J. and Wolfowitz, J.: J. and Wolfowitz, J.: Asymptotic minimaxcharacter of esample distribution function and the classical. *Ann. Math. Statist.* **27** (1956),642–669.

[3] M. Ehrgott: Multicriteria Optimization. Springer, Berlin, 2005.

[4] A. M. Geoffrion: Proper efficiency and the theory of vector maximization. J. Math. Annal. Appl. **22** (1968), 3, 618–630.

[5] Houda, M., and Kaňková, V.: Empirical estimates in economic and financial optimization problems. *Bulletin of the Czech Econometric Society* **19** (2012), 29, 50–69.

[6] V. Kaňková: Stability in stochastic programming – the case of unknown location parameter. Kybernetika **29** (1993), 1, 97–112.

[7] Kaňková, V. and Houda M.: Empirical estimates in stochastic programming. In: *Proceedings of Prague Stochastics 2006* (M. Hušková and M. Janžura, eds.), MATFYZPRESS, Prague 2006, 426–436.

[8] V. Kaňková: Empirical estimates in stochastic optimization via distribution tails. *Kybernetika* **46** (2010), 3, 459–471.

[9] Kaňková, V.: Empirical estimates in optimization problems: survey with special regard to heavy tails and dependent samples Bulletin of the Czech Econometric Society **19** (2012), 30, 92–111.

[10] Klebanov, L.B.: *Heavy Tailed Distributions.* MATFYZPRESS, Prague, 2003.

[11] Kozubowski, T.J., Panorska A. K., and Rachev, S. T.: Statistical issues in modelling stable portfolios. In: *Handbook of Heavy Tailed Distributions in Finance* (S.T. Rachev, ed.), Elsevier, Amsterdam 2003, 134–168.

[12] Meerschaert, M.M., and Scheffler, H.P.: Portfolio modelling with heavy tailed random vectors. In: *Handbook of Heavy Tailed Distributions in Finance* (S.T. Rachev, ed.), Elsevier, Amsterdam 2003, 595–640.

[13] Rachev, S.T., and Mitting, S.: *Stable Paretian Models in Finance (Series in Financial Economics and Quantitative Analysis)* John Wiley & Sons, Chichester, 2000.

[14] R. T. Rockafellar, and R. J. B. Wets: Variational Analysis. Springer, Berlin, 1983.

[15] G. R. Shorack and J. A. Wellner: Empirical Processes with Applications to Statistics. John Wiley & Sons, New York, 1986.