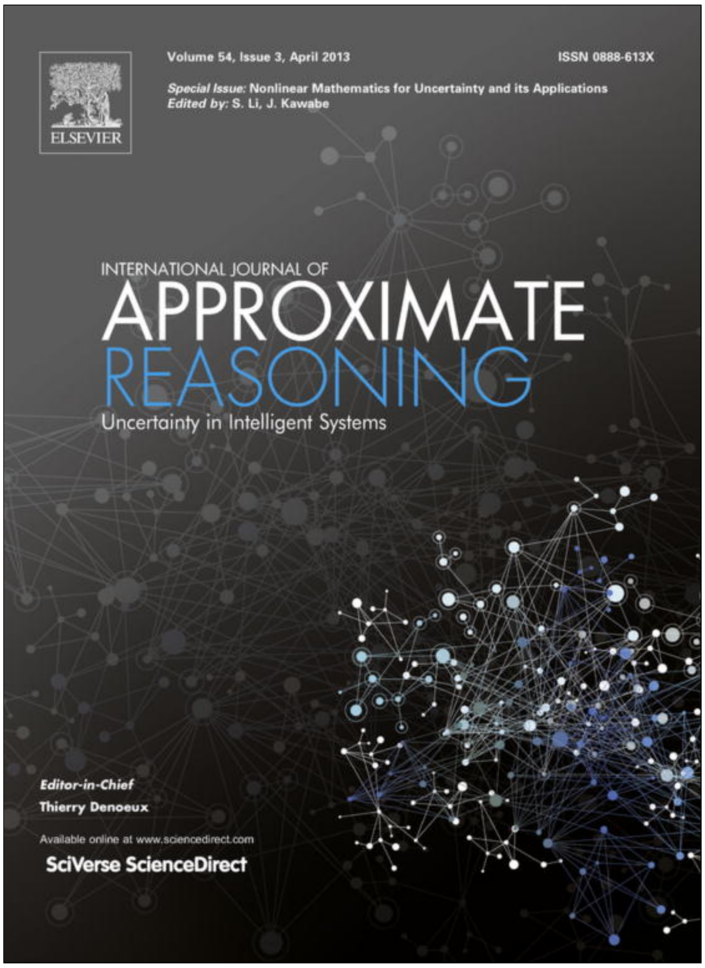
Volume 54, Issue 3, April 2013
ISSN 0888-613X
Special Issue: Nonlinear Mathematics for Uncertainty and its Applications
Edited by: S. Li, J. Kawabe
ELSEVIER
INTERNATIONAL JOURNAL OF
APPROXIMATE
REASONING
Uncertainty in Intelligent Systems
Editor-in-Chief
Thierry Denoeux
Available online at www.sciencedirect.com
SciVerse ScienceDirect

International Journal of Approximate Reasoning 54 (2013) 357–364

Contents lists available at SciVerse ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Discrete pseudo-integrals

Radko Mesiar [a,b,*], Jun Li [c], Endre Pap [d,e]

[a] Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia
[b] UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
[c] School of Sciences, Communication University of China, Beijing 100024, China
[d] Faculty of Natural Sciences and Mathematics, 21 000 Novi Sad, Serbia
[e] Obuda University, H-1034 Budapest, Hungary

ARTICLE INFO

Article history:
Available online 21 August 2012

Keywords:
Pseudo-addition
Pseudo-multiplication
Pseudo-concavity
Concave integral
Choquet integral
Sugeno integral

ABSTRACT

Integration of simple functions is a corner stone of general integration theory and it covers integration over finite spaces discussed in this paper. Different kinds of decomposition and subdecomposition of simple functions into basic functions sums, as well as different kinds of pseudo-operations exploited for integration and summation result into several types of integrals, including among others, Lebesgue, Choquet, Sugeno, pseudo-additive, Shilkret, PAN, Benvenuti and concave integrals. Some basic properties of introduced discrete pseudo-concave integrals are discussed, and several examples of new integrals are given.

1. Introduction

Discrete integrals are extensively used in decision theory based on a fixed number n of criteria and in game theory with n players. For a deeper discussion and references we recommend [3,5,7,15,22]. In the additive approach, Lebesgue integral is applied and the resulting integral (understood as an expected utility) is a weighted sum, i.e., an additive functional. Non-additive approaches were based mostly on Choquet integral [1] and on Sugeno integral [19].

Considering in this paper $N = \{1, \ldots, n\}$ as a fixed finite space, a basic function $b(c, E) : N \to [0, \infty]$ is given by

$$b(c, E)(i) = \begin{cases} c & \text{if } i \in E, \\ 0 & \text{else,} \end{cases}$$

where $E \subseteq N$ and $c \in [0, \infty]$. All above mentioned integrals assign to a basic function $b(c, E)$ a value dependent on c and measure value $m(E)$ only. This approach was recently exploited as a basic axiom for the universal integrals [5], generalizing all above mentioned integrals. However, there are integrals violating this rule, namely PAN-integrals [22,23] and recently introduced concave integral of Lehrer [7]. For these approaches, the integral of a basic function $b(c, E)$ depends on c and measure $m(F)$ of all non-empty subsets $F \subseteq E$. Moreover, here we can get the same integral for different measures and thus it is suitable to find a distinguished representative among all such measures.

