

# Measuring capital market efficiency: Long-term memory, fractal dimension and approximate entropy

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**Abstract.** We utilize long-term memory, fractal dimension and approximate entropy as input variables for the Efficiency Index [Kristoufek & Vošvrda (2013), *Physica A* 392]. This way, we are able to comment on stock market efficiency after controlling for different types of inefficiencies. Applying the methodology on 38 stock market indices across the world, we find that the most efficient markets are situated in the Eurozone (the Netherlands, France and Germany) and the least efficient ones in the Latin America (Venezuela and Chile).

**Keywords:** capital market efficiency, long-range dependence, fractal dimension, approximate entropy

## 1 Introduction

Efficient markets hypothesis (EMH) is one of the cornerstones of the modern financial economics. Since its introduction in 1960s [10, 24, 11], EMH has been a controversial topic. Nonetheless, the theory still remains a stable part of the classical financial economics. Regardless of its definition via a random walk [10] or a martingale [24], the main idea of EMH is that risk-adjusted returns cannot be systematically predicted and there can be no long-term profits above the market profits assuming the same risk. The EMH definition is also tightly connected with a notion of rational homogenous agents and Gaussian distribution of returns. Both these assumptions have been widely disregarded in the literature [6].

There are several papers ranking various financial markets with respect to their efficiency. Research group around Di Matteo [8, 9, 7] show that the correlations structure of various assets (stocks, exchange rates and interest rates) is connected to the development of the specific countries and stock markets. In the series of papers, Cajueiro & Tabak [3, 4, 2, 5] study the relationship between the long-term memory parameter  $H$  and development stages of the countries' economy. Both groups find interesting results connecting persistent (long-term correlated) behavior to the least developed markets but also anti-persistent behavior for the most developed ones. Lim [20] investigates how the ranking of stock markets based on Hurst exponent evolves in time and reports that the behavior can be quite erratic. Zunino *et al.* [28] utilize entropy to rank stock markets to show that the emergent/developing markets are indeed less efficient than the developed ones. Even though the ranking is provided in these studies, the type of memory taken into consideration (either long-term memory or entropy/complexity) is limited and treated separately. In this paper, we utilize the Efficiency Index proposed by Kristoufek & Vošvrda [19] incorporating long-term memory, fractal dimension and entropy to control for various types of correlations and complexity using a single measure. Basing the definition of the market efficiency simply on no correlation structure, we can state the expected values of long-term memory, fractal dimension and entropy to construct an efficiency measure based on a distance from the efficient market state. The procedure is then applied on 38 stock indices from different parts of the world and we show that the most efficient markets are indeed the most developed ones – the Western European markets and the US markets – and the least efficient ones are situated in the Latin America and South-East Asia.

The paper is structured as follows. In Section 2, we provide brief description of used methodology focusing on long-term memory, fractal dimension, entropy and efficiency measure. Section 3 introduces the dataset and describes the results. Section 4 concludes.

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## 2 Methodology

### 2.1 Long-term memory

Long-term memory (long-range dependence) is usually characterized in time domain by a power-law decay of autocorrelation function and in frequency domain by a power-law divergence of spectrum close to the origin. More specifically, the autocorrelation function  $\rho(k)$  with lag  $k$  of a long-range correlated process decays as  $\rho(k) \propto k^{2H-2}$  for  $k \rightarrow +\infty$ , and the spectrum  $f(\lambda)$  with frequency  $\lambda$  of a long-range correlated process diverges as  $f(\lambda) \propto \lambda^{1-2H}$  for  $\lambda \rightarrow 0+$ . The characteristic parameter of the long-term memory Hurst exponent  $H$  ranges between  $0 \leq H < 1$  for stationary processes. The breaking value of 0.5 indicates no long-term memory so that the autocorrelations decay rapidly (exponentially). For  $H > 0.5$ , the series is persistent with strong positive correlations characteristic by a trend-like behavior while still remaining stationary. For  $H < 0.5$ , the series is anti-persistent and it switches the direction of increments more frequently than a random process does. As the inputs to the Efficiency Index, we utilize two estimators from the frequency domain – the local Whittle [23] and GPH estimators [13] – which are more appropriate for rather short financial series with a possible weak short-term memory [25, 26], which can easily bias the time domain estimators [27, 1, 17]. Moreover, the frequency domain estimators have well-defined asymptotic properties and the selected two are consistent and asymptotically normal estimators.

