A New Approach to Interval-Valued Choquet Integrals and the Problem of Ordering in Interval-Valued Fuzzy Set Applications

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***Abstract*—We consider the problem of choosing a total order between intervals in multiexpert decision making problems. To do so, we first start researching the additivity of interval-valued aggregation functions. Next, we briefly treat the problem of preserving admissible orders by linear transformations. We study the construction of interval-valued ordered weighted aggregation operators by means of admissible orders and discuss their properties. In this setting, we present the definition of an interval-valued Choquet integral with respect to an admissible order based on an admissible pair of aggregation functions. The importance of the definition of the Choquet integral, which is introduced by us here, lies in the fact that if the considered data are pointwise (i.e., if they are not proper intervals), then it recovers the classical concept of this aggregation. Next, we show that if we make use of intervals in multiexpert decision making problems, then the solution at which we arrive may depend on the total order between intervals that has been chosen. For this reason, we conclude the paper by proposing two new algorithms such that the second one allows us, by means of the Shapley value, to pick up the best alternative from a set of winning alternatives provided by the first algorithm.**

***Index Terms*—Interval-valued Choquet integral, interval-valued decision making, interval-valued fuzzy set, interval-valued linear order, interval-valued ordered weighted aggregation (OWA) operators, Shapley value.**

I. Introduction

In recent years, there has been an increasing interest in the use of extensions of fuzzy sets such as interval-valued fuzzy sets [37], [49] and Atanassov's intuitionistic fuzzy sets [6], [7] in many different fields such as image processing [5], [10], [18], classification [29], or consensus [9]. For many of these applications, such as, for instance, multiexpert decision making [40], there exists an order relation that should be fixed and that plays a crucial role for the proposed results. Here, for illustrative purposes, we choose a multiexpert decision making example, but we could use any other example from the cited fields.

A multiexpert decision making problem [9] consists of finding from a set $X = \{x_1, \ldots, x_p\}, (p \geq 2)$ the alternative which is the most accepted by a set of n experts $E = \{e_1, \ldots, e_n\}, (n > 2)$.

In [13], it is stated that the resolution of a multiexpert decision making problem consists of two steps:

1) Uniform representation of information. In this phase, the heterogeneous information for the problem (the information can be represented by means of preference orderings or utility functions or fuzzy preference relations) is translated into homogeneous information by means of different transformation functions [13], [20].
2) Application of a selection procedure. This procedure consists of two phases:
   a) Aggregation phase. A collective preference structure is built from the set of individual homogeneous preference structures.
   b) Exploitation phase. A given method is applied to the collective preference structure to obtain a selection of appropriate alternatives.

We often assume that the information provided by the n experts is homogeneous and represented by means of fuzzy preference relations. Otherwise, we apply the techniques developed in [13]. Moreover, when we use fuzzy preference relations, the following comes out.

1) In the aggregation phase, we often employ ordered weighted aggregation (OWA) operators to build the collective relation, and we use, hence, the usual total order between real numbers.
2) In the exploitation phase, we often aggregate the elements in each row by means of the Choquet integral. Once the elements of each row have been aggregated, we order the resulting numerical values in decreasing order, and we take as the best alternative the one corresponding to the row with the highest numerical value. Clearly, the usual total order between real numbers is again used; therefore, there are not noncomparable results.

There are problems for which the results these methods provide are not good enough. This fact can be due to different

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reasons. One of the most common ones is that the numerical values provided by the experts to describe their preference of one alternative versus the others are not the most suitable ones. Sometimes, an expert has doubts on the exact numerical value to be assigned. When this happens, from our point of view, it is advisable to ask the experts to describe their preferences by means of intervals [8]. That is, experts tell us that they prefer an alternative x_i versus alternative x_j with a numerical value which is between $\underline{a}$ and $\overline{a}$, and we represent this fact by means of the interval $[\underline{a}, \overline{a}]$.

However, when we use an interval-valued representation of preferences, the following problem, which does not exist for real numbers, arises: *choosing a linear (total) order between intervals*.

The choice of the linear order is a very difficult problem. It may happen that the application forces us to take a specific order, but usually, this is not the case. It is important to remark that, in general, this problem is not taken into account in the literature on the subject [2], [16], [35], and, for the most part, the order between intervals defined in [42] is considered. However, we think that the choice of the order in this kind of problems is determinant, since different orders can lead to completely different solutions of the same problem [12].

Moreover, in several aggregation techniques, linear orders of processed data are necessary, and thus, linear orders of intervals (refining their standard partial order) are of great interest. One possible approach solving this problem was recently proposed in [42]. In [12], a new approach to defining linear orders on the lattice of closed subintervals of the unit interval was proposed. It was based on the so-called admissible pairs of aggregation functions, i.e., pairs of aggregation functions satisfying some appropriate conditions. A crucial advantage of this approach is that it recovers the most of usual examples of linear orders that have appeared in the literature, such as that of Xu and Yager [42], as well as the lexicographic ones.

It is clear that in applications, not only the order is important, but also some other tools. In particular, aggregation functions have shown themselves as a very useful tool to deal with many different problems [19], [21].

These considerations have led us to consider the following main objectives for this paper:

1) to analyze interval-valued OWA (IVOWA) operators;
2) to present an approach to the interval-valued Choquet integral such that, whenever the considered intervals are pointwise (i.e., when we consider degenerate intervals), we recover the classical Choquet integral;
3) to present some examples in which the relevance of the order choice in interval-valued multiexpert decision making problems is made explicit;
4) to propose an algorithm for consensus between the different total orders that are used in a given problem.

Regarding objective 2, we focus on the specific case of interval-valued Choquet integrals [15], [22], since such aggregation techniques, as in the real case, provide the basis for properly defined averaging aggregation functions [19]. In this sense, we intend to carry on a study how interval-valued OWAs and Choquet integrals can be defined, and in particular, which of linear orders defined by means of admissible pairs are of interest for a meaningful definition of interval-valued Choquet integrals.

For the consensus algorithm in objective 4, we are going to use the Shapley value. The Shapley function, as one of the most important payoff indices, has been deeply researched in game theory, which satisfies several reasonable axioms; see [31]. Many researchers have noticed that the Shapley function is a powerful tool as an interaction index among players among coalitions. Therefore, whenever we have several winning alternatives in the exploitation phase of an interval-valued multiexpert decision making problem, this index allows us to know which is the most suitable one, as it takes into account all of the winning coalitions.

Nevertheless, we want to stress that we have written this algorithm as an illustration of the influence that the choice of a linear order for an application may have in its final output. It would also be possible to look for possible algorithms in many other fields such as image processing [5], [18] or classification [29].

The structure of this paper is the following. In the next section, we start by investigation of the additivity of interval-valued aggregation functions. In Section III, we briefly consider the problem of preserving admissible orders by linear transformations. Section IV is devoted to the construction of IVOWA operators by means of admissible orders and discussion of their properties. Section V contains the definition of an interval-valued Choquet integral with respect to an admissible order based on an admissible pair of aggregation functions, including the discussion of comonotone additivity of such integrals and two open problems. Then, we present a multiexpert decision making algorithm that makes use of interval-valued OWA operators and Choquet integrals. Next, we analyze the relevance of the linear order which has been chosen for our algorithm, and we prove that different orders lead to different results. We present a method to select a winning alternative in this case by means of Shapley values. We finish with some concluding remarks.

II. Additivity of Interval-Valued Aggregation Functions

The aim of this section is to investigate the additivity of interval-valued aggregation functions processing interval-valued inputs. We begin by recalling the notion of the aggregation function.

Let $(L, \preceq)$ be a bounded partially ordered set (poset) with a smallest element (bottom) 0_L and a greatest element (top) 1_L. A mapping $A : L^n \to L$ is an n-ary ($n \in \mathbb{N}$, $n \geq 2$) aggregation function on $(L, \preceq)$ if it is $\preceq$-increasing, i.e., for all $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in L^n$

$$A(\mathbf{x}) \preceq A(\mathbf{y}) \text{ whenever } x_1 \preceq y_1, \ldots, x_n \preceq y_n$$

and satisfies the boundary conditions

$$A(0_L, \ldots, 0_L) = 0_L, \quad A(1_L, \ldots, 1_L) = 1_L.$$

If $L = [0, 1]$ and $\preceq = \leq$ is the standard order of reals, we get the usual definition of an aggregation function on the unit interval (see, e.g., [19]). For more details on aggregation functions on posets, we refer, e.g., to [24].

As we intend to study interval-valued aggregation functions processing interval-valued inputs, consider L to be the set $L([0,1])$, i.e., the set of all closed subintervals of the unit interval:

$$L([0,1]) = \{[a,b] \mid 0 \leq a \leq b \leq 1\}.$$

Throughout this section, consider on $L([0,1])$ the standard partial order of intervals, i.e., the binary relation $\leq_2$ defined by

$$[a,b] \leq_2 [c,d] \Leftrightarrow a \leq c \wedge b \leq d\,. \tag{1}$$

$(L([0,1]),\leq_2)$ is a poset with the bottom $[0,0]$ and top $[1,1]$. We are interested in finding conditions under which aggregation functions on $(L([0,1]),\leq_2)$ are additive.