The aim of this paper is to find a common framework for all above mentioned integrals. The paper is organized as follows. In the next section, pseudo-operations $\oplus$ and $\odot$ are introduced, possessing some possible relations. Section 3 is focused on monotone measures and some relations between them. In Section 4, vertical, horizontal and general pseudo-integrals are introduced and their relationship to the above mentioned integrals is clarified. Section 5 is devoted to discrete pseudo-concave integrals. Finally, some concluding remarks are added.

* Corresponding author at: Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia.
E-mail addresses: Radko.Mesiar@stuba.sk (R. Mesiar), lijun@cuc.edu.cn (J. Li), pap@im.ns.ac.yu (E. Pap).

0888-613X/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.ijar.2012.07.008

2. Pseudo-operations

Several generalizations of the standard addition + and multiplication $\cdot$ on $[0,\infty]$ have been considered in generalized integration theory, so far. Pseudo-addition $\oplus : [0,\infty]^2 \to [0,\infty]$ generalizing + is standardly assumed to be characterized by the next axioms [2,4,14,16,20].

Definition 1. A binary operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called a *pseudo-addition* whenever it is

(i) increasing in both coordinates, i.e., $x \oplus y \leqslant x' \oplus y'$ for all $x, y, x', y' \in [0,\infty]\,,\ x \leqslant x',\ y \leqslant y'$;
(ii) 0 is its neutral element, $0 \oplus x = x \oplus 0 = x$ for all $x \in [0,\infty]$;
(iii) associative, i.e., $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in [0,\infty]$;
(iv) it is continuous, i.e.,

$$\lim_{m\to\infty} (x_m \oplus y_m) = x \oplus y$$

whenever $\lim_{m\to\infty} x_m = x$ and $\lim_{m\to\infty} y_m = y$, $x, y, x_m, y_m \in [0,\infty]\,,$ $m = 1, 2, \ldots .$

Note that due to [13] each pseudo-addition $\oplus$ is also commutative, $x \oplus y = y \oplus x$ for all $x, y \in [0,\infty]$. Moreover, for each $x, y \in [0,\infty]$ it holds $x \oplus y \geq x \vee y$ (i.e., $\vee$ is the smallest pseudo-addition, $x \vee y = \max\{x, y\}$). Formally, pseudo-additions are special I-semigroups and their structure is completely described in [13], see also [4].

Concerning the pseudo-multiplication $\odot : [0,\infty]^2 \to [0,\infty]$ generalizing $\cdot$, there are several different approaches, depending on the discussed approach to the integration.

Definition 2. An operation $\odot : [0,\infty]^2 \to [0,\infty]$ is called a *pseudo-multiplication* whenever it is

(i) increasing in both coordinates, i.e., $x \odot y \leqslant x' \odot y'$ for all $x, y, x', y' \in [0,\infty]\,,\ x \leqslant x',\ y \leqslant y'$;
(ii) 0 is its annihilator, $0 \odot x = x \odot 0 = 0$ for all $x \in [0,\infty]$;
(iii) for each $x \in\,]0,\infty[$ there are $y, z \in\,]0,\infty[$ so that $0 < x \odot y < \infty$ and $0 < z \odot x < \infty$.

Let $\oplus : [0,\infty]^2 \to [0,\infty]$ be a given fixed pseudo-addition. Recall that then a pseudo-multiplication $\odot : [0,\infty]^2 \to [0,\infty]$ is *left-$\oplus$-distributive* if

$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

for all $x, y, z \in [0,\infty]$. Similarly, it is *right-$\oplus$-distributive* if

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

for all $x, y, z \in [0,\infty]$, and *$\oplus$-distributive* if it is both right-$\oplus$-distributive and left-$\oplus$-distributive.

Example 1

(i) Each pseudo-multiplication $\odot$ is $\vee$-distributive.
(ii) The only +-distributive pseudo-multiplications belong to the family

$$\left(\odot_{(\alpha)}\right)_{\alpha\in]0,\infty[}\,,\quad x \odot_{(\alpha)} y = \alpha \cdot x \cdot y.$$

Define a pseudo-multiplications $\odot_1, \odot_2 : [0,\infty]^2 \to [0,\infty]$ by

$$x \odot_1 y = x \cdot y^2, \quad x \odot_2 y = x^2 \cdot y.$$

Then $\odot_1$ is left-+-distributive and $\odot_2$ is right-+-distributive.

3. Monotone measures

Monotone measure $\mu : 2^N \to [0, \infty]$ can be seen as a source of weights of criteria groups.

Definition 3. A set function $\mu : 2^N \to [0, \infty]$ is called

(i) a monotone measure if $\mu(\emptyset) = 0$ and $\mu(E) \leqslant \mu(F)$ whenever $E \subseteq F \subseteq N$.
(ii) A monotone measure μ such that $\mu(N) = 1$ is called a *capacity*.
(iii) If a monotone measure μ satisfies $\mu(E \cup F) = \mu(E) \oplus \mu(F)$ for all pairs $E, F \subseteq N, \ E \cap F = \emptyset$, then μ is called a $\oplus$-*additive* measure.
(iv) If $\mu(E \cup F) \oplus \mu(E \cap F) \geqslant \mu(E) \oplus \mu(F)$ for all pairs $E, F \subseteq N$, then μ is called $\oplus$-*supermodular* measure.