### 2.2 Fractal dimension

Fractal dimension  $D$  is a measure of roughness of the series and can be taken as a measure of local memory of the series [19]. For a univariate series, it holds that  $1 < D \leq 2$ . For self-similar processes, the fractal dimension is connected to the long-term memory of the series so that  $D + H = 2$ . This can be attributed to a perfect reflection of a local behavior (fractal dimension) to a global behavior (long-term memory). However, the relation usually does not hold perfectly for the financial series so that both  $D$  and  $H$  give different insights on the dynamics of the series. In general,  $D = 1.5$  holds for a random series with no local trending or no local anti-correlations. For a low fractal dimension  $D < 1.5$ , the series is locally less rough and thus resembles a local persistence. Reversely, a high fractal dimension  $D > 1.5$  is characteristic for rougher series with local anti-persistence. For purposes of the Efficiency Index, we utilize Hall-Wood and Genton estimators [14, 15, 16, 12].

### 2.3 Approximate entropy

Entropy can be taken as a measure of complexity of the system. The systems with high entropy can be characterized by no information and are thus random and reversely, the series with low entropy can be seen as deterministic [22]. The efficient market can be then seen as the one with maximum entropy and the lower the entropy, the less efficient the market is. For purposes of the Efficiency Index, we need an entropy measure which is bounded. Therefore, we utilize the approximate entropy introduced by Pincus [21] which is bounded between 0 (completely deterministic behavior) and 1 (completely random behavior).

### 2.4 Capital market efficiency measure

According to Kristoufek & Vosvrda [18, 19], the Efficiency Index (EI) is defined as

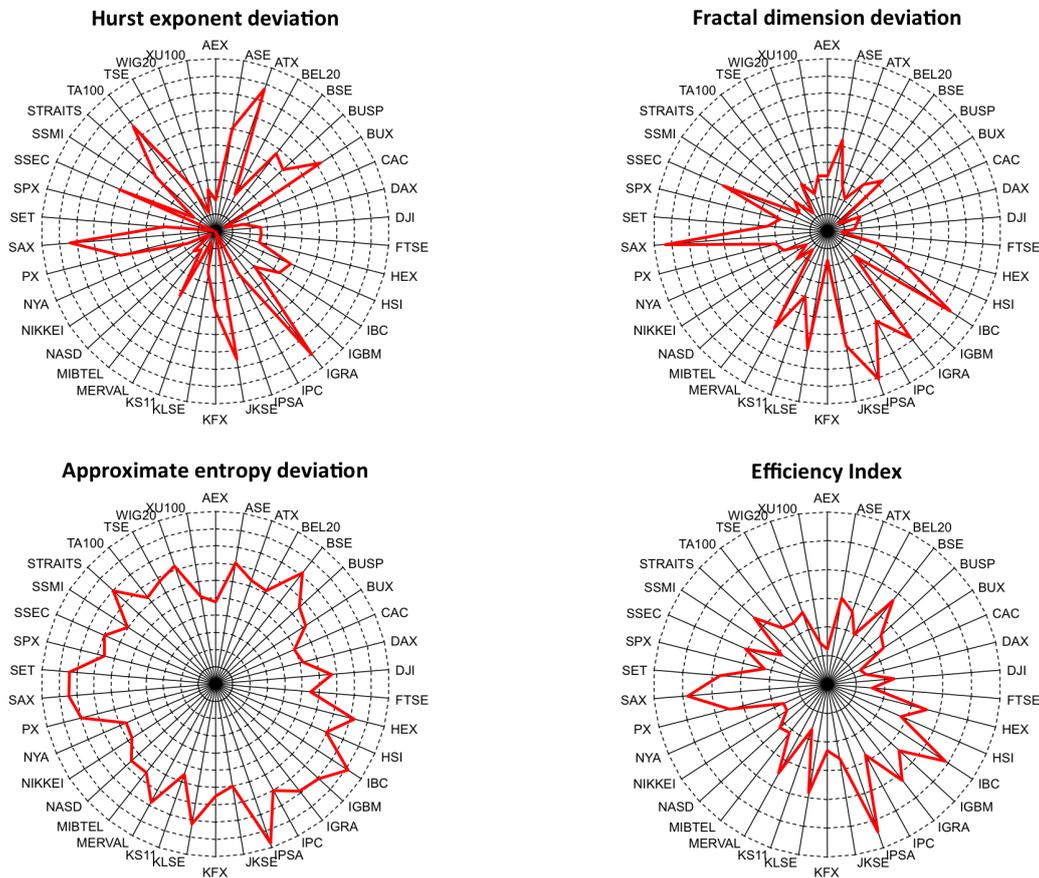
$$EI = \sqrt{\sum_{i=1}^n \left( \frac{\widehat{M}_i - M_i^*}{R_i} \right)^2},$$