First assume that $\varphi : (L([0,1]),\leq_2) \to (L([0,1]),\leq_2)$ is an additive increasing function, i.e., for all $[a,b],\ [c,d] \in L([0,1])$:
1) if $[a,b]+[c,d] \in L([0,1])$, then $\varphi([a,b]+[c,d]) = \varphi([a,b]) + \varphi([c,d])$;
2) if $[a,b] \leq_2 [c,d]$, then $\varphi([a,b]) \leq_2 \varphi([c,d])$.

The values of φ can be written as

$$\varphi([a,b]) = [f([a,b]), g([a,b])] \tag{2}$$

where $f,\ g : (L([0,1]),\leq_2) \to ([0,1],\leq)$ are additive and increasing scalar-valued functions of intervals that satisfy, for each $[a,b] \in L([0,1])$, the property $f([a,b]) \leq g([a,b])$.

For all intervals $[p,q],\ [r,s] \in L([0,1])$ whose sum is also in $L([0,1])$, we have

$$f([p+r,q+s]) = f([p,q]+[r,s]) = f([p,q]) + f([r,s]).$$

Thus, putting $p=a, q=a, r=0$, and $s=b-a$, we obtain

$$f([a,b]) = f([a,a]) + f([0,b-a]) = f_1(a) + f_2(b-a)$$

where f_1 and f_2 are additive $[0,1] \to [0,1]$ functions. As f_1 satisfies the Cauchy equation $f_1(x+y) = f_1(x) + f_1(y)$ whenever x, y, and $x+y$ are in $[0,1]$, it is of the form $f_1(x) = \alpha x$ with $\alpha \geq 0$ (see [1]). Similarly, $f_2(x) = \beta x$, where $\beta \geq 0$. Thus

$$f([a,b]) = \alpha a + \beta(b-a) = (\alpha - \beta)a + \beta b.$$

Consider $a > 0$. Then, from $f\left(\left[\frac{a}{2},b\right]\right) \leq f([a,b])$, it follows that $\alpha \geq \beta$, i.e., $\gamma = \alpha - \beta \geq 0$. Therefore, $f([a,b]) = \gamma a + \beta b$, where $\gamma,\ \beta \geq 0$. Similarly, $g([a,b]) = \delta a + \epsilon b$, with $\delta,\ \epsilon \geq 0$. As for each interval $[a,b]$, $f([a,b]) \leq g([a,b])$, considering $a=0$ and $b>0$, we can derive $\beta \leq \epsilon$, and if $a=b=1$, we obtain $\gamma + \beta \leq \delta + \epsilon$.

Summarizing, for the functions f and g in (2), we have

$$f([a,b]) = \gamma a + \beta b, \quad g([a,b]) = \delta a + \epsilon b$$

where

$$\gamma,\ \beta,\ \delta,\ \epsilon \geq 0,\ \ \beta \leq \epsilon,\ \ \gamma + \beta \leq \delta + \epsilon. \tag{3}$$

Proposition 1: Consider a mapping $A : (L([0,1]))^n \to L([0,1])$. Then
1) A is additive and increasing (w.r.t. $\leq_2$) if and only if

$$A([a_1,b_1],\ldots,[a_n,b_n]) \left[\sum_{i=1}^{n}(\gamma_i a_i + \beta_i b_i), \sum_{i=1}^{n}(\delta_i a_i + \epsilon_i b_i)\right] \tag{4}$$

where $\beta_i,\ \gamma_i,\ \delta_i,\ \epsilon_i \geq 0$, such that $\beta_i \leq \epsilon_i,\ \gamma_i + \beta_i \leq \delta_i + \epsilon_i$, and $\sum_{i=1}^{n}(\delta_i + \epsilon_i) \leq 1$.
2) A is an additive aggregation function on $(L([0,1]),\leq_2)$ if and only if the coefficients in (4) satisfy

$$\beta_i,\ \gamma_i,\ \delta_i,\ \epsilon_i \geq 0$$
$$\beta_i \leq \epsilon_i,\ \ \gamma_i + \beta_i = \delta_i + \epsilon_i$$
$$\text{and } \sum_{i=1}^{n}(\delta_i + \epsilon_i) = 1.$$

3) A is an additive aggregation function on $(L([0,1]),\leq_2)$ with idempotent element $[0,1]$ if and only if

$$A([a_1,b_1],\ldots,[a_n,b_n]) = \left[\sum_{i=1}^{n} w_i a_i, \sum_{i=1}^{n} w_i b_i\right]$$

where $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$.

The proof of this claim is simple and therefore omitted.

III. Admissible Orders

A. Admissible Orders Generated by Aggregation Functions

A crucial property for defining some types of aggregation functions on $[0,1]$, e.g., OWA operators [44], is the linearity of the standard order of reals which makes possible to compare any two inputs. The order $\leq_2$ considered in the previous section is only a partial order on $L([0,1])$.

In [12], the notion of admissible orders on $L([0,1])$ was introduced and studied. Recall that a binary relation $\preceq$ on $L([0,1])$ is an *admissible order* if it is a linear order on $L([0,1])$ refining $\leq_2$. The latter property means that for all $[a,b],\ [c,d] \in L([0,1])$, if $[a,b] \leq_2 [c,d]$, then $[a,b] \preceq [c,d]$ as well. As shown in [12], admissible orders on $L([0,1])$ can be generated by means of pairs of aggregation functions on $[0,1]$. For the convenience of the reader, we repeat from [12] that information concerning admissible orders generated by aggregation functions, which is relevant for our next work.

Let $K([0,1]) = \{(a,b) \in [0,1]^2 \mid a \leq b\}$. Intervals from $L([0,1])$ are in a one-to-one correspondence with points from $K([0,1])$, and a partial (linear) order $\preceq$ on one of these sets induces a partial (linear) order on the other, i.e., $[a,b] \preceq [c,d] \Leftrightarrow (a,b) \preceq (c,d)$.

Proposition 2 (see [12]): Let $A,\ B : [0,1]^2 \to [0,1]$ be two aggregation functions, such that for all $(x,y),\ (u,v) \in K([0,1])$, the equalities $A(x,y) = A(u,v)$ and $B(x,y) = B(u,v)$ can hold only if $(x,y) = (u,v)$. Define the relation $\preceq_{A,B}$ on $L([0,1])$ by

$[x,y] \preceq_{A,B} [u,v]$ if and only if

$$A(x,y) < A(u,v)$$
$$\text{or } A(x,y) = A(u,v) \text{ and } B(x,y) \leq B(u,v). \tag{5}$$

Then, $\preceq_{A,B}$ is an admissible order on $L([0,1])$.

We say that a pair (A, B) of aggregation functions described in Proposition 2 generates the order $\preceq_{A,B}$. It is called an *admissible pair of aggregation functions*. In this study, we will consider admissible orders generated by continuous aggregation functions only. Moreover, as proved in [12], if (A, B) is an admissible pair of continuous aggregation functions, then there exists an admissible pair of aggregation functions (A', B') such that A', B' are idempotent continuous aggregation functions, and the orders generated by the pairs (A, B) and (A', B') coincide.

Example 1: Consider the following relations on $L([0,1])$:

1) $[a, b] \preceq_{Lex1} [c, d] \;\;\Leftrightarrow\;\; a < c$ or $(a = c$ and $b \leq d)$;
2) $[a, b] \preceq_{Lex2} [c, d] \;\;\Leftrightarrow\;\; b < d$ or $(b = d$ and $a \leq c)$.
   It is clear that both these relations are admissible orders on $L([0,1])$. The order $\preceq_{Lex1}$ is generated by the pair (P_1, P_2), where $P_i, i = 1, 2$, is the projection to the ith coordinate, and similarly, $\preceq_{Lex2}$ is generated by (P_2, P_1).

The orders $\preceq_{Lex1}$ and $\preceq_{Lex2}$ are called the lexicographical orders with respect to the first or second coordinate, respectively.