Recall that a +-additive capacity μ is a classical discrete probability measure on N. A $\vee$-additive (i.e., maxitive) capacity μ is a possibility measure, see [25]. For more details on monotone measures we recommend monograph [22].

For any monotone measure μ and pseudo-addition $\oplus$, we can assign a monotone measure $\mu_\oplus$ introduced in [9] and called $\oplus$-optimal measure there.

Definition 4. Let a monotone measure $\mu : 2^N \to [0, \infty]$ and a pseudo-addition $\oplus : [0, \infty]^2 \to [0, \infty]$ be given. Then the set function $\mu_\oplus : 2^N \to [0, \infty]$ given by

$$\mu_\oplus(E) = \sup \left\{ \bigoplus_{i\in I} \mu(E_i) \mid \bigcup_{i\in I} E_i = E, \ E_i \cap E_j = \emptyset \text{ whenever } i \neq j \right\}$$

is called a $\oplus$-*optimal measure assigned to* μ.

Observe that $\oplus$-optimal measures are exactly super-$\oplus$-additive measures, i.e., $\mu_\oplus(E \cup F) \geq \mu(E) \oplus \mu(F)$ whenever $E, F \subseteq N, \ E \cap F = \emptyset$.

These measures play a key role by PAN-integrals [22,23]. In fact, for a given monotone measure μ, $\mu_\oplus$ is the smallest super-$\oplus$-additive measure bounded by μ from below.

For a given pseudo-addition $\oplus$ and a left-$\oplus$-distributive pseudo-multiplication $\odot$ with a neutral element $e \in]0, \infty]$ (we suppose further on that $\oplus$ and $\odot$ satisfy these properties), we introduce for each monotone measure μ a set function $\mu_\oplus^\odot$.

Definition 5. Let $\oplus : [0, \infty]^2 \to [0, \infty]$ and $\odot : [0, \infty]^2 \to [0, \infty]$ be a given pseudo-addition and a left-$\oplus$-distributive pseudo-multiplication with neutral element $e \in]0, \infty]$, respectively. For a monotone measure $\mu : 2^N \to [0, \infty]$, define a set function $\mu_\oplus^\odot : 2^N \to [0, \infty]$ by

$$\mu_\oplus^\odot(E) = \sup \left\{ \bigoplus_{i\in I} (a_i \odot \mu(E_i)) \ \mid \ \bigoplus_{i\in I} b\,(a_i, E_i) = b(e, E) \right\}.$$

Then $\mu_\oplus^\odot$ is called $\oplus$-$\odot$-*totally balanced measure assigned to* μ.

Observe that left-distributivity of $\odot$ over $\oplus$ allows to consider $E_i \neq E_j$ for $i \neq j$ in the above definition. Note that for standard arithmetical operations + and $\cdot$, $\mu_+^\cdot$ is a totally balanced measure as considered in [6]. These measures play a key role by the concave integrals.

Example 2

(i) Each monotone measure is $\vee$-optimal, i.e., $m = m_\vee$, and $\vee$-$\odot$-totally balanced, i.e., $m = m_\vee^\odot$ independently of pseudo-multiplication $\odot$.
(ii) Consider the standard arithmetic operations + and $\cdot$ and $N = \{1, 2, 3\}$. Let a set function $\mu : 2^N \to [0, \infty]$ be given by $\mu(\emptyset) = 0, \ \mu(E) = 0.1$ if $|E| = 1$, $\mu(E) = 1$ if $|E| = 2$ and $\mu(N) = a$. Then:
  (a) μ is a monotone measure if $a \geqslant 1$;
  (b) μ is a capacity if $a = 1$;
  (c) μ is +-supermodular if $a \geqslant 1.9$;
  (d) $\mu_+ = \mu$ (i.e., μ is superadditive) if $a \geq 1.1$ (if $a \in [1, 1.1[$, then $\mu_+(N) = 1.1$ and $\mu_+(E) = \mu(E)$ for $E \neq N$);
  (e) $\mu_+^\cdot = \mu$ (i.e., μ is totally balanced) if $a \geqslant 1.5$ (if $a \in [1, 1.5[$, then $\mu_+^\cdot(N) = 1.5$ and $\mu_+^\cdot(E) = \mu(E)$ for $E \neq N$).

4. New integrals

Each function $f : N \to [0, \infty]$ can be decomposed (subdecomposed) in different manners into basic functions.

Definition 6. Let $\oplus : [0, \infty]^2 \to [0, \infty]$ be a fixed pseudo-addition. Let $f : N \to [0, \infty]$ be given. A system $(b\,(a_i, E_i))_{i \in I}$ of basic functions is called

(i) a $\oplus$-vertical subdecomposition of f if $f \geqslant \bigoplus_{i \in I} b\,(a_i, E_i)$ and $E_i \cap E_j = \varnothing$ for all $i, j \in I,\ \ i \neq j$;
(ii) a $\oplus$-horizontal decomposition of f if $f = \bigoplus_{i \in I} b\,(a_i, E_i)$ and $(E_i)_{i \in I}$ is a chain;
(iii) a $\oplus$-decomposition of f if $f = \bigoplus_{i \in I} b\,(a_i, E_i)$.