where  $M_i$  is the  $i$ th measure of efficiency,  $\widehat{M}_i$  is an estimate of the  $i$ th measure,  $M_i^*$  is an expected value of the  $i$ th measure for the efficient market and  $R_i$  is a range of the  $i$ th measure. In words, the efficiency measure is simply defined as a distance from the efficient market specification based on various measures of the market efficiency. In our case, we consider three measures of market efficiency – Hurst exponent  $H$  with an expected value of 0.5 for the efficient market ( $M_H^* = 0.5$ ), fractal dimension  $D$  with an expected

value of 1.5 ( $M_D^* = 1.5$ ) and the approximate entropy with an expected value of 1 ( $M_{AE}^* = 1$ ). The estimate of Hurst exponent is taken as an average of estimates based on GPH and the local Whittle estimators. The estimate of the fractal dimension is again taken as an average of the estimates based on the Hall-Wood and Genton methods. For the approximate entropy, we utilize the estimate described in the corresponding section. However, as the approximate entropy ranges between 0 (for completely deterministic market) and 1 (for random series), we need to rescale its effect, i.e. we use  $R_{AE} = 2$  for the approximate entropy and  $R_H = R_D = 1$  for the other two measures so that the maximum deviation from the efficient market value is the same for all measures.

### 3 Application and discussion

We analyze 38 stock indices from various locations – the complete list is given in Tab. 1 – between January 2000 and August 2011. Various phases of the market behavior – DotCom bubble, bursting of the bubble, stable growth of 2003-2007 and the current financial crisis – are thus covered in the analyzed period. The indices cover stock markets in both North and Latin Americas, Western and Eastern Europe, Asia and Oceania so that markets at various levels of development are included in the study. The logarithmic returns are asymptotically stationary (according to the KPSS test), leptokurtic and returns of majority of the indices are negatively skewed (the results are available upon request).



**Figure 1:** Hurst exponent, fractal dimension, approximate entropy and efficiency index for analyzed indices.

Let us now turn to the results. In Fig. 1, all the results are summarized graphically. For the utilized three measures – Hurst exponent, fractal dimension and approximate entropy – we present the absolute deviations from the expected values of the efficient market for comparison. For the Hurst exponent estimates, we observe huge diversity – between practically zero (for IPSA of Chile) and 0.18 (for Peruvian IGRA). Interestingly, for some of the most developed markets, we observe Hurst exponents well below 0.5 (Tab. 1 gives the specific estimates) which is, however, in hand with results of other authors [9, 7]. The results for the fractal dimension again vary strongly across the stock indices. The highest deviation

is observed for the Slovakian SAX (0.19) and the lowest for the FTSE of the UK (0.02). In Tab. 1, we observe that apart from FTSE, all the other stock indices possess the fractal dimension below 1.5 which indicates that the indices are locally persistent, i.e. in some periods, the indices experience significant positively autocorrelated behavior which is well in hand with expectations about the herding behavior during critical events. The approximate entropy estimates are more stable across indices compared to the previous two cases. The highest deviation from the expected value for the efficient market is observed for the Chilean IPSA (0.98) and the lowest for the Dutch AEX (0.48). Evidently, all the analyzed stock indices are highly complex as the approximate entropy is far from the ideal (efficient market) value of 1 and such complexity is not sufficiently covered by the other two applied measures. The inclusion of the approximate entropy into the Efficiency Index thus proves its worth.