A particular way of obtaining admissible orders on $L([0,1])$ is defining them by means of K_α mappings. For $\alpha \in [0,1]$, define the mapping $K_\alpha : [0,1]^2 \to [0,1]$ by

$$K_\alpha(a, b) = a + \alpha(b - a). \tag{6}$$

The values of K_α can be written as $K_\alpha(a, b) = (1 - \alpha)a + \alpha b$; thus, K_α is a weighted mean. If for $\alpha, \beta \in [0,1], \alpha \neq \beta$, the relation $\preceq_{\alpha,\beta}$ on $L([0,1])$ is given by

$$\begin{aligned} [a, b] \preceq_{\alpha,\beta} [c, d] \;\; &\Leftrightarrow K_\alpha(a, b) < K_\alpha(c, d) \\ &\text{or } (K_\alpha(a, b) = K_\alpha(c, d) \text{ and } K_\beta(a, b) \leq K_\beta(c, d)) \end{aligned} \tag{7}$$

then it is an admissible order on $L([0,1])$ generated by an admissible pair of aggregation functions (K_α, K_β) [12]. The following important property of orders $\preceq_{\alpha,\beta}$ was also proved in [12].

Proposition 3 (see [12]):

1) Let $\alpha \in [0, 1[$. Then, all admissible orders $\preceq_{\alpha,\beta}$ with $\beta > \alpha$ coincide. This admissible order will be denoted by $\preceq_{\alpha+}$.
2) Let $\alpha \in]0, 1]$. Then, all admissible orders $\preceq_{\alpha,\beta}$ with $\beta < \alpha$ coincide. This admissible order will be denoted by $\preceq_{\alpha-}$.

Remark 1

1) The lexicographical orders $\preceq_{Lex1}$ and $\preceq_{Lex2}$ are recovered by orders $\preceq_{\alpha,\beta}$ as the orders $\preceq_{0,1} = \preceq_{0+}$ and $\preceq_{1,0} = \preceq_{1-}$, respectively.
2) Xu and Yager defined the order $\preceq_{XY}$ on $L([0,1])$ by

$$\begin{aligned} [a, b] \preceq_{XY} [c, d] \;\; &\Leftrightarrow \;\; a + b < c + d \\ &\text{or} \;\; a + b = c + d \;\wedge\; b - a \leq d - c \end{aligned}$$

   see [42]. $\preceq_{XY}$ is an admissible order which corresponds to the order $\preceq_{0.5+}$. From the statistical point of view, this order corresponds to the ordering of random variables based on the expected value as the primary criterion, and on the variance as the secondary criterion (in the case of uniform distributions this is a linear order over their supports).

B. Admissible Orders Preserved by Linear Transformations

In this section, we will discuss admissible orders $\preceq_{A,B}$ on $L([0,1])$ which are preserved by any increasing linear transformation, i.e., orders satisfying for any increasing linear transformation R, given by $R(x) = px + q$, with $p > 0$, the property

$$[a, b] \preceq_{A,B} [c, d] \;\;\Rightarrow\;\; R([a, b]) \preceq_{A,B} R([c, d]) \tag{8}$$

provided that all intervals are in $L([0,1])$. Note that $R([a, b])$ means $[R(a), R(b)]$.

Theorem 1: An order $\preceq_{A,B}$ on $L([0,1])$ generated by a continuous admissible pair (A, B) of aggregation functions is preserved by any increasing linear transformation, i.e., satisfies (8), if and only if $\preceq_{A,B}$ coincides with $\preceq_{\alpha-}$ or $\preceq_{\alpha+}$ for some $\alpha \in [0,1]$.

Proof: The sufficiency is only a matter of processing. Consider an admissible pair (A, B) of continuous aggregation functions such that the linear order $\preceq_{A,B}$ generated by (A, B), as mentioned in Proposition 2, is preserved by any increasing linear transformation. For simplifying, the notation put $\preceq_{A,B} = \preceq$. As mentioned earlier, we may suppose that both A and B are idempotent. As we are only interested in inputs $(a, b) \in K([0,1])$, we may suppose, without loss of generality, that both A and B are symmetric. Suppose that for intervals $[a, b], [c, d] \in L(]0, 1[)$ (closed subintervals of the interval $]0, 1[$), it holds $A(a, b) < A(c, d)$. Then, for any increasing linear transformation R such that $R([a, b]), R([c, d]) \in L([0,1])$, it holds that $R([a, b]) \prec R([c, d])$. If $A(R(a), R(b)) = A(R(c), R(d))$, then, due to the continuity of A, there exists an $\epsilon > 0$ such that $[a + \epsilon, b + \epsilon] \in L([0,1])$ and $A(a + \epsilon, b + \epsilon) < A(c, d)$ and, consequently, $A(R(a), R(b)) = A(R(a + \epsilon), R(b + \epsilon)) = A(R(c), R(d))$. However, A is constant on the rectangle determined by points (a, b) and $(a + \epsilon, b + \epsilon)$. This contradicts the fact that it cannot exist a continuous order isomorphism between the unit square with a linear order that extends $\leq_2$ and the $[0, 1]$ interval with the usual order (see [12]). Thus, necessarily $A(R(a), R(b)) < A(R(c), R(d))$. As a consequence, we obtain that $A(a, b) = A(c, d)$ implies $A(R(a), R(b)) = A(R(c), R(d))$, and from the continuity of A, this result holds on $K([0,1])$, i.e., increasing linear transformations preserve level lines of $A|K([0,1])$. However, this means that these level lines are necessarily parallel segments, and thus, we have $A = K_\alpha$ for all $(a, b) \in K([0,1])$ and some $\alpha \in [0,1]$.

Next, if $A|K([0,1]) = K_0 = Min$, then since $\preceq$ refines $\leq_2$, we obtain $\preceq = \preceq_{0+}$. Similarly, if $A|K([0,1]) = K_1$, then necessarily, $\preceq = \preceq_{1-}$.

Let $A|K([0,1]) = K_\alpha$ for some $\alpha \in]0, 1[$. The aggregation function B should be injective on each level line of A in K([0,1]). Suppose that $B(0, 1) < B(\alpha, \alpha)$. Due to continuity (and injectivity on the segment connecting $(0, 1)$ and (α, α), which is just the α-level line of A), it holds $B(0, 1) < B(c\alpha, c\alpha + 1 - \alpha) < B(\alpha, \alpha)$ for each $c \in]0, 1[$, which corresponds to the admissible order $\preceq_{\alpha-}$. Due to the preservation of the order $\preceq$ by any increasing linear transformation, this result can be extended to any level line of A, and thus, $\preceq = \preceq_{\alpha-}$.

Similarly, if $B(0, 1) > B(\alpha, \alpha)$, we get $\preceq = \preceq_{\alpha+}$. □

IV. Interval-Valued Ordered Weighted Aggregation Operators

One type of aggregation functions, that are very often discussed in the literature and applied in practice, are OWA operators introduced by Yager [44]. Their definition strongly depends on the fact that the interval $[0, 1]$ with the usual order between real numbers is a linearly ordered set.

Definition 1: Let $\mathbf{w} = (w_1, \ldots, w_n) \in [0, 1]^n$ with $w_1 + \cdots + w_n = 1$ be a weighting vector. An OWA operator $OWA_{\mathbf{w}}$ associated with $\mathbf{w}$ is a mapping $OWA_{\mathbf{w}} : [0, 1]^n \to [0, 1]$ defined by

$$OWA_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{(i)} \tag{9}$$

where $x_{(i)}, i = 1, \ldots, n$, denotes the ith greatest component of the input $(x_1, \ldots, x_n)$.

It is clear that this definition in the case of real weights can be extended straightforwardly to the interval-valued setting.

The concept of OWA has been extended to the interval-valued setting (or more generally, to the type-2 fuzzy sets setting) by Zhou *et al*. ([51], see also [52] for a fast implementation of the method, and [14]), where weights are given by means of type-1 fuzzy sets. In these works, the authors, taking into account the concept of alpha-level aggregation, define OWA operators with linguistic weights. Our definition, which takes as weights real numbers in [0,1] and focuses on the choice of the linear order between intervals, can be seen as a particular case of the Zhou *et al*. definition.

Definition 2: Let $\preceq$ be an admissible order on $L([0, 1])$, and $\mathbf{w} = (w_1, \ldots, w_n) \in [0, 1]^n, w_1 + \cdots + w_n = 1$, a weighting vector. An IVOWA operator associated with $\preceq$, and $\mathbf{w}$ is a mapping $IVOWA^{\preceq}_{\mathbf{w}} : (L([0, 1]))^n \to L([0, 1])$ defined by

$$IVOWA^{\preceq}_{\mathbf{w}}([a_1, b_1], \ldots, [a_n, b_n]) = \sum_{i=1}^{n} w_i \cdot [a_{(i)}, b_{(i)}] \tag{10}$$

where $[a_{(i)}, b_{(i)}], i = 1, \ldots, n$, denotes the ith greatest interval of the input intervals with respect to the order $\preceq$.

Note that the arithmetic operations on intervals are given as follows:

$$w \cdot [a, b] = [wa, wb] \quad \text{and} \quad [a, b] + [c, d] = [a + c, b + d].$$

Observe that IVOWA operators in Definition 2 are well defined, since

$$w_1 a_{(1)} + \cdots + w_n a_{(n)} \leq w_1 + \cdots + w_n = 1$$

and analogously for the upper bound. The increasing monotonicity of real-valued weighted arithmetic means ensures that the resulting set on the right-hand side of (10) is an interval $[a, b]$, $a \leq b$.

