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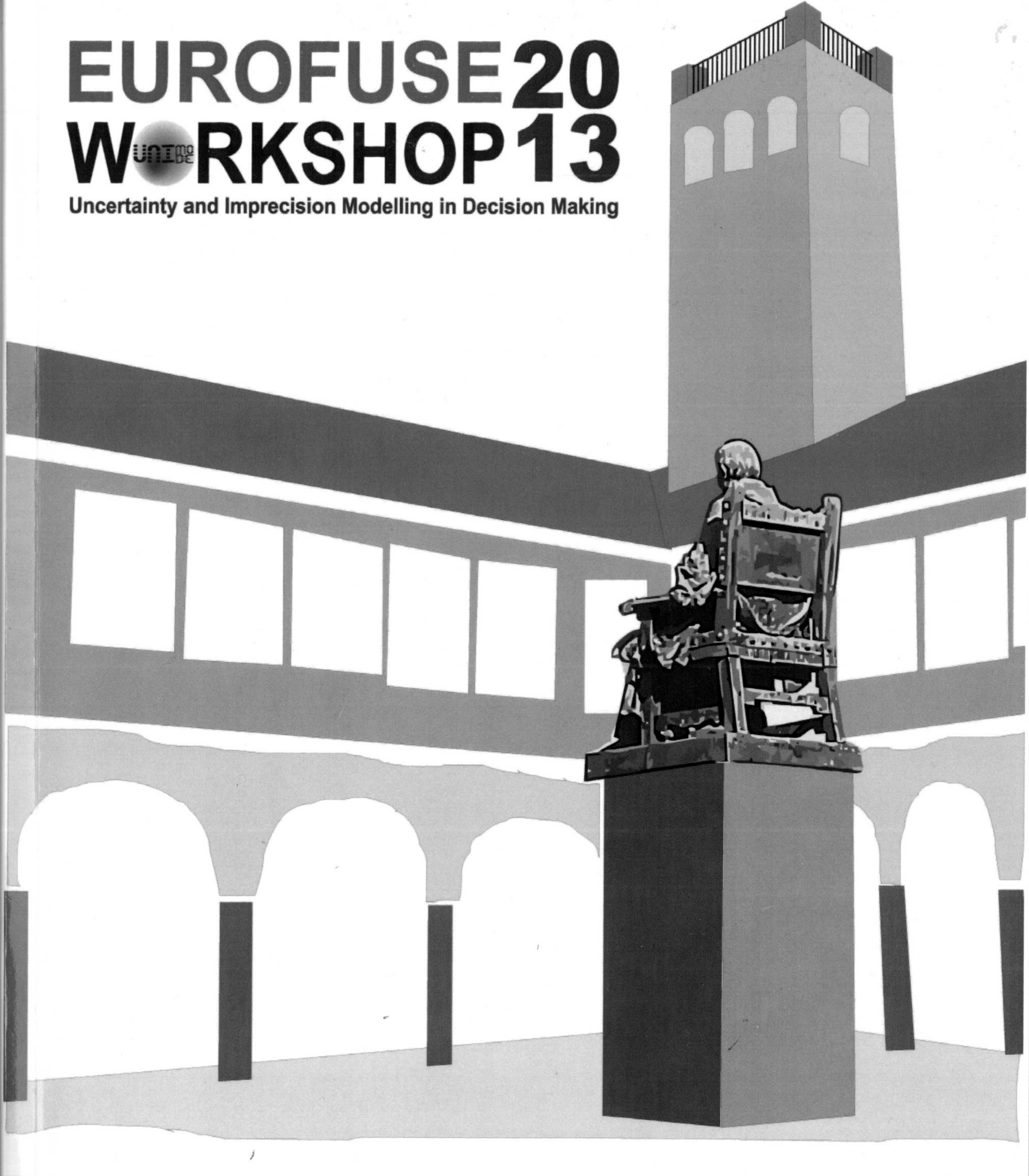
Uncertainty and Imprecision Modelling in Decision Making



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Contributions

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Some notions of internal operators

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Abstract. In this contribution we consider a first approach to the notion of internal operator as a mapping which gives back as output one of the inputs. We present the definition of such operators and their first properties.

Keywords: Internal operator, aggregation function, idempotency.

1 Introduction

In recent years aggregation operators have attracted a growing interest both from theoretical and applied researchers due to their huge applicability in many different fields [6]. In particular, such operators are a crucial tool for any procedure in which the fusion of information coming from either homogeneous or heterogeneous sources is required [4].

