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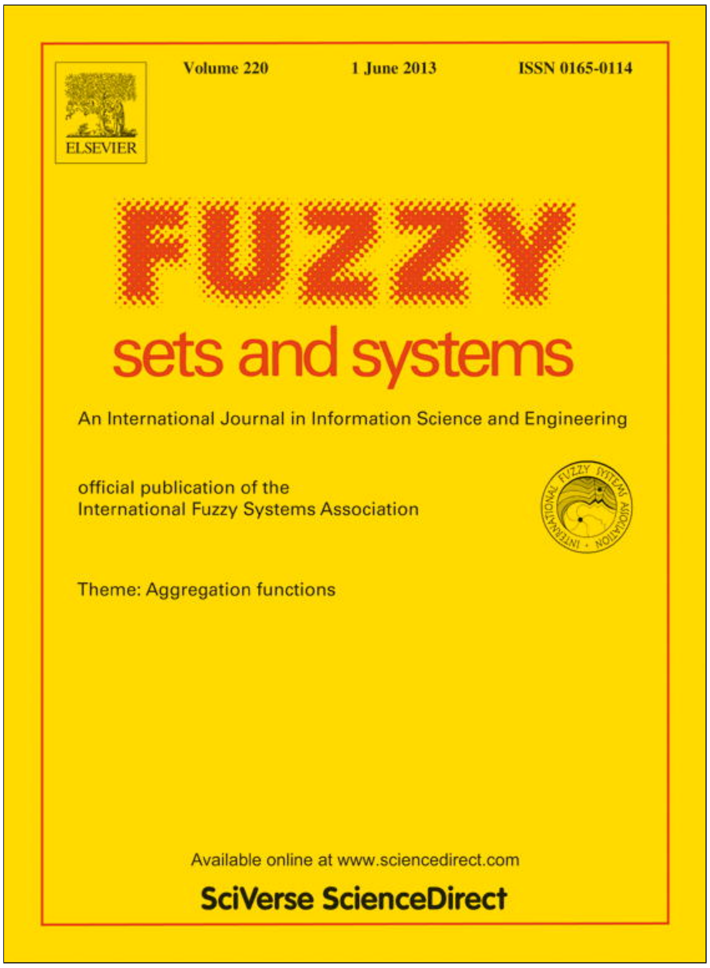

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Generation of linear orders for intervals by means of aggregation functions

H. Bustince[a,*], J. Fernandez[a], A. Kolesárová[b], R. Mesiar[c,d]

[a] Departamento de Automática y Computación, Universidad Pública de Navarra, Campus Arrosadia s/n, P.O. Box 31006, Pamplona, Spain
[b] Institute of Information Engineering, Automation and Mathematics, Slovak University of Technology, 81237 Bratislava, Slovakia
[c] Department of Mathematics, Slovak University of Technology, Radlinskeho 11, Bratislava, Slovakia
[d] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 18208 Prague, Czech Republic

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Abstract

The problem of choosing an appropriate total order is crucial for many applications that make use of extensions of fuzzy sets. In this work we introduce the concept of an admissible order as a total order that extends the usual partial order between intervals. We propose a method to build these admissible orders in terms of two aggregation functions and we prove that some of the most used examples of total orders that appear in the literature are specific cases of our construction.
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1. Introduction

In recent years there has been an increasing interest in the use of extensions of fuzzy sets such as interval-valued fuzzy sets [14] and Atanassov's intuitionistic fuzzy sets [1] in many different fields such as image processing [4,7], decision making [11], classification [10] or consensus [3]. For many of these applications there is required an order relation that should be fixed and that plays a crucial role for the proposed results. In decision making, for instance, [6] a set of alternatives $\{A_1, \ldots, A_p\}$ and a set of criteria $\{C_1, \ldots, C_q\}$ are given and it is necessary to determine which is the alternative that best suits the criteria. In order to do so, a very widely used method consists of assigning a value (either numerical or of some other type) to each of the alternatives, and to use this value to create a ranking of alternatives. If the values that are assigned to the alternatives are interval-valued (or L-valued, in general, for some lattice L) two problems arise:

1. How to fix a linear order, namely, an order that allows to compare any possible two intervals arising in the valuation of alternatives, and
2. To determine in which sense this order is appropriate for dealing with the considered problem.