Moreover, although the choice of a permutation (.) in formula (10) need not be unique (this may happen only if some inputs are repeated), it has no influence on the resulting output interval.

It is worth saying that Xu and Da in [41, Def. 3.1] also present a notion of an IVOWA operator, which they call the uncertain OWA operator, and they use it for a linear objective-programming model. Their construction differs from ours mainly in the fact that the authors consider a specific, fixed order for the intervals rather than a general one. Moreover, Xu [39] carries on a detailed study of OWA operators in Atanassov's intuitionistic setting.

Definition 2 extends the usual definition of OWA operators, as shown in the next proposition.

Proposition 4: Let $\preceq$ be an admissible order on $L([0, 1])$, and let $\mathbf{w} = (w_1, \ldots, w_n) \in [0, 1]^n$ with $w_1 + \cdots + w_n = 1$ be a weighting vector. Then

$$OWA_{\mathbf{w}}(x_1, \ldots, x_n) = IVOWA^{\preceq}_{\mathbf{w}}([x_1, x_1], \ldots, [x_n, x_n]).$$

Proof: Observe that $x_{(1)} \geq \ldots \geq x_{(n)}$ implies $[x_{(1)}, x_{(1)}] \geq_2 \ldots \geq_2 [x_{(n)}, x_{(n)}]$ and, if the order $\preceq$ is admissible, also $[x_{(1)}, x_{(1)}] \succeq \ldots \succeq [x_{(n)}, x_{(n)}]$. Therefore

$$\begin{aligned} &OWA_{\mathbf{w}}(x_1, \ldots, x_n) \\ &\quad = [OWA_{\mathbf{w}}(x_1, \ldots, x_n), OWA_{\mathbf{w}}(x_1, \ldots, x_n)] \\ &\quad = \left[\sum_{i=1}^{n} w_i x_{(i)}, \sum_{i=1}^{n} w_i x_{(i)}\right] \\ &\quad = \sum_{i=1}^{n} w_i [x_{(i)}, x_{(i)}] \\ &\quad = IVOWA^{\preceq}_{\mathbf{w}}([x_1, x_1], \ldots, [x_n, x_n]). \end{aligned}$$

□

However, in general, the representability of IVOWA operators in the form

$$\begin{aligned} &IVOWA^{\preceq}_{\mathbf{w}}([a_1, b_1], \ldots, [a_n, b_n]) \\ &\quad = [OWA_{\mathbf{w}}(a_1, \ldots, a_n), OWA_{\mathbf{w}}(b_1, \ldots, b_n)] \end{aligned} \tag{11}$$

does not hold, as shown in the following example.

Example 2: Consider the weighting vector $\mathbf{w} = (1, 0, 0)$ and the lexicographical order $\preceq_{Lex1}$. For the intervals $\left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{1}{3}, \frac{1}{2}\right]$, and $\left[\frac{1}{3}, 1\right]$, it holds that

$$\left[\frac{1}{3}, \frac{1}{2}\right] \preceq_{Lex1} \left[\frac{1}{3}, 1\right] \preceq_{Lex1} \left[\frac{1}{2}, \frac{3}{4}\right].$$

Therefore

$$IVOWA^{\preceq_{Lex1}}_{\mathbf{w}}\left(\left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{3}, 1\right]\right) = \left[\frac{1}{2}, \frac{3}{4}\right]$$

and on the other hand

$$\left[OWA_{\mathbf{w}}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right), OWA_{\mathbf{w}}\left(\frac{3}{4}, \frac{1}{2}, 1\right)\right] = \left[\frac{1}{2}, 1\right].$$

Note that Xu [39] presents a specific example, for Atanassov's intuitionistic case, in which representability holds.

Now, let us investigate several properties of IVOWA operators.

Example 3: Consider the Xu and Yager's order $\preceq_{XY}$ (i.e., the order $\preceq_{0.5+}$), here simply denoted by $\preceq$, and the weighting vector $\mathbf{w} = (0.8, 0.2)$. Then, for intervals

$$\mathbf{x} = [0.5, 0.5], \quad \mathbf{y} = [0.1, 1] \quad \text{and} \quad \mathbf{z} = [0.6, 0.6]$$

it holds that $\mathbf{x} \preceq \mathbf{y} \preceq \mathbf{z}$, and therefore

$$IVOWA_{\mathbf{w}}^{\preceq}(\mathbf{x},\mathbf{y}) = 0.8 \cdot [0.1, 1] + 0.2 \cdot [0.5, 0.5] = [0.18, 0.9]$$

$$IVOWA_{\mathbf{w}}^{\preceq}(\mathbf{z},\mathbf{y}) = 0.8 \cdot [0.6, 0.6] + 0.2 \cdot [0.1, 1] = [0.5, 0.68].$$

Observe that although $\mathbf{x} = [0.5, 0.5] \leq_2 [0.6, 0.6] = \mathbf{z}$ (i.e., we have increased the first input interval with respect to the order $\leq_2$), the obtained values of the $IVOWA_{\mathbf{w}}^{\preceq}$ operator are not comparable in the order $\leq_2$, i.e., $IVOWA_{\mathbf{w}}^{\preceq}$ is not an aggregation function with respect to $\leq_2$.

Example 4: Consider the order $\preceq_{A,B}$ generated by an admissible pair (A, B) of aggregation functions, where $A(x, y) = (\sqrt{x} + \sqrt{y})/2$ and $B(x, y) = y$, and the IVOWA operator associated with the weighting vector $\mathbf{w} = \left(\frac{2}{3}, \frac{1}{3}\right)$. Let

$$\mathbf{x} = [0.25, 0.25], \quad \mathbf{y} = [0, 1], \quad \mathbf{z} = [0.25, 0.28].$$

Then, $\mathbf{x} \preceq_{A,B} \mathbf{y} \preceq_{A,B} \mathbf{z}$ and

$$IVOWA_{\mathbf{w}}^{\preceq_{A,B}}(\mathbf{x},\mathbf{y}) = \frac{2}{3}\mathbf{y} + \frac{1}{3}\mathbf{x} = \left[\frac{1}{12}, \frac{3}{4}\right],$$

$$IVOWA_{\mathbf{w}}^{\preceq_{A,B}}(\mathbf{z},\mathbf{y}) = \frac{2}{3}\mathbf{z} + \frac{1}{3}\mathbf{y} = \left[\frac{1}{6}, 0.52\right].$$

Next, $A\left(\frac{1}{12}, \frac{3}{4}\right) = 0.57735$ and $A\left(\frac{1}{6}, 0.52\right) = 0.5646679$, which means that $IVOWA_{\mathbf{w}}^{\preceq_{A,B}}(\mathbf{x},\mathbf{y}) \succ_{A,B} IVOWA_{\mathbf{w}}^{\preceq_{A,B}}(\mathbf{z},\mathbf{y})$, and this contradicts the $\preceq_{A,B}$- increasing monotonicity of $IVOWA_{\mathbf{w}}^{\preceq_{A,B}}$ operator.

In the next part, the notation $K_\alpha([a, b])$ means that we have assigned to an interval $[a, b] \in L([0, 1])$ the same value as to the corresponding point $(a, b) \in K([0, 1])$ by the mapping K_α, i.e., $K_\alpha([a, b]) = a + \alpha(b - a)$.

Proposition 5: Let $\preceq$ be an admissible order on $L([0, 1])$ generated by a pair (K_α, B), and let $IVOWA_{\mathbf{w}}^{\preceq}$ be an IVOWA operator defined by (10). Then

$$K_\alpha\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_n, b_n])\right) = OWA_{\mathbf{w}}\left(K_\alpha([a_1, b_1]), \ldots, K_\alpha([a_n, b_n])\right) \quad (12)$$

independently of B.

Proof: It is enough to observe that if $[a_{(1)}, b_{(1)}] \succeq \ldots \succeq [a_{(n)}, b_{(n)}]$, then $K_\alpha([a_{(1)}, b_{(1)}]) \geq \ldots \geq K_\alpha([a_{(n)}, b_{(n)}])$ as well. Next

$$\begin{aligned} K_\alpha&\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_n, b_n])\right) \\ &= K_\alpha\left(\sum_{i=1}^{n} w_i \cdot [a_{(i)}, b_{(i)}]\right) \\ &= K_\alpha\left(\left[\sum_{i=1}^{n} w_i a_{(i)}, \sum_{i=1}^{n} w_i b_{(i)}\right]\right) \\ &= \sum_{i=1}^{n} w_i K_\alpha([a_{(i)}, b_{(i)}]) \\ &= OWA_{\mathbf{w}}\left(K_\alpha([a_1, b_1]), \ldots, K_\alpha([a_n, b_n])\right). \end{aligned}$$

□

Corollary 1: Let $\preceq_{\alpha,\beta}$ be an admissible order on $L([0, 1])$ introduced in (7). Then, the IVOWA operator $IVOWA_{\mathbf{w}}^{\preceq_{\alpha,\beta}}$ is an aggregation function on $L([0, 1])$ with respect to the order $\preceq_{\alpha,\beta}$.