For some applications, however, it is important that the resulting output does not incorporate any new information from that already contained in the inputs. In image fusion algorithms ([1], but see also [5]), for instance, it is natural that the value of the intensity of a given pixel corresponds to the value of the same pixel for some of the considered images. In this sense, it is natural to require that the operator which is chosen for the fusion displays some kind of internality, in the sense of providing an output which is equal to one of the inputs.

These considerations have led us to consider in the present contribution the notion of internal operator. In particular, we present its definition (which is very close to that of locally internal operator, [7]) and we make a first analysis of its first properties.

The structure of this contribution is as follows. We first present the notion of internal operator. Then, in Section 3, we analyze a construction method for internal operators, and in Section 4 we consider those internal operators which are also aggregation functions. We finish with some conclusions and references.

2 Internal operators

As we have stated in the introduction, we are interested in those operators whose output is exactly equal to one if its inputs. It is important to recall that such notion, in the case of aggregation functions, was already considered by G. Mayor and J. Martin in [7], where they introduced the concept of locally internal operator as follows.

Definition 1 *$f : [0,1]^n \to [0,1]$ is a locally internal aggregation function if:*

1. *f is continuous.*
2. *f is non-decreasing.*
3. *$f(x, \ldots, x) = x$ for every $x \in [0,1]$.*
4. *f is locally internal; that is, $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for every $(x_1, \ldots, x_n) \in [0,1]^n$.*

In the same work, the authors provide a complete characterization of such locally internal aggregation operators in terms of the min, the max and the projection operators.

However, the requirements of continuity and monotonicity may be too demanding for some specific applications [1]. For this reason, we propose the following definition of internal operator.

Definition 2 *An internal operator is a mapping $F : [0,1]^n \to [0,1]$ such that*

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$$

for every $(x_1, \ldots, x_n) \in [0,1]^n$.

Note that only local internality is demanded in our definition.

Example 1 *1. Let π_j denote the j-th projection given by*

$$\pi_j(x_1, \ldots, x_n) = x_j$$

Then, for every $j \in \{1, \ldots, n\}$, the operator π_j is an internal operator. This operator is also a locally internal operator in the sense of Mayor and Martin.

2. *Both the minimum and the maximum are internal operators.*
3. *Fix $a \in [0,1]$. Consider the operator F such that $F(x_1, \ldots, x_n) = a$ if $a \in \{x_1, \ldots, x_n\}$ or x_1 in other case. This is an operator which is not continuous and not even monotone, so it is not a locally internal operator.*

On the other hand, the averaging character of the internal operators is straight.

Proposition 1 *Let F be an internal operator. The following items hold:*

i) $F(x, \ldots, x) = x$ for all $x \in [0,1]$;

ii) $min(x_1, \ldots, x_n) \leq F(x_1, \ldots, x_n) \leq max(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in [0,1]^n$.

Proof: Direct since we always recover one of the inputs. *QED*

Notice that, on the other hand, not every averaging aggregation function is an internal operator, as the case of the arithmetic mean, for instance, shows.

In the following example we show that internal operators need not to satisfy well known properties as monotonocity, homogeneity, invariance under translation and others.

Example 2 *1. An internal operator needs not to be monotone non-decreasing. Consider the mode, which recovers the most frequent value. The mode is an internal operator but needs not to be monotone non-decreasing.*

2. An internal operator needs not to be homogeneous. Consider the F operator defined as:

$$F(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$$

if $\max(x_1, \ldots, x_n) \leq \frac{1}{4}$ *and*

$$F(x_1, \ldots, x_n) = x_1$$

in other case. Then, if we take $n = 3$ *and* $\lambda = 0.5$*, we have that* $F\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right) = \frac{1}{2}$ *and* $F\left(\lambda\frac{1}{2}, \lambda\frac{1}{3}, \lambda\frac{1}{4}\right) = \frac{1}{8} \neq \lambda F\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right) = \frac{1}{4}$.

3. An internal operator needs not to be shift-invariant. Consider the same example as below. Then for $n = 3$ *and* $r = \frac{1}{2}$ *we have that* $F\left(\frac{1}{4}, \frac{1}{6}, \frac{1}{8}\right) = \frac{1}{8}$ *and* $F\left(\frac{1}{2} + r, \frac{1}{3} + r, \frac{1}{4} + r\right) = 1 \neq F\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right) + r$.