Example 3. Let $f : \{1, 2, 3\} \to [0, \infty]$ be given by $f(1) = 1,\ \ f(2) = f(3) = 2$. Then $(b(1, \{1\}), b(2, \{2, 3\}))$ and $(b(1, \{1\}), b(2, \{2\}), b(2, \{3\}))$ are $\oplus$-vertical decompositions of f, independently of $\oplus$. Further, $(b(1, \{1, 2, 3\}), b(1, \{2, 3\}))$ is a +-horizontal decomposition of f. Moreover, $(b(0.5, \{1, 2\}),\ b(0.5, \{1, 3\}),\ b(1.5, \{2, 3\}))$ is a +-decomposition of f which is neither +-vertical nor +-horizontal.

The three introduced (sub-)decompositions allow us to introduce three different types of pseudo-integrals.

Definition 7. Let $\oplus, \odot : [0, \infty]^2 \to [0, \infty]$ be a fixed pseudo-addition and pseudo-multiplication, respectively. For any function $f : N \to [0, \infty]$ and monotone measure $\mu : 2^N \to [0, \infty]$ denote

$$I_{\oplus,\odot}(f, \mu) = \sup \left\{ \bigoplus_{i \in I} (a_i \odot \mu(E_i)) \ \middle|\ (b\,(a_i, E_i))_{i \in I} \text{ is a } \oplus\text{-vertical subdecomposition of } f \right\},$$

$$I^{\oplus,\odot}(f, \mu) = \sup \left\{ \bigoplus_{i \in I} (a_i \odot \mu(E_i)) \ \middle|\ (b\,(a_i, E_i))_{i \in I} \text{ is a } \oplus\text{-horizontal decomposition of } f \right\},$$

$$I_{\oplus}^{\odot}(f, \mu) = \sup \left\{ \bigoplus_{i \in I} (a_i \odot \mu(E_i)) \ \middle|\ (b\,(a_i, E_i))_{i \in I} \text{ is a } \oplus\text{-decomposition of } f \right\}.$$

Then $I_{\oplus,\odot}(f, \mu)$ is called a *vertical* $(\oplus, \odot)$-integral, $I^{\oplus,\odot}(f, \mu)$ is called a *horizontal* $(\oplus, \odot)$-integral and $I_{\oplus}^{\odot}(f, \mu)$ is called a $(\oplus, \odot)$-integral of f with respect to μ.

Note that all introduced integrals are monotone both in functions and in measures. Observe that we cannot replace the $\oplus$-vertical subdecompositions in the definition of the vertical $(\oplus, \odot)$-integral $I_{\oplus,\odot}$ by $\oplus$-vertical decompositions if we want to keep the monotonicity of this integral with respect to the integrated functions. For example, consider $N = \{1, 2\}$ and a capacity $\mu : 2^N \to [0, \infty]$ given by $\mu(N) = 1$ and $\mu(E) = 0$ for $E \neq N$. Let $f(\{1\}) = 1,\ f(\{2\}) = 2$ and $g(\{1\}) = g(\{2\}) = 1$. Then the only $\oplus$-vertical decomposition of f is $(b(1, \{1\}), b(2, \{2\}))$, and $1 \odot \mu(\{1\}) \oplus 2 \odot \mu(\{2\}) = 0$. However, $(b(1, N))$ is $\oplus$-vertical decomposition of g and $1 \odot \mu(N) = 1 \odot 1$. On the other side, using $\oplus$-vertical subdecompositions, $I_{\oplus,\odot}(f, \mu) = I_{\oplus,\odot}(g, \mu) = 1 \odot 1$.

In general, $\oplus$ and $\odot$ used in Definition 6 may be unrelated. However, then the resulting integrals may be rather peculiar. For example, consider $f : N \to [0, \infty],\ f = 1_{\{1\}}$. Then $(b(\alpha, \{1\}))_{\alpha \in [0,1]}$ are the unique $\oplus$-vertical subdecompositions of f (neglecting trivial basic functions $b(0, E)$ and $b(a, \varnothing)$) which all equals to zero functions. Thus for any monotone measure μ we have $I_{\oplus,\odot}(1_{\{1\}}, \mu) = 1 \odot \mu\,(\{1\})$. However, each system $(b\,(a_i, \{1\}))_{i \in I}$ such that $\bigoplus_{i \in I} a_i = 1$ is $\oplus$-horizontal decomposition of f and

$$I^{\oplus,\odot}(1_{\{1\}}, \mu) = \sup \left\{ \bigoplus_{i \in I} (a_i \odot \mu(\{1\})) \ \middle|\ \bigoplus_{i \in I} a_i = 1 \right\}.$$

Obviously, if the pseudo-multiplication $\odot$ is left-continuous in the first coordinate and left $\oplus$-distributive, then also $I^{\oplus,\odot}(1_{\{1\}}, \mu) = 1 \odot \mu\,(\{1\})$. Consider a pseudo-multiplication $\odot : [0, \infty]^2 \to [0, \infty]$ given by $x \odot y = \sqrt{x} \cdot y$. Then $I^{+,\odot}(1_{\{1\}}, \mu) = \infty$ whenever $\mu(\{1\}) > 0$. On the other hand, $I^{\vee,\odot}(1_{\{1\}}, \mu) = 1 \odot \mu(\{1\})$, independently of $\odot$.