Proof: To simplify notation, write in the proof $\preceq$ instead of $\preceq_{\alpha,\beta}$. We have to show the increasing monotonicity of $IVOWA_{\mathbf{w}}^{\preceq}$ operators with respect to the order $\preceq$. If we increase any input $[a_i, b_i]$ to $[a_i', b_i']$ in $\preceq$-order, then, certainly, $K_\alpha([a_i, b_i]) \leq K_\alpha([a_i', b_i'])$, and thus, by Proposition 5, we have

$$K_\alpha\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_i, b_i], \ldots, [a_n, b_n])\right) \leq K_\alpha\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_i', b_i'] \ldots, [a_n, b_n])\right).$$

If the inequality is strict, the result follows. If it turns into equality, it is possible only if $K_\alpha([a_i, b_i]) = K_\alpha([a_i', b_i'])$, but then $K_\beta([a_i, b_i]) < K_\beta([a_i', b_i'])$. If the increase from $[a_i, b_i]$ to $[a_i', b_i']$ does not influence the ordinal relation of single inputs, then

$$\begin{aligned} K_\beta&\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_i, b_i], \ldots, [a_n, b_n])\right) \\ &= \sum_{i=1}^{n} w_i K_\beta\left([a_{(i)}, b_{(i)}]\right) \leq w_1 K_\beta\left([a_{(1)}, b_{(1)}]\right) \\ &\quad + \cdots + w_i K_\beta\left([a'_{(i)}, b'_{(i)}]\right) + \cdots + w_n K_\beta\left([a_{(n)}, b_{(n)}]\right) \\ &= K_\beta\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_i', b_i'], \ldots, [a_n, b_n])\right). \end{aligned}$$

If this is not a case, then necessarily there are some inputs $[a_j, b_j]$ with

$$K_\alpha([a_i, b_i]) = K_\alpha([a_j, b_j])$$

but

$$K_\beta([a_i, b_i]) < K_\beta([a_j, b_j]) < K_\beta([a_i', b_i'])$$

(observe the freedom in the determination of permutation (.) in Definition 2 if there are some ties). Then

$$\begin{aligned} r = K_\beta&\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_i', b_i'], \ldots, [a_n, b_n])\right) \\ &- K_\beta\left(IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_i, b_i], \ldots, [a_n, b_n])\right) \end{aligned}$$

depends on inputs $[a_j, b_j]$ satisfying $K_\alpha([a_j, b_j]) = K_\alpha([a_i, b_i])$, and $[a_i, b_i], [a_i', b_i']$ only, and due to the linearity of K_β, the result follows. □

Note that IVOWA operators can be seen as modified and particular cases of intuitionistic OWA operators (see, e.g., [26], [39], and [45]). However, the approaches in all mentioned papers are different from the presented one, as the aggregation of intervals is splitted into the aggregation of their left bounds (membership functions of intuitionistic fuzzy sets) and aggregation of right bounds (complements to nonmembership functions).

Recently, OWA operators on complete lattices were proposed and discussed in [25]. As a particular case, OWA operators on intervals in the form (11) are obtained.

V. Interval-Valued Choquet Integral

A. *Interval-Valued Choquet Integral Based on Aumann's Approach*

OWA operators are a particular case of more general aggregation functions called Choquet integrals. In this section, we introduce discrete interval-valued Choquet integrals of interval-valued fuzzy sets based on admissible orders $\preceq_{A,B}$. However, in the first subsection, we recall an extension of the Choquet integral to the interval-valued setting, which has been discussed, e.g., in [22] and [50]. A similar idea led Aumann [4] to introduce his integral of set-valued functions. These concepts are of the same nature as is the Zadeh extension principle [49].

Let $U \neq \emptyset$ be a finite set. Recall that a fuzzy measure m is a set function $m : 2^U \to [0,1]$ such that

$$m(\emptyset) = 0, \ \ m(U) = 1, \ \text{and} \ m(A) \leq m(B)$$
$$\text{whenever} \ \ A \subseteq B.$$

The discrete Choquet integral (or expectation) of a fuzzy set $f : U \to [0,1]$ with respect to m is defined by

$$C_m(f) = \sum_{i=1}^{n} f(u_{\sigma(i)}) \left(m\left(\{u_{\sigma(i)}, \ldots, u_{\sigma(n)}\}\right) - m\left(\{u_{\sigma(i+1)}, \ldots, u_{\sigma(n)}\}\right)\right) \tag{13}$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that

$$f(u_{\sigma(1)}) \leq f(u_{\sigma(2)}) \leq \cdots \leq f(u_{\sigma(n)})$$

and $\{u_{\sigma(n+1)}, u_{\sigma(n)}\} = \emptyset$, by convention. The discrete Choquet integral can be extended to the interval-valued setting as follows.

Definition 3: Let $F : U \to L([0,1])$ be an interval-valued fuzzy set and $m : 2^U \to [0,1]$ a fuzzy measure. The discrete Choquet integral $\mathbf{C}_m(F)$ of an interval-valued fuzzy set F with respect to m is given by

$$\{C_m(f) \mid f : U \to [0,1], \ f(u_i) \in F(u_i)\}. \tag{14}$$

From the properties of the standard Choquet integral of fuzzy sets, it follows that

$$\mathbf{C}_m(F) = [C_m(f_*), C_m(f^*)] \tag{15}$$

where $f_*, f^* : U \to [0,1]$ are given by $f_*(u_i) = a_i$ and $f^*(u_i) = b_i$, and $[a_i, b_i] = F(u_i)$.

Several properties of the discrete interval-valued Choquet integral $\mathbf{C}_m$ are discussed in [22] and [50]. For example, this integral is comonotone additive, i.e.,

$$\mathbf{C}_m(F + G) = \mathbf{C}_m(F) + \mathbf{C}_m(G)$$

whenever $F, G : U \to L([0,1])$ are such that interval $F(u_i) + G(u_i) \subseteq [0,1]$ for each $u_i \in U$, and F, G are comonotone, i.e.,

$$(f^*(u_i) - f^*(u_j))(g^*(u_i) - g^*(u_j)) \geq 0$$

and

$$(f_*(u_i) - f_*(u_j))(g_*(u_i) - g_*(u_j)) \geq 0$$

for all $u_i, u_j \in U$.

B. Interval-Valued Choquet Integral With Respect to $\preceq_{A,B}$-Orders

The basic idea of the original Choquet integral [15] is based on the linear order of reals allowing two different looks at functions. The vertical look is based on function values and is a background of the Lebesgue integral, while the horizontal look is linked to level cuts and is a basis not only for the Choquet integral but for several other types of integrals as well (see [23]), including among others, the Sugeno integral [32]. In this section, we introduce a discrete interval-valued Choquet integral of interval-valued fuzzy sets based on an (admissible) order of intervals in $L([0,1])$ directly, without using the notion of the Choquet integral of scalar-valued fuzzy sets.

Let $\preceq_{A,B}$ be an admissible order on $L([0,1])$ given by a generating pair of aggregation function (A, B) as explained in Proposition 2. The discrete interval-valued Choquet with respect to the order $\preceq_{A,B}$ is defined as follows.

Definition 4: Let $F : U \to L([0,1])$ be an interval-valued fuzzy set and $m : 2^U \to [0,1]$ a fuzzy measure. The discrete interval-valued Choquet integral with respect to an admissible order $\preceq_{A,B}$ ($\preceq_{A,B}$-Choquet integral for short) of an interval-valued fuzzy set F with respect to m, with the notation $\mathbf{C}_m^{\preceq_{A,B}}(F)$, is given by

$$\mathbf{C}_m^{\preceq_{A,B}}(F) = \sum_{i=1}^{n} F(u_{\sigma_{A,B}(i)}) \left(m\left(\{u_{\sigma_{A,B}(i)}, \ldots, u_{\sigma_{A,B}(n)}\}\right) - \left(m\left(\{u_{\sigma_{A,B}(i+1)}, \ldots, u_{\sigma_{A,B}(n)}\}\right)\right. \tag{16}$$

where $\sigma_{A,B} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that

$$F(u_{\sigma_{A,B}(1)}) \leq F(u_{\sigma_{A,B}(2)}) \leq \ldots \leq F(u_{\sigma_{A,B}(n)})$$

and $\{u_{\sigma_{A,B}(n+1)}, u_{\sigma_{A,B}(n)}\} = \emptyset$, by convention.