**Table 1:** Ranked stock indices according to the Efficiency Index

Index	Country	Hurst exponent	Fractal dimension	Approximate entropy	Efficiency index
AEX	Netherlands	0.5358	1.4356	0.5246	0.0619
CAC	France	0.5118	1.4592	0.5059	0.0628
DAX	Germany	0.5334	1.4646	0.4807	0.0698
XU100	Turkey	0.5493	1.4350	0.4870	0.0724
FTSE	UK	0.4470	1.5171	0.4500	0.0787
NYA	USA	0.5348	1.4457	0.4418	0.0821
NIKKEI	Japan	0.5063	1.4716	0.4285	0.0825
KS11	South Korea	0.5137	1.4204	0.4473	0.0829
SSMI	Switzerland	0.5297	1.4617	0.3983	0.0929
BEL20	Belgium	0.5481	1.4574	0.3869	0.0981
MIBTEL	Italy	0.5267	1.4728	0.3525	0.1063
NASD	USA	0.5340	1.4526	0.3428	0.1114
SPX	USA	0.5026	1.4437	0.3405	0.1119
KFX	Denmark	0.5927	1.4665	0.3516	0.1148
DJI	USA	0.4477	1.4685	0.3284	0.1165
BUX	Hungary	0.6448	1.4844	0.3811	0.1170
TSE	Canada	0.5626	1.4375	0.3272	0.1210
TA100	Israel	0.6536	1.4739	0.3648	0.1251
BUSP	Brazil	0.6055	1.4142	0.3435	0.1262
JKSE	Indonesia	0.6505	1.3657	0.3986	0.1311
WIG20	Poland	0.5232	1.4545	0.2790	0.1326
ATX	Austria	0.6744	1.4455	0.3669	0.1336
HSI	Hong-Kong	0.5945	1.4033	0.3033	0.1396
IPC	Mexico	0.5550	1.3817	0.2991	0.1398
ASE	Greece	0.6210	1.3926	0.2911	0.1518
SSEC	China	0.6205	1.3698	0.3019	0.1533
IGBM	Spain	0.5615	1.4581	0.1912	0.1691
STRAITS	Singapore	0.5937	1.4500	0.2027	0.1702
PX	Czech Rep	0.6124	1.4386	0.2053	0.1743
MERVAL	Argentina	0.5850	1.3729	0.2225	0.1745
HEX	Finland	0.5524	1.4385	0.1747	0.1768
BSE	India	0.6139	1.4313	0.1842	0.1841
SET	Thailand	0.5591	1.4311	0.1590	0.1851
KLSE	Malaysia	0.5489	1.3620	0.1773	0.1906
IGRA	Peru	0.6806	1.3435	0.2160	0.2108
SAX	Slovakia	0.6673	1.3132	0.1534	0.2421
IBC	Venezuela	0.5881	1.3308	0.0890	0.2439
IPSA	Chile	0.4997	1.3187	0.0239	0.2711

Putting the estimates of the three measures together, we get the Efficiency Index which is also graphically presented in 1. For the ranking of indices according to their efficiency, we present Tab. 1. The most efficient stock market turns out to be the Dutch AEX closely followed by the French CAC and the German DAX. We can observe that the most efficient markets are usually the EU (or rather Eurozone) countries followed by the US markets and other developed markets from the rest of the world – Japanese NIKKEI, Korean KS11, Swiss SSMI. The least efficient part of the ranking is dominated by the Asian

and the Latin American countries. At the very end, we have the Slovakian SAX, Venezuelan IBC and Chilean IPSA. The efficiency of the stock markets is thus strongly geographically determined which is connected to the stage of development of the specific markets.

## 4 Conclusions

We have utilized long-term memory, fractal dimension and approximate entropy as input variables for the Efficiency Index [19]. This way, we are able to comment on stock market efficiency after controlling for different types of inefficiencies. Applying the methodology on 38 stock market indices across the world, we find that the most efficient markets are situated in the Eurozone (the Netherlands, France and Germany) and the least efficient ones in the Latin America (Venezuela and Chile). The Efficiency Index thus well corresponds to the expectation that the stock market efficiency is connected to the development of the market.

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