Bipolar semicopulas

Salvatore Greco[1], Radko Mesiar[2], and Fabio Rindone[1]

[1] Department of Economics and Business
University of Catania , Italy
{salgreco,frindone}@unict.it
[2] Department of Mathematics and Descriptive Geometry
Slovak University of Technology, Bratislava, Slovakia;
Institute of Theory of Information and Automation
Czech Academy of Sciences, Prague, Czech Republic
radko.mesiar@stuba.sk

Abstract. The concept of semicopula plays a fundamental role in the definition of a universal integral. We present an extension of semicopula to the case of symmetric interval $[-1,1]$. We call this extension bipolar semicopula. The last definition can be used to obtain a simplified definition of the bipolar universal integral. Moreover bipolar semicopulas allow for extension of theory of copulas to the interval $[-1,1]$.

1 Bipolar semicopulas

Definition 1. *A semicopula is a function* $\otimes : [0,1] \times [0,1] \to [0,1]$, *which is nondecreasing and has 1 as neutral element, i.e.*

- *if* $a_1 \leq a_2$ *and* $b_1 \leq b_2$, *then* $a_1 \otimes b_1 \leq a_2 \otimes b_2$; *and*
- $1 \otimes a = a \otimes 1 = a$.

Note that a semicopula has 0 as annihilator. Indeed $0 \leq a \otimes 0 \leq 1 \otimes 0 = 0$ and $0 \leq 0 \otimes a \leq 0 \otimes 1 = 0$.

Definition 2. *A bipolar semicopula is a function*

$$\otimes_b : [-1,1]^2 \to [-1,1]$$

that is "absolute-nondecreasing", has 1 as neutral element and -1 *as opposite-neutral element, and preserves the sign rule, i.e*

(A1) *if* $|a_1| \leq |a_2|$ *and* $|b_1| \leq |b_2|$ *then* $|a_1 \otimes_b b_1| \leq |a_2 \otimes_b b_2|$;
(A2) $a \otimes_b \pm 1 = \pm 1 \otimes_b a = \pm a$; *and*
(A3) $sign\,(a \otimes_b b) = sign(a) \otimes_b sign(b)$.

Let us note that a bipolar semicopula also satisfies the following additional properties

(A4) $a \otimes_b 0 = 0 \otimes_b a = 0$;
(A5) $sign(a) \otimes_b sign(b) = sign(a \cdot b)$; and
(A6) $|a \otimes_b b| = |a| \otimes_b |b|$.

Indeed, $0 \leq |a \otimes_b 0| \leq |\pm 1 \otimes_b 0| = |\pm 0| = 0$ and $0 \leq |0 \otimes_b a| \leq |0 \otimes_b \pm 1| = |\pm 0| = 0$. $(A5)$ is true by $(A4)$ if $a = sign(a) = 0$ or $b = sign(b) = 0$, while is true by $(A2)$ and $(A3)$ if $a = sign(a), b = sign(b) \in \{-1, 1\}$. Regarding $(A6)$, it is sufficient to note that for all $a \in [0,1]$, $|-a| \leq |a| \leq |-a|$, then $|\pm a \otimes_b (\pm b)| = |a \otimes_b b|$.

Let us consider the binary operation $*$ on $[-1,1]$ given by

$$a * b = \begin{cases} -ab \; if \; (a,b) \; \in]-1,1[^2 \\ ab \quad else. \end{cases}$$

This satisfies axioms $(A1)$ and$(A2)$, but not $(A3)$ (think to $a = -1/3 = b$), then the additional axiom $(A3)$ is necessary in order to consider bipolar semicopulas as symmetric extensions of standard semicopulas in the sense of product. Note that this approach preserves commutativity and associativity.
Notable examples of bipolar semicopulas are the standard product, $a \cdot b$ and the symmetric minimum [1, 2],

$$a \oslash b = sign(a \cdot b)(|a| \wedge |b|).$$

Proposition 1. $\otimes_b : [-1,1]^2 \to [-1,1]$ *is a bipolar semicopula if and only if there exists a semicopula* $\otimes : [0,1]^2 \to [0,1]$ *such that for all* $a, b \in [-1,1]$