Observe that if $F(u_i) = [a_i, b_i], i = 1, \ldots, n$, then (16) can be written as

$$\left[\sum_{i=1}^{n} a_{\sigma_{A,B}(i)} \left(m\left(\{u_{\sigma_{A,B}(i)}, \ldots, u_{\sigma_{A,B}(n)}\}\right) - m\left(\{u_{\sigma_{A,B}(i+1)} - u_{\sigma_{A,B}(n)}\}\right)\right) \right.$$
$$\left. \sum_{i=1}^{n} b_{\sigma_{A,B}(i)} \left(m\left(\{u_{\sigma_{A,B}(i)}, \ldots, u_{\sigma_{A,B}(n)}\}\right) - m\left(\{u_{\sigma_{A,B}(i+1)} - u_{\sigma_{A,B}(n)}\}\right)\right) \right].$$

Next, for any fixed $F : U \to L([0,1])$ such that the corresponding f_* and f^* are comonotone, i.e., for all $u_i, u_j \in U$

$$(f_*(u_i) - f_*(u_j))(f^*(u_i) - f^*(u_j)) \geq 0$$

it holds that for any admissible pair (A, B) of aggregation functions, the Choquet integrals of F introduced in Definitions 3 and 4 coincide, i.e., $\mathbf{C}_m^{\preceq_{A,B}}(F) = \mathbf{C}_m(F)$.

The concept of an interval-valued Choquet integral $\mathbf{C}_m^{\preceq_{A,B}}$ introduced in Definition 4 extends the standard discrete Choquet integral given by (13). Indeed, if $F : U \to L([0,1])$ is singleton-valued, i.e., it is a fuzzy subset of U, then

$$\mathbf{C}_m^{\preceq_{A,B}}(F) = \mathbf{C}_m(F) = C_m(F)$$

independently of A and B.

Moreover, observe that if m is a symmetric fuzzy measure [36], then, similarly to the classical case, $\mathbf{C}_m^{\preceq_{A,B}} =$

$IVOWA_m^{\preceq_{A,B}}$, where $\mathbf{w} = (w_1, \ldots, w_n), w_i = m(\{i, i+1, \ldots, n\}) - m(\{i+1, \ldots, n\}), i = 1, \ldots, n$, with convention $\{n+1, n\} = \emptyset$.

C. Comonotone Additivity Based on $\preceq_{A,B}$-Orders

Recall that in [30], a comonotone additive aggregation function $H : [0,1]^n \to [0,1]$ is just the Choquet integral with respect to a fuzzy measure m given by $m(Y) = H(1_Y)$. It can easily be seen that the Choquet integral based on an order $\preceq_{A,B}$, where $\preceq_{A,B}$ is generated by an admissible pair (A, B) of aggregation functions, is comonotone additive. To simplify notation, put $\preceq_{A,B} = \preceq$. The comonotonicity of two interval vectors $\mathbf{x} = ([a_1, b_1], \ldots, [a_n, b_n])$ and $\mathbf{y} = ([c_1, d_1], \ldots, [c_n, d_n])$ means that there is a common permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that

$$[a_{\sigma(1)}, b_{\sigma(1)}] \leq_2 \ldots \leq_2 [a_{\sigma(n)}, b_{\sigma(n)}]$$

and

$$[c_{\sigma(1)}, d_{\sigma(1)}] \leq_2 \ldots \leq_2 [c_{\sigma(n)}, d_{\sigma(n)}].$$

In general, for two different linear orders $\preceq_1$ and $\preceq_2$, the corresponding Choquet integrals differ, and thus, the comonotone additivity cannot be a sufficient condition for defining the $\preceq$-Choquet integral. This integral is also $\leq_2$-increasing and, thus, a $\leq_2$-aggregation function on $L([0,1])$. However, in general, a $\preceq$-Choquet integral is not $\preceq$-comonotone additive.

Example 5: Let $U = \{1, 2\}$. Consider the weakest fuzzy measure m_* on U (i.e., $m_*(Y) = 0$ for each $Y \subsetneq U$) and the aggregation functions A, B on $[0,1]$, $A(u,v) = \frac{u^2+v^2}{2}$ and $B(u,v) = uv$. Let $\preceq$ be the linear order on $L([0,1])$ generated by the pair (A, B).

As $[0.5, 0.5] \preceq [0.2, 0.7]$, the interval vectors

$$\mathbf{x} = ([0.5, 0.5], [0.2, 0.7]),\ \mathbf{y} = ([0.2, 0.2], [0.2, 0.2])$$

are clearly $\preceq$-comonotone.

Note that any input interval vector $\mathbf{u} = ([a_1, b_1], [a_2, b_2])$ is an interval-valued function on U, given by $\mathbf{u}(i) = [a_i, b_i]$. By Definition 4, $C_{m_*}^{\preceq}(\mathbf{x})$ is the $\preceq$-minimal input, and hence

$$C_{m_*}^{\preceq}(\mathbf{x}+\mathbf{y}) = C_{m_*}^{\preceq}([0.7, 0.7], [0.4, 0.9]) = [0.4, 0.9]$$

which violates the comonotone additivity of $C_{m_*}^{\preceq}$.

Hence, $\preceq$-comonotone additivity is not a necessary condition to define the $\preceq$-Choquet integral axiomatically.

Open problem 1: Is there an axiomatic characterization of the $\preceq$-Choquet integral?

Proposition 6: Let $\preceq_{\alpha,\beta}$ be a linear order on $L([0,1])$ introduced in (7). Then, the $\preceq_{\alpha,\beta}$-Choquet integral is $\preceq_{\alpha,\beta}$-comonotone additive.

Proof: Recall that $\preceq_{\alpha,\beta}$ is a linear order generated by the aggregation functions $A = K_\alpha$ and $B = K_\beta$, where $K_\alpha(a,b) = a + \alpha(b-a) = (1-\alpha)a + \alpha b$ and, analogously, K_β.

We first prove that if $[a_1, b_1] \preceq_{\alpha,\beta} [c_1, d_1]$ and $[a_2, b_2] \preceq_{\alpha,\beta} [c_2, d_2]$, then, in addition

$$[a_1 + a_2, b_1 + b_2] \preceq_{\alpha,\beta} [c_1 + c_2, d_1 + d_2].$$

If $K_\alpha(a_1, b_1) = (1-\alpha)a_1 + \alpha b_1 < (1-\alpha)c_1 + \alpha d_1 = K_\alpha(c_1, d_1)$, then due to $K_\alpha(a_2, b_2) \leq K_\alpha(c_2, d_2)$, it holds that

$$K_\alpha(a_1 + a_2, b_1 + b_2) = K_\alpha(a_1, b_1) + K_\alpha(a_2, b_2)$$
$$< K_\alpha(c_1, d_1) + K_\alpha(c_2, d_2) = K_\alpha(c_1 + c_2, d_1 + d_2).$$

Thus

$$[a_1 + a_2, b_1 + b_2] \preceq_{\alpha,\beta} [c_1 + c_2, d_1 + d_2].$$

A similar result can be shown in all remaining cases, i.e., when $K_\alpha(a_1, b_1) = K_\alpha(c_1, d_1)$ and $K_\alpha(a_2, b_2) < K_\alpha(c_2, d_2)$, or when

$$K_\alpha(a_1, b_1) = K_\alpha(c_1, d_1),\ \ K_\alpha(a_2, b_2) = K_\alpha(c_2, d_2)$$
$$K_\beta(a_1, b_1) \leq K_\beta(c_1, d_1),\ \ K_\beta(a_2, b_2) \leq K_\beta(c_2, d_2).$$

Then, if $\mathbf{x}, \mathbf{y} \in (L([0,1]))^n$ are $\preceq_{\alpha,\beta}$-comonotone interval vectors, for a permutation σ related to $\preceq_{\alpha,\beta}$-comonotonicity, it also holds that $[a_{\sigma(1)} + c_{\sigma(1)}, b_{\sigma(1)} + d_{\sigma(1)}] \preceq_{\alpha,\beta} \ldots \preceq_{\alpha,\beta} [a_{\sigma(n)} + c_{\sigma(n)}, b_{\sigma(n)} + d_{\sigma(n)}]$, and the $\preceq_{\alpha,\beta}$-comonotone additivity of the $\preceq_{\alpha,\beta}$-Choquet integral follows from its definition. □

Open problem 2: Is it true that a $\preceq_{\alpha,\beta}$-comonotone additive aggregation function C on $L([0,1])$, where the order $\preceq_{\alpha,\beta}$ is defined by (7), is necessarily the $\preceq_{\alpha,\beta}$-Choquet integral?

VI. Application to Multiexpert Decision Making

Consider n experts $E = \{e_1, \ldots, e_n\}, (n > 2)$ and a set of p alternatives $X = \{x_1, \ldots, x_p\}, (p \geq 2)$. Our goal is to find the alternative which is the most accepted one by the n experts.