Integrals introduced in Definition 7 include as special cases many of integrals known from the literature.

Example 4. Vertical integrals cover:

(i) Lebesgue integral, $(\oplus, \odot) = (+, \cdot)$ and μ is additive;
(ii) PAN integral, $\oplus$ and $\odot$ is PAN-addition and PAN-multiplication, respectively, see [22,23], μ is arbitrary;
(iii) pseudo-additive integral of Sugeno and Murofushi [20], $\oplus$ is pseudo-addition, $\odot$ is a right-$\oplus$-fitting pseudo-multiplication and μ is $\oplus$-additive;
(iv) generalized Lebesgue integral introduced in [24].

Example 5. Horizontal integrals cover:

(i) Choquet integral [1], $(\oplus, \odot) = (+, \cdot)$ and μ is arbitrary (if μ is additive, then the Lebesgue integral is recovered);
(ii) Sugeno integral [19] and its extended form introduced in [17], $(\oplus, \odot) = (\vee, \wedge)$ and μ is arbitrary;
(iii) Shilkret integral [18], $(\oplus, \odot) = (\vee, \cdot)$ and μ is arbitrary (in its original version, μ was supposed to be $\vee$-additive);
(iv) N-integral (introduced in [26]), $\oplus = \vee$) and μ is arbitrary.

$(\oplus, \odot)$-integral can be seen as a generalization of Lehrer concave integral introduced in [7], which is, indeed, exactly $(+, \cdot)$-integral.

It is not difficult to check that, supposing the left-continuity of the pseudo-multiplication $\odot$ and its left-$\oplus$-distributivity, the $\oplus$-horizontal integral $I^{\oplus,\odot}(\cdot, \mu_1)$ differ from $I^{\oplus,\odot}(\cdot, \mu_2)$ whenever $\mu_1 \neq \mu_2$. This is no more true for the integrals $I_{\oplus,\odot}$ and $I_{\oplus}^{\odot}$.

Proposition 1. *Let $\oplus : [0, \infty]^2 \to [0, \infty]$ be a fixed pseudo-addition and let $\odot : [0, \infty]^2 \to [0, \infty]$ be right-$\oplus$-distributive pseudo-multiplication. Let μ be a fixed monotone measure. Then $I_{\oplus,\odot}(\cdot, \mu) = I_{\oplus,\odot}(\cdot, \eta)$ for each monotone measure η such that $\mu \leq \eta \leq \mu_{\oplus}$.*

Proof. Due to the monotonicity of vertical $(\oplus, \odot)$-integrals in measures it is enough to show that for each $f \in \mathcal{F}$ it holds $I_{\oplus,\odot}(f, \mu) \geqslant I_{\oplus,\odot}(f, \mu_{\oplus})$ (recall that $\mu \leq \mu_{\oplus}$). To see this inequality, consider $\oplus$-vertical subdecomposition $(b(a_i, E_i))_{i\in I}$ of f. Each subset $E_i,\ i \in I$, is finite and thus there is a partition $(E_{i,j})_{j\in J_i}$ of E_i such that $\mu_{\oplus}(E_i) = \bigoplus_{j\in J_i} \mu(E_{i,j})$.

Due to the right-distributivity of $\odot$ over $\oplus$ it holds

$$\bigoplus_{i\in I}(a_i \odot \mu_{\oplus}(E_i)) = \bigoplus_{i\in I}\left(\bigoplus_{j\in J_i}(a_i \odot \mu(E_{i,j}))\right).$$

As far as $\big(b(a_i, E_{i,j})\big)_{i\in I, j\in J_i}$ is a $\oplus$-vertical subdecomposition of f, it follows that $\bigoplus_{i\in I}(a_i \odot \mu_{\oplus}(E_i)) \leq I_{\oplus,\odot}(f, \mu)$, and the result follows. $\square$

Similarly, the next result can be shown.

Proposition 2. *Let $\oplus : [0, \infty]^2 \to [0, \infty]$ be a fixed pseudo-addition and let $\odot : [0, \infty]^2 \to [0, \infty]$ be $\oplus$-distributive pseudo-multiplication with neutral element e which is left-continuous. Let μ be a fixed monotone measure. Then*

$$I_{\oplus}^{\odot}(\cdot, \mu) = I_{\oplus}^{\odot}(\cdot, \eta)$$

for any monotone measure η such that $\mu \leqslant \eta \leqslant \mu_{\oplus}^{\odot}$.