* Corresponding author. Tel.: +34 94816 9254; fax: +34 948169281.
E-mail addresses: bustince@unavarra.es (H. Bustince), fcojavier.fernandez@unavarra.es (J. Fernandez), anna.kolesarova@stuba.sk (A. Kolesárová), mesiar@math.sk (R. Mesiar).

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Focusing on interval-valued fuzzy sets, in most part of the literature, only one linear order, namely, that of Xu and Yager [13] is considered. The main goal of the present work is to present a general method that allows to build different linear orders, and that in particular, covers the most widely known and used linear orders in the literature, such as Xu and Yager's or the lexicographical ones. We do not consider here the crucial second question, i.e., which is the most appropriate linear order for a given application.

The construction method that we propose here is based on the use of two aggregation functions. In this sense, it provides an easy-to-use tool in order to generate new total orders between intervals. Moreover, our method can be easily extended to use either more aggregation functions or functions other than aggregation functions as long as some basic properties are fulfilled.

The structure of this paper is the following: we start with some preliminaries and, in Section 3, we present the main results of the paper, including the definition of admissible orders, its connection with K_α operators and some other properties. We finish with some conclusions and the references.

2. Preliminary notions

We start recalling some well-known concepts that will be useful for subsequent developments in the paper.

An n-ary aggregation function ($n \in \mathbb{N},\ n\geq 2$) on the unit interval is a function $A\colon [0,1]^n \to [0,1]$ that is increasing, i.e., for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $A(\mathbf{x})\leq A(\mathbf{y})$ whenever $\mathbf{x}\leq\mathbf{y}$, and satisfies the boundary conditions $A(0,\ldots,0)=0$, $A(1,\ldots,1)=1$. For more details on aggregation functions we refer, e.g., to [8].

In this definition the usual partial order of real n-tuples is considered, i.e.,

$$\mathbf{x}=(x_1,\ldots,x_n)\leq\mathbf{y}=(y_1,\ldots,y_n) \quad \text{if and only if } x_1\leq y_1,\ldots,x_n\leq y_n.$$

It is clear that this definition also makes sense for any non-empty set L endowed with a partial order $\preceq$ and having a greatest and a smallest element with respect to $\preceq$.

Recall that given a non-empty set L, a partial order $\preceq$ on the set L is a binary relation on L which is reflexive, antisymmetric and transitive, i.e.,

- for each $a \in L$, $a \preceq a$ (reflexivity),
- for all $a, b \in L$, if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry),
- for all $a, b, c \in L$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

Note that we will write $a \prec b$ if a couple (a, b) is in a relation $\preceq$ but $a\neq b$. A set L with a partial order $\preceq$ is called a partially ordered set (poset) and denoted by $(L, \preceq)$. If in a poset $(L, \preceq)$ any two elements a, b are comparable, i.e., either $a \preceq b$ or $b \preceq a$, the partial order $\preceq$ is called a linear order (and then L is called a chain).

If there is an element $a \in L$ such that for each $x \in L$ it holds $x \preceq a$, then a is called the greatest element (top) of a poset $(L, \preceq)$ and denoted by 1_L. Similarly the smallest element (bottom) 0_L of a poset $(L, \preceq)$ is defined.

Given a poset $(L, \preceq)$ with a bottom 0_L and top 1_L, an aggregation function A on L with respect to an order $\preceq$ ($\preceq$-aggregation function) is defined as a mapping $A\colon L^n \to L$ with properties

- $A(0_L,\ldots,0_L)=0_L, \quad A(1_L,\ldots,1_L)=1_L$,
- $A(x_1,\ldots,x_n) \preceq A(y_1,\ldots,y_n)$ whenever $(x_1,\ldots,x_n) \preceq (y_1,\ldots,y_n)$,

where $(x_1,\ldots,x_n) \preceq (y_1,\ldots,y_n)$ if and only if for each $i = 1,\ldots,n$, $x_i \preceq y_i$. For more details on aggregation functions on posets we refer to [9].

Finally, recall that if $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are two posets, a mapping $\varphi\colon L_1 \to L_2$ is an order-preserving mapping (order homomorphism) if for all $a, b \in L_1$,

$$a \preceq_1 b \Rightarrow \varphi(a) \preceq_2 \varphi(b).$$

An order homomorphism $\varphi\colon L_1 \to L_2$ is called an order isomorphism whenever it is bijective and its inverse $\varphi^{-1}\colon L_2 \to L_1$ is also an order homomorphism.