Many times experts have difficulties to determine the exact value of the preference of an alternative x_i against x_j for each $i, j \in \{1, \ldots, p\}$. When this happens, they usually give their preferences by means of elements in $L([0,1])$, i.e., by means of intervals. In these cases, we say that the preference of the expert is given by a numerical value inside the interval.

Interval-valued fuzzy preference relations have been studied by many authors, as, for instance, Turksen and Biljic [34] or Xu [38]. In this paper, as already stated in Section I, we use them to illustrate the importance that the choice of a linear order has for many applications.

A. Interval-Valued Preference Relations

We know that an interval-valued fuzzy binary relation R_{IV} on X is defined as an interval-valued fuzzy subset of $X \times X$, i.e., $R_{IV} : X \times X \to L([0,1])$. The interval $R_{IV}(x_i, x_j) = R_{IV_{ij}}$ denotes the degree to which elements x_i and x_j are related in the relation R_{IV} for all $x_i, x_j \in X$ [17], [38]. Particularly, in preference analysis, $R_{IV_{ij}}$ denotes the degree to which an alternative x_i is preferred to alternative x_j.

Each expert e provides his/her preferences by means of an interval-valued fuzzy relation R_{IV_e} with p rows and p columns and where the elements in the diagonal are not considered, i.e., R_{IV_e}, shown at the bottom of the next page,

To find the solution alternative for the problem, we propose the following algorithm:

Aggregation phase:

$IVD1$) Choose a linear order $\preceq$ between intervals.

$IVD2$) Choose a weighting vector $\mathbf{w}$.

$IVD3$) Calculate the interval-valued collective fuzzy relation $R_{IV\,c}$ using the operators $IVOWA^{\preceq}_{\mathbf{w}}$.

Exploitation phase:

$IVD4$) For each row i in $R_{IV\,c}$, build the fuzzy measure m_i:

$$m_i(\{x_{ij}\})_{i\neq j} = \left(\frac{\underline{R}_{ij} + \overline{R}_{ij}}{\sum_{\substack{l=1\\ l\neq i}}^{p} (\underline{R}_{il} + \overline{R}_{il})}\right)^2$$

$$m_i(\{x_{ij}, x_{ik}\})_{\substack{i\neq j\\ i\neq k\\ j<k}} = \left(\frac{\underline{R}_{ij} + \overline{R}_{ij} + \underline{R}_{ik} + \overline{R}_{ik}}{\sum_{\substack{l=1\\ l\neq i}}^{p} (\underline{R}_{il} + \overline{R}_{il})}\right)^2$$

$$\cdots \tag{17}$$

that is, given $i \in \{1, \ldots, p\}$, for every $A \subseteq \{1, \ldots, n\}\backslash\{i\}$

$$m_i(\{x_{ij} \mid j \in A\}) = \left(\frac{\sum_{j\in A} \underline{R}_{ij} + \overline{R}_{ij}}{\sum_{\substack{l=1\\ l\neq i}}^{p} (\underline{R}_{il} + \overline{R}_{il})}\right)^2.$$

$IVD5$) For each row of $R_{IV\,c}$, aggregate the intervals by means of the interval-valued Choquet integral constructed with the order $\preceq$ chosen in step $IVD1$) and the measure built in step $IVD4$).

$IVD6$) Take as solution the alternative corresponding to the row with the biggest interval with respect to the order $\preceq$ chosen in step $IVD1$).

Algorithm 1

Remarks:

$I.-$ Note that if the preference relations provided by the experts are numerical, then with this algorithm, we recover the classical methods which are used for multiexpert decision making and which make we use of the Choquet integrals in the exploitation phase [46].

$II.-$ In step $IVD4$, for each row, i.e., for each alternative, we use (17) since the values obtained with this measure are proportional to the preferences provided by the experts for one alternative against the others. This way, with this measure, we take into account all the information originally provided by the experts.

Proposition 7: The measure defined in (17) is superadditive, i.e., for any two nonintersecting subsets $A, B \in X, A \cap B = \emptyset$

$$m_i(A \cup B) \geq m_i(A) + m_i(B)$$
$$\text{for each row } i = 1, \ldots, p \tag{18}$$

Proof: The fact follows from the superadditivity of the quadratic function $f(x) = x^2$ on $[0, 1]$. □

$III.-$ Note that we do not require that R_{IV_e} is reciprocally additive [17], [27], i.e., we do not demand the following property:

$$\underline{R}_{e_{ij}} + \overline{R}_{e_{ji}} = 1 \text{ and } \underline{R}_{e_{ji}} + \overline{R}_{e_{ij}} = 1.$$

The advantage of not demanding it is that we do not modify the preferences provided by the experts in order to ensure additivity.

B. Choice of the Best Alternative Using the Shapley Value

The result of Algorithm 1 depends on the order $\preceq$ and the weighting vector $\mathbf{w}$ that we use. In both cases, the choice we make is linked to the application in which we are working. Usually, the choice of the weighting vector is easier, since the weights are often related to the quantifiers given in [44], and it is the application which determines that we have to consider aggregations of the type: *most of the experts* say ... or *at least one half of the experts* say ... etc.

The choice of the order is more complicated. Both the application and the experts should be taken into account. For instance, if the experts are considered to be optimistic, it may be logical to use the order $\preceq_{Lex2}$. On the contrary, if they are considered to be pessimistic, the order $\preceq_{Lex1}$ might be more suitable. However, in many cases, we do not have this information. Clearly, if the application determines the order to be used, we apply Algorithm 1 straight.

If we do not know which is the most appropriate order, we propose to run Algorithm 1 with different orders, for instance, with s different orders. If for all the considered orders we obtain the same result, i.e., the same alternative, then we have finished and we choose as the winning alternative that one. However, if we obtain different winning alternatives, then we propose the following algorithm (i.e., Algorithm 2):

$SC1$ Run Algorithm 1 for each of the s selected orders.

$$R_{IV_e} = \begin{pmatrix} & x_1 & x_2 & \cdots, & \cdots, & x_p \\ x_1 & - & [\underline{R}_{e_{12}}, \overline{R}_{e_{12}}] & [\underline{R}_{e_{13}}, \overline{R}_{e_{13}}] & \cdots, & [\underline{R}_{e_{1p}}, \overline{R}_{e_{1p}}] \\ x_2 & [\underline{R}_{e_{21}}, \overline{R}_{e_{21}}] & - & [\underline{R}_{e_{23}}, \overline{R}_{e_{23}}] & \cdots, & [\underline{R}_{e_{2p}}, \overline{R}_{e_{2p}}] \\ & \cdots & \cdots & \cdots & - & \cdots \\ x_p & [\underline{R}_{e_{p1}}, \overline{R}_{e_{p1}}] & [\underline{R}_{e_{p2}}, \overline{R}_{i_{p2}}] & \cdots & \cdots & - \end{pmatrix}.$$

$SC2$ For each interval-valued collective fuzzy relation ${R_{IV}}_c{}^l$ with $l = 1, \ldots, s$, calculate the fuzzy preference relation such that each of its elements is obtained as the midpoint of the corresponding interval in the relation ${R_{IV}}_c{}^l$.

$SC3$ Calculate the arithmetic mean matrix MP of the s fuzzy matrices obtained in Step $SC2$:

$$MP = \begin{pmatrix} - & a_{12} & a_{13} & \cdots, & a_{1p} \\ a_{21} & - & a_{23} & \cdots, & a_{2p} \\ \ldots & \ldots & \ldots & - & \ldots \\ a_{p1} & a_{p2} & \cdots & a_{p(p-1)} & - \end{pmatrix}.$$

$SC4$ Build the measure (19), shown at the bottom of the page, that is, for each $A \subseteq \{1, \ldots, p\}$

$$m(\{x_i \mid i \in A\}) = \left(\frac{\sum_{i \in A} \sum_{j \in \{1,\ldots,p\} \setminus \{i\}} a_{ij}}{\sum_{i=1}^{n} \sum_{j \in \{1,\ldots,p\} \setminus \{i\}} a_{ij}} \right)^2.$$

$SC5$ Using the measure m from step $SC4$, calculate the Shapley value:

$$\varphi(x_i) = \sum_{A \subseteq X \setminus \{x_i\}} \frac{1}{n \begin{pmatrix} n & -1 \\ |A| \end{pmatrix}} (m(A \cup \{x_i\}) - m(A)) \quad (20)$$

for each of the solutions obtained in step $SC1$.

$SC6$ Take as solution the alternative corresponding to the highest Shapley value.

Algorithm 2

Remarks:

$I.-$ We use the Shapley value φ since, once the winning alternatives x_i have been calculated with Algorithm 1 ($i = 1, \ldots, s$), $\varphi(x_i)$ measures the relevance of alternative x_i in possible coalitions with other alternatives.