4. An internal operator needs not to be migrative [2]. Any projection provides an example of internal operator which is not migrative.

Next, we study the structure of internal operators. Let's denote by $\mathcal{F}(n)$ the class of all internal operators defined over $[0,1]^n$.

Theorem 1 *$(\mathcal{F}(n), max_{\mathcal{F}}, min_{\mathcal{F}})$ is a bounded lattice, with the operations $max_{\mathcal{F}}$ and $min_{\mathcal{F}}$ defined as*

$$\begin{aligned} max_{\mathcal{F}}(F,G)(x_1, \ldots, x_n) &= \max(F(x_1, \ldots, x_n), G(x_1, \ldots, x_n)) \\ min_{\mathcal{F}}(F,G)(x_1, \ldots, x_n) &= \min(F(x_1, \ldots, x_n), G(x_1, \ldots, x_n)) \end{aligned}$$

for every $F, G \in \mathcal{F}(n)$ and every $(x_1, \ldots, x_n) \in [0,1]$.

Proof: It's enough to notice that $max_{\mathcal{F}}(F(x_1, \ldots, x_n), G(x_1, \ldots, x_n))$ and $min_{\mathcal{F}}(F(x_1, \ldots, x_n), G(x_1, \ldots, x_n)) \in \{x_1, \ldots, x_n\}$ since $F(x_1, \ldots, x_n)$, $G(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$. *QED*

Proposition 2 *Let's denote by*

$$\begin{aligned} F_0 &= \inf\{F : [0,1]^n \to [0,1] | F \in \mathcal{F}(n)\} \\ F_\infty &= \sup\{F : [0,1]^n \to [0,1] | F \in \mathcal{F}(n)\} \end{aligned}$$

Then $F_0(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$ *and* $F_0(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)$.

Proof: It's straight forward from Theorem 1. *QED*

Regarding composition, we can also say the following.

Proposition 3 *Let* $F_0, F_1, \ldots, F_n : [0,1]^n \to [0,1]$ *be* $n+1$ *internal operators. Then,* $F : [0,1]^n \to [0,1]$ *given by:*

$$F(x_1, \ldots, x_n) = F_0(F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$$

is also an internal operator.

Proof: It is just a straight comprobation. *QED*

3 A characterization of internal operators

In this section we provide an easy characterization of internal operators in terms of topological partitions. Let's recall the definition of the latter in the sense that we consider in the present work.

Definition 3 *Consider a family of indexes I. A family $\{\varphi_i\}_{i\in I}$ with $\varphi_i : [0,1]^n \to \{0,1\}$ for every $i \in I$ is a partition if for every $(x_1,\ldots,x_n) \in [0,1]^n$ there exists $i_0 \in I$ such that $\varphi_{i_0}(x_1,\ldots,x_n) = 1$ and $\varphi_i(x_1,\ldots,x_n) = 0$ for every $i \neq i_0$.*

Example 3 *1. Let's take $\varphi_1(x_1,\ldots,x_n) = 1$ if $x_1 \leq \frac{1}{2}$ and 0 in other case and $\varphi_2(x_1,\ldots,x_n) = 1 - \varphi_1(x_1,\ldots,x_n)$. Then $\{\varphi_1,\varphi_2\}$ is a partition of $[0,1]^n$.*

Theorem 2 *A mapping $F : [0,1]^n \to [0,1]$ is an internal operator of dimension n if and only if there exists a partition $\{\varphi_1,\ldots,\varphi_n\}$ of $[0,1]^n$ such that*

$$F(x_1,\ldots,x_n) = \varphi_1(x_1,\ldots,x_n)\pi_1(x_1,\ldots,x_n) + \cdots + \varphi_n(x_1,\ldots,x_n)\pi_n(x_1,\ldots,x_n). \quad (1)$$

Proof: First, observe that F defined as in Expression 1 is an internal operator. Now, assume that F is an internal operator. For $j = 1,\ldots,n$ we define:

$$A_j = \{(x_1,\ldots,x_n) \in [0,1]^n | F(x_1,\ldots,x_n) = x_j \text{ and there is not } k < j \text{ such that } F(x_1,\ldots,x_n) = x_k\}$$