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3. Orderings of intervals

3.1. Admissible orders

As mentioned above, this paper is devoted to the study of possible orderings of intervals. It is clear that to each interval $[a, b] \subset \mathbb{R}$ we can assign uniquely a point $(a, b) \in \mathbb{R}^2$ and that intervals can be ordered by means of orders of points in $\mathbb{R}^2$. The usual partial order in $\mathbb{R}^2$, given by $(a, b) \leq_2 (c, d) \Leftrightarrow a \leq c \wedge b \leq d$, induces the partial order of intervals which will be also denoted by $\leq_2$,

$$[a, b] \leq_2 [c, d] \Leftrightarrow a \leq c \wedge b \leq d. \tag{1}$$

Let us denote by $L([0, 1])$ the set of all closed subintervals of the unit interval,

$$L([0, 1]) = \{[a, b] | 0 \leq a \leq b \leq 1\}.$$

Observe that if

$$K([0, 1]) = \{(x, y) \in [0, 1]^2 | x \leq y\},$$

there is a natural bijection i from $L([0, 1])$ onto $K([0, 1])$, given by $i([a, b]) = (a, b)$. A partial (linear) order $\preceq$ on one of these sets induces a partial (linear) order on the other, $[a, b] \preceq [c, d] \Leftrightarrow (a, b) \preceq (c, d)$.

The set $L([0, 1])$ with the relation $\leq_2$ given in (1), is a poset with the smallest element $0_L = [0, 0]$ and the greatest element $1_L = [1, 1]$. However, some situations, e.g., the definition of interval-valued ordered weighted means [12], require a linear order of intervals. In the next part, we are interested in extending the partial order $\leq_2$ to a linear order (in several ways). We first define the notion of an admissible order.

Definition 3.1. Let $(L([0, 1]), \preceq)$ be a poset. The order $\preceq$ is called an admissible order, if

(i) $\preceq$ is a linear order on $L([0, 1])$,
(ii) for all $[a, b], [c, d] \in L([0, 1])$, $[a, b] \preceq [c, d]$ whenever $[a, b] \leq_2 [c, d]$.

Simply said, an order $\preceq$ on $L([0, 1])$ is admissible, if it is linear and refines the order $\leq_2$.

In the same sense we will use the notion of an admissible order on $K([0, 1])$.

Example 3.1. Motivated by the lexicographical order of points in $\mathbb{R}^2$, define the relations on $L([0, 1])$ as follows:

(i) $[a, b] \preceq_{Lex1} [c, d] \Leftrightarrow a < c \vee a = c \wedge b \leq d$,
(ii) $[a, b] \preceq_{Lex2} [c, d] \Leftrightarrow b < d \vee b = d \wedge a \leq c$.
It is clear that both these orders are admissible. Note that the order $\preceq_{Lex1}$ applied on points in $\mathbb{R}^2$, is the standard lexicographical order in $\mathbb{R}^2$.
(iii) Consider the order $\preceq_{YX}$ on $L([0, 1])$ introduced by Xu and Yager in [13]:

$$[a, b] \preceq_{YX} [c, d] \Leftrightarrow a + b < c + d \vee a + b = c + d \wedge b - a \leq d - c.$$

The order $\preceq_{YX}$ is also a linear order refining the order $\leq_2$, thus, an admissible order.

Proposition 3.1. *Let $\preceq$ be an admissible order on $L([0, 1])$. Then $1_L = [1, 1]$ and $0_L = [0, 0]$ are the greatest and the smallest elements of $(L([0, 1]), \preceq)$, respectively.*

Proof. The result follows from the fact that $[1, 1]$ is the greatest element of the poset $(L([0, 1]), \leq_2)$, and $[0, 0]$ is its smallest element, and that $\preceq$ is an admissible order, that is, a refinement of $\leq_2$. $\square$

Observe that any aggregation function $A : [0, 1]^2 \to [0, 1]$ induces on $K([0, 1])$ (and thus by isomorphism also on $L([0, 1])$) a relation $\leq_A$ given by $(a, b) \leq_A (c, d)$ if and only if $A(a, b) \leq A(c, d)$. This relation is reflexive and transitive, and it refines the standard partial order $\leq_2$, i.e., $(a, b) \leq_2 (c, d)$ implies $(a, b) \leq_A (c, d)$. However, $\leq_A$ needs not be antisymmetric, in general.