$II.-$ The advantage of using the measure given in (19) is that it takes into account all the preference values provided by all the experts. This way, the Shapley value is calculated using the same matrix MP for all the winning alternatives. This is the main difference between the measure given in (17) and the one given in (19). Note that the measure in (19) is superadditive as well.

If with Algorithm 2 we get the same Shapley value for different alternatives and we cannot decide which is the best one, then we can take as solution the one which appears most times as winner when we run Algorithm 1 with the s different orders.

In [43], Xu proposes an example that we develop next. Xu makes use of Atanassov's multiplicative intuitionistic fuzzy sets in the range $[\frac{1}{9}, 9]$. We adapt this example to the interval-valued setting [3] in the lattice $[0, 1]$ by means of the linear transformation $f(x) = \frac{80x+1}{9}$.

TABLE I
RANKINGS OF OBTAINED ALTERNATIVES

order	$\preceq_{XY}$	$\preceq_{Lex1}$	$\preceq_{Lex2}$	$\preceq_{\frac{1}{3},\frac{2}{3}}$
	Altx_4	Altx_3	Altx_4	Altx_3
	Altx_3	Altx_4	Altx_3	Altx_4
	Altx_1	Altx_1	Altx_1	Altx_1
	Altx_2	Altx_2	Altx_2	Altx_2

Example 6: Four university students share a house, where they intend to have broadband internet connection installed. There are four options available to choose from, which are provided by three internet service providers:

1) x_1: 1 Mb/s broadband;
2) x_1: 2 Mb/s broadband;
3) x_1: 3 Mb/s broadband;
4) x_1: 4 Mb/s broadband.

Since the internet service and its monthly bill will be shared among the four students $\{e_1, e_2, e_3, e_4\}$ with the weight vector $\mathbf{w} = (0.3, 0.3, 0.2, 0.2)$, they decide to perform a multiexpert decision making problem. Suppose that the students reveal their preference relations for the options independently and anonymously as in (21), shown at the bottom of the next page.

First, we run Algorithm 1 with the order $\preceq_{XY}$ in step $IVD1$) and the operator $IVOWA_{\mathbf{w}}^{\preceq_{XY}}$ for step $IVD3$). Then, the collective matrix ${R_{IV}}_c$ is given as the second expression at bottom of the next page.

Using the measure given in (17) for the exploitation phase in Algorithm 1, we have

$$\begin{aligned} &Altx_1 \ [0.0493545, 0.865581] \\ &Altx_2 \ [0.0282378, 0.656260] \\ &Altx_3 \ [0.1158570, 0.881090] \\ &Altx_4 \ [0.0954420, 0.909448]. \end{aligned} \quad (22)$$

As the considered order is $\preceq_{XY}$, we have the following ranking of preferences: $Altx_4 \succeq_{XY} Altx_3 \succeq_{XY} Altx_1 \succeq_{XY} Altx_2$.

We repeat Algorithm 1 for $\preceq_{Lex1}$, $\preceq_{Lex2}$, and $\preceq_{\alpha,\beta}$ with $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$. In Table I, we present the rankings of alternatives that we have obtained.

From Table I, we deduce that depending on the considered order relation, the winning alternative may be the third or the fourth. To decide which one of them we choose, we use Algorithm 2. After calculating the midpoints of the intervals for

$$\begin{aligned} m(\{x_i\}) &= \left(\frac{a_{i1} + \cdots + a_{i(i-1)} + a_{i(i+1)} + \cdots a_{ip}}{a_{12} + \cdots + a_{1p} + \cdots + a_{i1} + \cdots + a_{ip} + \cdots + a_{p1} + \cdots + a_{i(p-1)}} \right)^2 \\ m(\{x_i, x_j\}) &= \left(\frac{a_{i1} + \cdots + a_{i(i-1)} + a_{i(i+1)} + \cdots a_{ip} + a_{j1} + \cdots + a_{j(j-1)} + a_{j(j+1)} + \cdots a_{jp}}{a_{12} + \cdots + a_{1p} + \cdots + a_{i1} + \cdots + a_{ip} + \cdots + a_{p1} + \cdots + a_{i(p-1)}} \right)^2 \\ &\cdots \end{aligned} \quad (19)$$

each of the collective matrices, we have the following arithmetic mean matrix:

$$MP = \begin{pmatrix} - & 0.466625 & 0.441875 & 0.498125 \\ 0.564125 & - & 0.3498125 & 0.37990625 \\ 0.57078125 & 0.662 & - & 0.486875 \\ 0.5121875 & 0.63209375 & 0.5196875 & - \end{pmatrix}.$$

Using the measure given in (19), we have the following Shapley values:

$$\varphi(x_3) = 0.2626754310172201$$
$$\varphi(x_4) = 0.25479505453479. \quad (23)$$

Therefore, we have to pick up alternative x_3.

Clearly, it would also be possible to choose other decision making methods, as, for instance, the one proposed by Xu [38]. However, in this method, a real-valued compatibility is used to determine the ranking between alternatives, whereas in our case,

$$R_{IV_1} = \begin{pmatrix} - & [0.01, 0.675] & [0.025, 0.9] & [0.04375, 0.9] \\ [0.325, 0.99] & - & [0.015625, 0.7875] & [0.025, 0.7875] \\ [0.1, 0.975] & [0.2125, 0.984375] & - & [0.025, 0.9] \\ [0.1, 0.95625] & [0.2125, 0.975] & [0.1, 0.975] & - \end{pmatrix}$$

$$R_{IV_2} = \begin{pmatrix} - & [0.025, 0.7875] & [0.015625, 0.7875] & [0.025, 0.9] \\ [0.2125, 0.975] & - & [0.01, 0.675] & [0.015625, 0.7875] \\ [0.2125, 0.984375] & [0.325, 0.99] & - & [0.04375, 0.9] \\ [0.1, 0.975] & [0.2125, 0.984375] & [0.1, 0.95625] & - \end{pmatrix}$$

$$R_{IV_3} = \begin{pmatrix} - & [0.2125, 0.975] & [0.025, 0.7875] & [0.1, 0.975] \\ [0.025, 0.7875] & - & [0.015625, 0.675] & [0.01, 0.675] \\ [0.2125, 0.975] & [0.325, 0.984375] & - & [0.1, 0.95625] \\ [0.025, 0.9] & [0.325, 0.99] & [0.04375, 0.9] & - \end{pmatrix}$$

$$R_{IV_4} = \begin{pmatrix} - & [0.025, 0.9] & [0.04375, 0.9] & [0.04375, 0.95625] \\ [0.1, 0.975] & - & [0.01, 0.5625] & [0.015625, 0.675] \\ [0.1, 0.95625] & [0.4375, 0.99] & - & [0.04375, 0.9] \\ [0.04375, 0.95625] & [0.325, 0.984375] & [0.1, 0.95625] & - \end{pmatrix}. \quad (21)$$

$$\begin{pmatrix} - & [0.07825, 0.855] & [0.02875, 0.855] & [0.056875, 0.939375] \\ [0.18625, 0.942] & - & [0.013375, 0.68625] & [0.0173125, 0.7425] \\ [0.1675, 0.9740625] & [0.33625, 0.98775] & - & [0.056875, 0.916875] \\ [0.07375, 0.950625] & [0.28, 0.9841875] & [0.08875, 0.950625] & - \end{pmatrix}.$$

we look for an interval-valued valuation of each alternative, since our main objective is to make clear the importance of the choice of the order between intervals for the applications. Nevertheless, in future works, we intend to carry on an analysis of possible combinations of our ideas with Xu's method.

VII. Concluding Remarks

In this paper, starting from the notion of admissible order built by means of admissible pairs of aggregation functions, we have proposed the construction of interval-valued Choquet integrals. To do so, we have analyzed several properties of admissible orders, with a special focus on their preservation by linear transformations, which is a crucial characteristic for defining Choquet integrals.

Our study of interval-valued Choquet integrals has allowed us to define IVOWA operators. The interest of this definition lies in the fact that admissible orders enable us to build many different OWA operators, that, on one hand, extend usual operators, but, on the other hand, leave some free space for choosing the most appropriate one for the problem under consideration. The question of determining the most suitable linear order for a given problem is of great interest, as we have exhibited for multiexpert decision making when we use intervals to represent the alternatives. The theoretical studies in this paper have allowed us to present an algorithm (i.e., Algorithm 1) similar to the classical ones for decision making but using intervals. Running this algorithm for different orders shows that depending on the order, the winning alternative may change. For this reason, we have presented another algorithm (i.e., Algorithm 2) to select the best winning alternative. To do so, we have made use of the Shapley value.

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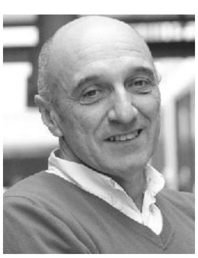

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