Corollary 1. *Under conditions of Proposition 1 (Proposition 2),*

$$I_{\oplus,\odot}(\cdot, \mu) = I_{\oplus,\odot}(\cdot, \eta)$$

if and only if $\mu_{\oplus} = \eta_{\oplus}$ ($I_{\oplus}^{\odot}(\cdot, \mu) = I_{\oplus}^{\odot}(\cdot, \eta)$ if and only if $\mu_{\oplus}^{\odot} = \eta_{\oplus}^{\odot}$).

5. Pseudo-concave integrals

Recall that Lehrer has introduced his concave integral in [7] as minimum over all concave and positively homogeneous functionals $H : \mathcal{F} \to [0, \infty]$ satisfying $H(1_E) \geqslant \mu(E)$ for all $E \subseteq N$, where $\mathcal{F}$ is the set of all $N \to [0, \infty]$ functions, $I_{+}^{\cdot}(f, \mu) = \min\{H(f)\}$, see Lemma 1 in [7].

The concavity of H, i.e.,

$$H(\lambda f + (1 - \lambda) g) \geq \lambda H(f) + (1 - \lambda) H(g)$$

for all $\lambda \in [0, 1]$ and $f, g \in \mathcal{F}$, together with the positive homogeneity of H, i.e., $H(\alpha f) = \alpha H(f)$ for all $\alpha > 0$ and $f \in \mathcal{F}$ is equivalent to the superadditivity, i.e.,

$$H(f + g) \geqslant H(f) + H(g)$$

for all $f, g \in \mathcal{F}$, and positive homogeneity of H. This observation allows us to introduce an integral $I_{\odot}^{\oplus}$ linked to special functionals over $\mathcal{F}$.

Definition 8. Let $\oplus, \odot : [0, \infty]^2 \to [0, \infty]$ be a fixed pseudo-addition and pseudo-multiplication, respectively, such that e is unique left-neutral element of $\odot$. For any monotone measure $\mu : 2^N \to [0, \infty]$, the integral $I_{\odot}^{\oplus} : \mathcal{F} \to [0, \infty]$ is given by

$$I_{\odot}^{\oplus}(f, \mu) = \inf \left\{ H(f) \mid H : \mathcal{F} \to [0, \infty] \text{ is } \oplus\text{-superadditive and } \odot\text{-homogeneous}, H(e, E) \geqslant \mu(E) \text{ for all } E \subseteq N \right\}$$

where the $\oplus$-superadditivity of H means $H(f \oplus g) \geqslant H(f) \oplus H(g)$ for all $f, g \in \mathcal{F}$, and $\odot$-homogeneity of H means $H(\alpha \odot f) = \alpha \odot H(f)$ for all $\alpha > 0$ and $f \in \mathcal{F}$.

Remark 2. In general the integral $I_{\odot}^{\oplus}$ can differ from $I_{\oplus}^{\odot}$. Consider $\oplus = \vee$ and let $\odot : [0, \infty]^2 \to [0, \infty]$ be given by $x \odot y = x^2 \cdot y$, i.e., left-neutral element $e = 1$. For constant functions on N the $\odot$-homogeneity of a functional $H : \mathcal{F} \to [0, \infty]$ means

$$H\left(b(c, N)\right) = H\left(\sqrt{c} \odot b(1, N)\right) = \sqrt{c} \odot H\left(b(1, N)\right) = c \cdot H\left(b(1, N)\right).$$

Moreover, $H\left(b(1, N)\right) \geqslant \mu(N)$ should hold. Evidently the $\vee$-superadditivity of H is exactly the monotonicity of H (increasingness). Thus

$$I_{\odot}^{\vee}\left(b(c, N), \mu\right) = c \cdot \mu(N).$$

On the other hand $I_{\vee}^{\odot}\left(b(c, N), \mu\right) = c \odot \mu(N) = c^2 \cdot \mu(N)$.

As a proper generalization of Lehrer's concave integral we will consider those $(\oplus, \odot)$-integrals for which $I_{\odot}^{\oplus} = I_{\oplus}^{\odot}$ holds, independently of N, i.e., when both approaches based on integral sums (and supremum) and on special functionals (and infimum) coincide.

Theorem 1. *Let $([0, \infty], \oplus, \odot)$ be a semiring, i.e., $\oplus : [0, \infty]^2 \to [0, \infty]$ is a given pseudo-addition, and $\odot : [0, \infty]^2 \to [0, \infty]$ is an associative commutative $\oplus$-distributive pseudo-multiplication with neutral element $e \in\,]0, \infty]$ which is continuous on $[0, \infty]^2 \setminus \{(0, \infty), (\infty, 0)\}$. Then $I_{\odot}^{\oplus} = I_{\oplus}^{\odot}$ and this integral will be called pseudo-concave integral.*

Proof. Recall that due to [12]: (i) either $\oplus = \vee$ and $\odot$ is either isomorphic to the standard product, $x \odot y = g^{-1}\left(g(x) \cdot g(y)\right)$ for an automorphism $g : [0, \infty] \to [0, \infty]$, or $\odot$ is an I-semigroup operation on $[0, \infty]$ with neutral element $e = +\infty$. In both cases

$$\begin{aligned} I_{\vee}^{\odot}(f, \mu) &= I_{\odot}^{\vee}(f, \mu) \\ &= \sup \left\{ a_i \odot \mu(E_i) \mid b(a_i, E_i) \leqslant f \right\} \\ &= \sup \left\{ t \odot \mu(f \geqslant t) \mid t \in [0, \infty] \right\}. \end{aligned}$$