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Theorem 3.1. *Let $\preceq$ be an admissible order on $K([0,1])$. Then it cannot be induced by means of a single continuous function $f:[0,1]^2 \to [0,1]$.*

Proof. Let $\preceq$ be an admissible order on $K([0,1])$ and suppose that there is a continuous function $f:[0,1]^2 \to [0,1]$ such that $\preceq$ coincides with $\le_f$. Due to the linearity of $\preceq$, f restricted to $K([0,1])$ is a continuous bijective mapping. However, this fact is in contradiction with the classical result of Brouwer [2] concerning the isomorphism of the unit square and the unit interval, excluding the continuous bijections. □

3.2. Generation of admissible orders

If we focus on the orders provided in Example 3.1, we can see that they are generated by appropriate mappings acting on the bounds of the intervals. To be precise, let us first introduce the following definition.

Definition 3.2. Let $\preceq$ be an admissible order on $L([0,1])$. The order $\preceq$ is called a generated admissible order if there exist two continuous functions $f, g: K([0,1]) \to \mathbb{R}$ such that for all $[a,b],[c,d] \in L([0,1])$,

$$[a,b] \preceq [c,d] \Leftrightarrow [f(a,b), g(a,b)] \preceq_{Lex1} [f(c,d), g(c,d)]. \tag{2}$$

If $\preceq$ is a generated admissible order on $L([0,1])$ in the sense of the previous definition, the pair of functions (f,g) is called a generating pair of the order $\preceq$.

Example 3.2. The admissible orders introduced in Example 3.1 are generated admissible orders with the following generating functions:

(i) $f_{Lex1}(x,y) = x,\ g_{Lex1}(x,y) = y$,
(ii) $f_{Lex2}(x,y) = y,\ g_{Lex2}(x,y) = x$,
(iii) $f_{YX}(x,y) = x+y,\ g_{YX}(x,y) = y-x$.

Observe that if $\preceq$ is a generated admissible order, then it can be generated by different generating pairs. For example, it is easy to see that the pair $(\max\{x,y\},(x+y)/2)$ is another generating pair for $\preceq_{Lex2}$. If we take $f_1(x,y)=\max\{x,y\}$ and $g_1(x,y)=(x^2+2y^2)/3$, then the pair (f_1,g_1) also generates $\preceq_{Lex2}$.

Similarly, the pairs of functions $f_2(x,y)=xy$ and $g_2(x,y)=y-x$ and $f_3(x,y)=\sqrt{xy}$ and $g_3(x,y)=(y-x)^3$ generate the same admissible order on $L([0,1])$ (different from $Lex2$).

Let us summarize several properties of generating functions.

Theorem 3.2. *Let $\preceq$ be a generated admissible order on $L([0,1])$ with a generating pair (f,g). Then*

(i) $f(0,0) < f(1,1)$,
(ii) *the function f is jointly strictly increasing on $K([0,1])$ with respect to the order $\le_2$,*
(iii) *if $f(a,b)=f(c,d)$ and $g(a,b)=g(c,d)$, then $(a,b)=(c,d)$,*
(iv) *if the function g is increasing on $K([0,1])$ with respect to the order $\le_2$, then the function $f+g$ is strictly increasing.*

Proof. Assume that (f,g) is a generating pair of the given admissible order $\preceq$ on $L([0,1])$. From $[0,0]\le_2[1,1]$ it follows that either $f(0,0)<f(1,1)$ or $f(0,0)=f(1,1)$. However, in the latter case, a contradiction with Theorem 3.1 can be shown, thus (i) holds. Similarly, the joint strict monotonicity of f can be shown, proving (ii).

(iii) is straight from the definition of general admissible order.