$$a \otimes_b b = sign(a \cdot b)\left(|a| \otimes |b|\right). \tag{1}$$

Proof. Sufficient part. Suppose there exists a semicopula $\otimes$ such that (1) holds. If $|a_1| \leq |a_2|$ and $|b_1| \leq |b_2|$, then $|a_1| \otimes |b_1| \leq |a_2| \otimes |b_2|$, i.e. $|a_1 \otimes_b b_1| \leq |a_2 \otimes_b b_2|$. Moreover, $a \otimes_b \pm 1 = sign(a \cdot (\pm 1))(|a| \otimes 1) = \pm a$ and $\pm 1 \otimes_b a = sign(\pm 1 \cdot a)(1 \otimes |a|) = \pm a$. Proof of $(A3)$ is trivial and then, we conclude that $\otimes_b$ is a bipolar semicopula. Necessary part. Suppose $\otimes_b$ is a bipolar semicopula and define $a \otimes b = a \otimes_b b$ for all $a, b \in [0,1]$, then $a \otimes_b b = sign(a \cdot b)|a \otimes_b b| = sign(a \cdot b)(|a| \otimes_b |b|) = sign(a \cdot b)(|a| \otimes |b|)$.

We call $\otimes_b$ the bipolar semicopula induced by the semicopula $\otimes$ whenever the (1) holds. For example, the semicopula product induces the bipolar semicopula product, the semicopula minimum induces the bipolar semicopula symmetric minimum. Finally let us note that the concept of bipolar semicopula is closely related to that of *symmetric pseudo-multiplication* in [3].

2 Bipolar semicopula and bipolar universal integral

For the sake of simplicity in this note we present the result in a multiple criteria decision making setting. Let $N = \{1, \ldots, n\}$ be the set of criteria and let us identify the set of possible alternatives with $[-1,1]^n$. The definition of bipolar semicopulas can be used in order to define the bipolar universal integral [7] which is a generalization of the universal integral [9] from the scale $[0,1]$ to the symmetric scale $[-1,1]$. Let us consider the set of all disjoint pairs of subsets of N, i.e. $\mathcal{Q} = \left\{(A,B) \in 2^N \times 2^N \; : \; A \cap B = \emptyset\right\}$.

Definition 3. *A function* $m_b : \mathcal{Q} \to [-1,1]$ *is a normalized bi-capacity ([4], [5], [6]) on N if*

– $m_b(\emptyset,\emptyset) = 0$, $m_b(N,\emptyset) = 1$ *and* $m_b(\emptyset,N) = -1$;
– $m_b(A,B) \leq m_b(C,D)\ \forall\ (A,B),(C,D) \in Q\ :\ A \subseteq C$ *and* $B \supseteq D$.

Definition 4. *Let F_b be the set of functions $f : N \to [-1,1]$ and M_b the set of bi-capacities on Q. A function $I_b : M_b \times F_b \to [-1,1]$ is a bipolar universal integral on the scale $[-1,1]$ (or bipolar fuzzy integral) if the following axioms hold:*

(I1) $I_b(m_b, f)$ *is nondecreasing with respect to* m_b *and with respect to* f*;*
(I2) There exists a bipolar semicopula $\otimes_b$ *such that for any* $m_b \in M_b$, $c \in [0,1]$ *and* $(A,B) \in Q$, $I(m_b, c \cdot 1_{(A,B)}) = c \otimes_b m_b(A,B)$*;*
(I3) for all pairs $(m_{b_1}, f_1), (m_{b_2}, f_2) \in M_b \times F_b$*, such that for all* $t \in [0,1]$,
$m_{b_1}(\{i \in N : f_1(i) \geq t\}, \{i \in N : f_1(i) \leq -t\}) =$
$= m_{b_2}(\{i \in N : f_2(i) \geq t\}, \{i \in N : f_2(i) \leq -t\})$, $I(m_{b_1}, f_1) = I(m_{b_2}, f_2)$.

Clearly, in definition 4, F_b can be identified with $[-1,1]^n$, such that a function $f : N \to [-1,1]$ can be regarded as a vector $\boldsymbol{x} \in [-1,1]^n$. Note that the bipolar Choquet, Shilkret and Sugeno integrals [8] are bipolar universal integrals in the sense of Definition 4. Observe that the underlying bipolar semicopula $\otimes_b$ is the standard product in the case of the bipolar Choquet and Shilkret integrals, while $\otimes_b$ is the symmetric minimum for the bipolar Sugeno integral.

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