(ii) or both $\oplus$ and $\odot$ are generated by an automorphism $g : [0, \infty] \to [0, \infty]$, $x \odot y = g^{-1}\left(g(x) \cdot g(y)\right)$ (observe that neutral element e of $\odot$ is given by $e = g^{-1}(1)$), see [14,16]. Then

$$\begin{aligned} I_{\oplus}^{\odot}(f, \mu) &= \sup \left\{ \bigoplus_{i \in I} \left(a_i \odot \mu(E_i)\right) \mid \bigoplus_{i \in I} b(a_i, E_i) = f \right\} \\ &= \sup \left\{ g^{-1}\left(\sum_{i \in I} \left(g(a_i) \cdot g(\mu(E_i))\right) \right) \middle| \sum_{i \in I} b(g(a_i), E_i) = g \circ f \right\} \\ &= g^{-1}\left(I_{+}^{\cdot}(g \circ f, g \circ \mu) \right). \end{aligned}$$

On the other hand, $H : \mathcal{F} \to [0, \infty]$ is a $\oplus$-superadditive and $\odot$-homogeneous functional satisfying $H(b(e, E)) \geqslant \mu(E)$ if and only if the functional $g \circ H \circ g^{-1} : \mathcal{F} \to [0, \infty]$ satisfies the requirements of Lehrer's functional in definition of $I_{\cdot}^{+}$, and thus

$$\begin{aligned}
I_{\odot}^{\oplus}(f, \mu) &= \inf \Big\{ H(f) \mid H : \mathcal{F} \to [0, \infty] \text{ is } \oplus\text{-superadditive,} \\
&\qquad \text{positively homogeneous and } H(b(e, E)) \geqslant \mu(E) \text{ for all } E \subseteq N \Big\} \\
&= g^{-1} \Big(\inf \Big\{ G(g(f)) \mid G : \mathcal{F} \to [0, \infty] \text{ is superadditive,} \\
&\qquad \text{positively homogeneous and } G(b(1, E)) \geqslant g(\mu(E)) \Big\} \Big) \\
&= g^{-1} \left(I_{\cdot}^{+}(g \circ f,\ g \circ \mu) \right).
\end{aligned}$$

The result follows. □

Lehrer in [7] has shown also that his concave integral $I_{+}^{\cdot}$ is bounded from below by Choquet integral and that these two integrals coincide if and only if μ is supermodular (and, obviously, they coincide with Lebesgue integral if and only if μ is additive). Based on Theorem 1, it is not difficult to check the following result.

Corollary 3. *Denote by $Ch_{\oplus}^{\odot}$ the Choquet-like integral introduced by Mesiar [10]. Let $([0, \infty], \oplus, \odot)$ be a semiring. Then the pseudo-integral $I_{\oplus}^{\odot}$ has the following properties*

(i) $I_{\oplus}^{\odot} \geq Ch_{\oplus}^{\odot}$;

(ii) $I_{\oplus}^{\odot}(\cdot, \mu) = Ch_{\oplus}^{\odot}(\cdot, \mu)$ *if and only if μ is $\oplus$-supermodular;*

(iii) $I_{\oplus}^{\odot}(\cdot, \mu) = Ch_{\oplus}^{\odot}(\cdot, \mu) = SM_{\oplus}^{\odot}(\cdot, \mu)$ *if and only if μ is $\oplus$-additive, where $SM_{\oplus}^{\odot}$ is the pseudo-additive integral introduced by Sugeno and Murofushi in [20].*

Example 6. Let N and $\mu : 2^N \to [0, \infty]$ be considered as in Example 2 (ii). Let $f : N \to [0, \infty]$ be given by $f(1) = f(3) = 1$ and $f(2) = 2$. Then

$$I_{+,\cdot}(f, \mu) = a \vee 1.2,$$

(considering the systems $(b(1, \{1, 3\}), b(2, \{2\}))$ and $(b(1, \{1, 2, 3\}))$, respectively),

$$I^{+,\cdot}(f, \mu) = a + 0.1,$$

(attained for the system $(b(1, \{1, 2, 3\}), b(1, \{2\}))$),

$$I_{+}^{\cdot}(f, \mu) = I_{\cdot}^{+}(f, \mu) = (a + 0.1) \vee 2,$$

(considering the systems $(b(1, \{1, 2, 3\}), b(1, \{2\}))$ and $(b(1, \{1, 2\}), b(1, \{2, 3\}))$), i.e., the Choquet integral $I^{+,\cdot}(f, \mu)$ coincides with Lehrer integral $I_{+}^{\cdot}(f, \mu)$ only $a \geq 1.9$ (hence if μ is supermodular).