Finally, suppose that g is an increasing function on $K([0,1])$ with respect to the order $\le_2$. By (ii), f is also increasing, thus $f+g$ is increasing, i.e., for all points in $K([0,1])$, if $(x_1,y_1) <_2 (x_2,y_2)$ then $f(x_1,y_1)+g(x_1,y_1) \le f(x_2,y_2)+g(x_2,y_2)$. To show the strict monotonicity of $f+g$, suppose that $(x_1,y_1) <_2 (x_2,y_2)$ and $f(x_1,y_1)+g(x_1,y_1) = f(x_2,y_2)+g(x_2,y_2)$. If $f(x_1,y_1)=f(x_2,y_2)$, then also $g(x_1,y_1)=g(x_2,y_2)$, thus, by (iii) $(x_1,y_1)=(x_2,y_2)$, which is in contradiction with our assumption. So $f(x_1,y_1)<f(x_2,y_2)$, which ensures the property $(f+g)(x_1,y_1) < (f+g)(x_2,y_2)$. □

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Evidently, due to properties (i) and (ii) of a generating pair (f, g), there is an aggregation function $A: [0, 1]^2 \to [0, 1]$ such that $f(a, b) = f(0, 0) + (f(1, 1) - f(0, 0))A(a, b)$, i.e., f is a strictly increasing linear transform of an aggregation function A (restricted to $K([0, 1])$).

Moreover, observe that in general, the converse of property (iii) does not hold. For instance, consider the functions $f(x, y) = x$ and $g(x, y) = 1/(y + 1)$, which clearly satisfy the property (iii), because if $f(a, b) = f(c, d)$ then $a = c$ and $g(a, b) = g(c, d)$ implies $b = d$. Denote the relation generated by the pair (f, g) by $\preceq$, and consider intervals $[a, b]$ and $[a, d]$ with $b < d$. Then $[a, b] \leq_2 [a, d]$, but $[a, b] \preceq [a, d]$ if and only if $d \leq b$, which means that the relation $\preceq$ is not a refinement of $\leq_2$, i.e., $\preceq$ is not an admissible order.

We conjecture that not each admissible order can be generated by means of just two generating functions.

Example 3.3. Consider the functions $f, g: K([0, 1]) \to \mathbb{R}$, defined by $f(x, y) = x + y$ and $g(x, y) = (2x - y)^2$. Let $\preceq$ be the relation on $L([0, 1])$ defined by $[a, b] \preceq [c, d]$ if and only if

(i) $f(a, b) < f(c, d)$ or
(ii) $f(a, b) = f(c, d)$ and $g(a, b) < g(c, d)$ or
(iii) $f(a, b) = f(c, d)$ and $g(a, b) = g(c, d)$ and $a \leq c$.

The relation $\preceq$ is an admissible order on $L([0, 1])$, but it cannot be generated by functions f and g only, because, for example, (i) and (ii) cannot distinguish the intervals [0.3, 0.45] and [0.2, 0.55]. More general, observe that although

$$\left[\frac{2a}{3} - 0.1, 0.4 + \frac{a}{3}\right] \prec [0.3, a] \quad \text{if } a \in [0.3, 0.6[,$$

$$\left[\frac{2a}{3} - 0.1, 0.4 + \frac{a}{3}\right] \succ [0.3, a] \quad \text{if } a \in]0.6, 1],$$

for $a \in [0.3, 1]$ we have

$$f\left(\frac{2a}{3} - 0.1, 0.4 + \frac{a}{3}\right) = f(0.3, a) \quad \text{and} \quad g\left(\frac{2a}{3} - 0.1, 0.4 + \frac{a}{3}\right) = g(0.3, a).$$

3.3. Admissible orders generated by aggregation functions

In Definition 3.2 we have introduced generated admissible orders. Now, we restrict the class of generating functions to special aggregation functions.

Proposition 3.2. *Let $A, B: [0, 1]^2 \to [0, 1]$ be two continuous aggregation functions, such that for all $(x, y), (u, v) \in K([0, 1])$, the equalities $A(x, y) = A(u, v)$ and $B(x, y) = B(u, v)$ can only hold if $(x, y) = (u, v)$. Define the relation $\preceq_{A,B}$ on $L([0, 1])$ by*
$[x, y] \preceq_{A,B} [u, v]$ if and only if

$$A(x, y) < A(u, v) \text{ or } A(x, y) = A(u, v) \text{ and } B(x, y) \leq B(u, v). \tag{3}$$

Then $\preceq_{A,B}$ is an admissible order on $L([0, 1])$.

The proof of the claim is simple, therefore omitted.

From now on, we will only consider admissible orders generated by continuous aggregation functions A, B described in Proposition 3.2. A pair (A, B) of continuous aggregation functions A, B generating such order will be called an admissible pair of aggregation functions.