Notice that $A_i \cap A_j = \emptyset$ as long as $i \neq j$, since we are avoiding the problems that can arise from n-tuples with several equal components. Moreover, since F is internal, it follows that $\bigcup_{j=1}^n A_j = [0,1]^n$. Let's define as φ_j the characteristic function of the set A_j, that is, $\varphi_j : [0,1]^n \to [0,1]$ given by

$$\varphi_j(x_1,\ldots,x_n) = 1 \text{ if } (x_1,\ldots,x_n) \in A_j$$

and 0 otherwise. Then, the family $\{\varphi_1,\ldots,\varphi_n\}$ is a partition of $[0,1]^n$ and by definition of each φ_j we have that

$$F(x_1,\ldots,x_n) = \varphi_1(x_1,\ldots,x_n)\pi_1(x_1,\ldots,x_n) + \cdots + \varphi_n(x_1,\ldots,x_n)\pi_n(x_1,\ldots,x_n).QED$$

Corollary 1 *Let $F : [0,1]^n \to [0,1]$ be an internal operator and let $\{\varphi_j\}_{j=1,\ldots,n}$ and $\{\psi_j\}_{j=1,\ldots,n}$ be partitions of $[0,1]^n$. Then, if*

$$\begin{aligned} F(x_1,\ldots,x_n) &= \varphi_1(x_1,\ldots,x_n)\pi_1(x_1,\ldots,x_n) + \cdots + \varphi_n(x_1,\ldots,x_n)\pi_m(x_1,\ldots,x_n) \\ F(x_1,\ldots,x_n) &= \psi_1(x_1,\ldots,x_n)\pi_1(x_1,\ldots,x_n) + \cdots + \psi_n(x_1,\ldots,x_n)\pi_m(x_1,\ldots,x_n) \end{aligned}$$

holds for every $(x_1,\ldots,x_n) \in [0,1]^n$ such that $x_i \neq x_j$ whenever $i \neq j$, then $\varphi_j = \psi_j$ for every $j = 1,\ldots,n$.

Proof: Assume on the contrary that there exist $(x_1,\ldots,x_n) \in [0,1]^n$ and $j \in \{1,\ldots,n\}$ such that $\varphi_j(x_1,\ldots,x_n) \neq \psi_j(x_1,\ldots,x_n)$. Then we can assume that $\varphi_j(x_1,\ldots,x_n) = 1$ and $\psi_j(x_1,\ldots,x_n) = 0$. But then, on one hand

$$\begin{aligned} F(x_1,\ldots,x_n) &= \varphi_1(x_1,\ldots,x_n)\pi_1(x_1,\ldots,x_n) + \cdots + \varphi_n(x_1,\ldots,x_n)\pi_n(x_1,\ldots,x_n) = \\ &\quad \pi_j(x_1,\ldots,x_n) = x_j \end{aligned}$$

whereas on the other hand

$$F(x_1,\ldots,x_n) = \psi_1(x_1,\ldots,x_n)\pi_1(x_1,\ldots,x_n)+\cdots+\psi_n(x_1,\ldots,x_n)\pi_n(x_1,\ldots,x_n) = \pi_k(x_1,\ldots,x_n) = x_k$$

for some $k \neq j$. Since by hypothesis $x_k \neq x_j$, we arrive at a contradiction and the result follows. *QED*

4 Internal aggregation functions

In this section we study internal operators that are monotone non-decreasing. This property allows us to relate internal operators with aggregation functions.

Theorem 3 *Let $F : [0,1]^n \to [0,1]$ be an internal operator. If F is monotone non-decreasing in each variable, then F is an averaging aggregation function.*

Proof: Direct since F is idempotent and then $F(0,\ldots,0) = 0$ and $F(1,\ldots,1) = 1$. *QED*

Corollary 2 *Under conditions of Theorem 3, F is jointly strict monotone, that is*

$$x_i < y_i \text{ for all } i \in \{1,\ldots,n\} \text{ implies } F(x_1,\ldots,x_n) < F(y_1,\ldots,y_n).$$

Proof: Direct. *QED*

Proposition 4 *Let $F : [0,1]^n \to [0,1]$ be an aggregation functions with absorbing element $a \in [0,1]$. Then, if $a \in \{x_1,\ldots,x_n\}$ F is internal.*

5 Conclusions

In this work we have presented the idea of internal operator. Apart from considering a construction method, we have considered in particular the case of those internal operators which are also aggregation functions.

In the future we intend to go deeper into the analysis of such operators. In particular, we would like to consider those internal operators which could be of interest for specific applications.

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Reinterpret

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Abstract.
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