Considering $\oplus, \odot : [0, \infty]^2 \to [0, \infty]$ given by

$$u \oplus v = (u^p + v^p)^{1/p} \text{ and } u \oplus v = uv$$

for some $p \in]0, \infty[$, $([0, \infty], \oplus, \odot)$ form a semiring and we have

$$I_{\oplus,\odot}(f, \mu) = a \vee (1 + 0.2^p)^{1/p},$$

$$I^{\oplus,\odot}(f, \mu) = (a^p + 0.1^p)^{1/p},$$

$$I_{\oplus}^{\odot}(f, \mu) = I_{\odot}^{\oplus}(f, \mu) = (a^p + 1 + 0.1^p)^{1/p} \vee 2^{1/p},$$

and then Choquet-like integral $I^{\oplus,\odot}(f,\mu)$ coincides with the pseudo-concave integral $I^{\odot}_{\oplus}(f,\mu)$ whenever $a \geq (2-0.1^p)^{1/p}$ (and only then μ is super-$\oplus$-modular). PAN-integral $I_{\oplus,\odot}(f,\mu)$ is always smaller than the pseudo-concave integral $I^{\odot}_{\oplus}(f,\mu)$.

6. Concluding remarks

We have introduced four approaches how to define an integral on a finite space by means of basic functions, a pseudo-addition and a pseudo-multiplication. We have discussed some properties of these new integrals and we have shown many integrals known from the literature to be a special case of our integrals. As an example of an integral not covered by our approach recall special universal integrals based on copulas recently introduced in [5].

For the future research, our integrals over general abstract measurable spaces could be defined and studied, thus generalizing our results in similar manner as the works [6,21] did with the original concave integral defined on finite spaces in [7].

Note that during the finalisation of this paper, a closely related approach to integration (based on + and · and dealing with subdecomposition additivity) appeared in [8]. Finally, recall that the proposal of $(\oplus,\odot)$-integrals generalizing the Lehrer integral appeared for the first time in [11] and it was presented at NLMUA'2011 conference. We expect applications of our results in several domains, especially in decision making and game theory.

Acknowledgement

The research summarized in this paper was supported by the Grants APVV-0073-10, GAČR P402/11/0378, and MNPRS 174009. We are grateful to anonymous referees and editors for their valuable comments leading to an essential improvement of the original version of this paper.

References

[1] G. Choquet, Theory of capacities, Ann. Inst. Fourier (Grenoble) 5 (1953–1954) 131–292.
[2] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals, in: E. Pap (Ed.), Handbook of Measure Theory, Vol. II, Elsevier, 2002, pp. 1329–1379.
[3] D. Denneberg, Non-Additive Measure and Integral, Kluwer Academic Publisher, 1994.
[4] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Kluwer Academic Publishers., Dordrecht, 2000.
[5] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (2010) 178–187.
[6] E. Lehrer, R. Teper, The concave integral over large spaces, Fuzzy Sets Syst. 159 (2008) 2130–2144.
[7] E. Lehrer, A new integral for capacities, Econom. Theory 39 (2009) 157–176.
[8] Y. Even, E. Lehrer, Decomposition-Integral: Unifying Choquet and Concave Integrals, http://www.math.tau.ac.il/ lehrer/Papers/decomposition.pdf, 2011.
[9] J. Li, R. Mesiar, P. Struk, Pseudo-optimal measures, Inform. Sci. 180 (20) (2010) 4015–4021.
[10] R. Mesiar, Choquet-like integrals, J. Math. Anal. Appl. 194 (1995) 477–488.
[11] R. Mesiar, J. Li, E. Pap, Pseudo-concave integrals, NLMUA'2011, AISC 100, Springer-Verlag, Berlin, Heidelberg, 2011, pp. 43–49.
[12] R. Mesiar, J. Rybárik, Entropy of fuzzy partitions - a general model, Fuzzy Sets Syst. 74 (1995) 365–369.
[13] P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Ann. Math. 65 (1957) 117–143.
[14] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers., Dordrecht- Boston-London, 1995.
[15] E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002.
[16] E. Pap, Pseudo-Additive Measures and Their Aplications, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002, pp. 1403–1465.
[17] D. Ralescu, G. Adams, The fuzzy integral, J. Math. Anal. Appl. 75 (1980) 562–570.
[18] N. Shilkret, Maxitive measure and integration, Indag. Math. 33 (1971) 109–116.
[19] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. thesis, Tokyo Institute of Technology, 1974.
[20] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, J. Math. Anal. Appl. 122 (1987) 197–222.
[21] R. Teper, On the continuity of the concave integral, Fuzzy Sets Syst. 160 (2009) 1318–1326.
[22] Z. Wang, G.J. Klir, Generalized measure theory, Springer, Boston, 2009.
[23] Q. Yang, The pan-integral on fuzzy measure space, Fuzzy Math. 3 (1985) 107–114 (in Chinese).
[24] Q. Zhang, R. Mesiar, J. Li, P. Struk, Generalized Lebesgue integral, Int. J. Approx. Reason. 52 (2011) 427–443.
[25] L.A. Zadeh, Probability measures of fuzzy events, J. Math. Anal. Appl. 23 (1968) 421–427.
[26] R. Zhao, (N)-Fuzzy integrals, Math. Res. Exposit. 2 (1981) 55–82 (in Chinese).