Let us prove several properties of admissible pairs.

Proposition 3.3. *Let (A, B) be an admissible pair of aggregation functions and let $\phi, \eta: [0, 1] \to [0, 1]$ be two automorphisms. Then the orders generated by the pairs $(\phi \circ A, \eta \circ B)$ and (A, B) coincide.*

Proof. Observe that the increasing monotonicity of ϕ and η ensures that the composed functions $\phi \circ A$ and $\eta \circ B$ are aggregation functions. Due to the injectivity of ϕ and η, and the fact that (A, B) is an admissible pair of aggregation

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functions, for all $(x, y), (u, v) \in K([0, 1])$, the equalities $(\phi \circ A)(x, y) = (\phi \circ A)(u, v)$ and $(\phi \circ B)(x, y) = (\phi \circ B)(u, v)$ imply $(x, y) = (u, v)$. As automorphisms are strictly increasing mappings, the result follows. □

Proposition 3.4. *Let (A, B) be an admissible pair of aggregation functions. Then the function d_A: $[0, 1] \to [0, 1]$ given by $d_A(x) = A(x, x)$, is strictly increasing.*

The same holds for the function d_B: $[0, 1] \to [0, 1]$ given by $d_B(x) = B(x, x)$.

Proof. Although the result for d_A follows directly from Theorem 3.2(ii), we give here a more detailed proof. Assume that d_A is not strictly increasing. Then there exist $x, y \in [0, 1]$, $x < y$, such that $d_A(x) = d_A(y)$, i.e., $A(x, x) = A(y, y)$. From the monotonicity of A it follows that A is constant on the square $[x, y] \times [x, y]$. As the order defined by an admissible pair (A, B) is linear, B restricted to $[x, y] \times [x, y]$ must be an injective function, otherwise there would be non-distinguishable points. Let (u, v) be any point, such that $x < u \leq v < y$. Denote $c = B(x, x)$ and $d = B(y, y)$. Then $c < B(u, v) < d$. If we define the mapping φ: $K([0, 1]) \to [0, 1]$ by

$$\varphi(a, b) = \frac{1}{d - c}(B(x + a(y - x), x + b(y - x)) - c),$$

then φ is a continuous order isomorphism between $K([0, 1])$ and $[0, 1]$, which is in contradiction with Theorem 3.1. The result for d_B can be proved similarly. □

Proposition 3.5. *Let (A, B) be an admissible pair of continuous aggregation functions. Then there exists an admissible pair of aggregation functions (A', B') such that A', B' are idempotent continuous aggregation functions and the orders generated by the pairs (A, B) and (A', B') coincide.*

Proof. Given an admissible pair of aggregation functions (A, B), define the functions ϕ, η: $[0, 1] \to [0, 1]$ by $\phi(x) = d_A^{-1}(x)$ and $\eta(x) = d_B^{-1}(x)$, where d_A and d_B are functions introduced in Proposition 3.4. From this proposition it follows that the functions ϕ and η are well defined, and due to the continuity of A, B, they are automorphisms of the unit interval. Thus, by Proposition 3.3, the pair of aggregation functions $(\phi \circ A, \eta \circ B)$ generates an admissible order equivalent to the order $\preceq_{A,B}$. As for each $x \in [0, 1]$,

$$(\phi \circ A)(x, x) = \phi(A(x, x)) = d_A^{-1}(d_A(x)) = x,$$

and similarly $(\eta \circ B)(x, x) = x$, the aggregation functions $\phi \circ A$ and $\eta \circ B$ are idempotent and it is enough to take $A' = \phi \circ A$ and $B' = \eta \circ B$. □

Note that for some admissible orders we can characterize all pairs of aggregation functions (A, B) generating the discussed order.

Proposition 3.6. *Let (A, B) be an admissible pair of aggregation functions. Then $\preceq_{A,B} = \preceq_{Y,X}$ (i.e., we obtain the Xu–Yager order) if and only if the level lines in $K([0, 1])$ of A and the arithmetic mean are the same objects (i.e., $A(x, y) = A(u, v)$ for some $(x, y), (u, v) \in K([0, 1])$ if and only if $x + y = u + v$), and the sections B_c : $[(\max(0, c - 1) + \min(c, 1))/2, \min(c, 1)] \to [0, 1]$ given by $B_c(u) = B(c - u, u)$ are strictly increasing for each c in]0,2[.*

Proof. The sufficiency is a matter of an easy processing only. To see the necessity, recall that $[x, y] \preceq_{Y,X} [u, v]$ implies $x + y \leq u + v$, and if $\preceq_{Y,X} = \preceq_{A,B}$ also $A(x, y) \leq A(u, v)$. Clearly, if $x + y < u + v$ then necessarily $A(x, y) \leq A(u, v)$. Suppose that for some $(x, y), (u, v) \in K([0, 1])$ it holds $x + y < u + v$ but $A(x, y) = A(u, v)$. If $x < u$ and $y < v$, then A is constant on the rectangle $[x, u] \times [y, v]$, a contradiction with Theorem 3.1. If $x = u$ and thus $y < v$, then A is constant on the triangle connecting points (x, y), $((x + v)/2, (x + v)/2)$ and (x, v), again a contradiction with Theorem 3.1 (one can find nontrivial rectangle on which A is constant). The last case when $y = v$ and $x < u$ is similar. Consequently, $A(x, y) = A(u, v)$ if and only if $x + y = u + v$. The rest of the proof follows trivially from the fact that if $x + y = u + v = c$ and $y < v$, then $[x, y] \preceq_{Y,X} [u, v]$ and hence, due to $A(x, y) = A(u, v)$, necessarily $B(x, y) = B(c - y, y) < B(c - v, v) = B(u, v)$. □

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3.4. Admissible orders generated by mappings K_α

A particular way of obtaining admissible orders on $L([0,1])$ is defining them by means of K_α mappings [5].
For $\alpha \in [0,1]$, define the mapping $K_\alpha : [0,1]^2 \to [0,1]$ by

$$K_\alpha(a,b) = a + \alpha(b-a). \tag{4}$$

As the values of K_α can be written as $K_\alpha(a,b) = (1-\alpha)a + \alpha b$, K_α is a weighted mean. From the statistical point of view, $K_\alpha(a,b)$ can be seen as the α-quantile of a probability distribution uniformly distributed over the interval $[a,b]$.
Take $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$ and consider the relation $\preceq_{\alpha,\beta}$ on $L([0,1])$ such that

$$[a,b] \preceq_{\alpha,\beta} [c,d] \Leftrightarrow K_\alpha(a,b) < K_\alpha(c,d) \vee K_\alpha(a,b) = K_\alpha(c,d) \wedge K_\beta(a,b) \leq K_\beta(c,d). \tag{5}$$

Theorem 3.3. *Let $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then the relation $\preceq_{\alpha,\beta}$ defined by (5) is an admissible order on $L([0,1])$ generated by an admissible pair of aggregation functions (K_α, K_β).*

Proof. Let us show that if $K_\alpha(a,b) = K_\alpha(c,d)$ and $K_\beta(a,b) = K_\beta(c,d)$, then necessarily $(a,b) = (c,d)$, i.e., (K_α, K_β) is an admissible pair of aggregation functions.
The above equalities lead to the homogeneous system

$$(1-\alpha)(a-c) + \alpha(b-d) = 0,$$
$$(1-\beta)(a-c) + \beta(b-d) = 0,$$

which, if $\alpha \neq \beta$, has the only solution $a - c = 0$ and $b - d = 0$, i.e., $a = c$ and $b = d$. The relation $\preceq_{\alpha,\beta}$ defines a linear order on $L([0,1])$, and moreover, this order refines the order $\leq_2$, i.e., for all $[a,b],[c,d] \in L([0,1])$, if $[a,b] \leq_2 [c,d]$ then also $[a,b] \preceq_{\alpha,\beta} [c,d]$. □

Remark 3.1. The lexicographical orders with respect to the first and the second coordinate are recovered by orders $\preceq_{\alpha,\beta}$ as the orders $\preceq_{0,1}$ and $\preceq_{1,0}$, respectively.

Proposition 3.7. *Let $[a,b],[c,d] \in L([0,1])$ be two intervals which are not comparable by means of the order $\leq_2$. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ such that*

$$[a,b] \preceq_{\alpha_1,\beta_1} [c,d] \quad \textit{and} \quad [c,d] \preceq_{\alpha_2,\beta_2} [a,b].$$

Proof. If intervals $[a,b],[c,d]$ are not comparable by means of the order $\leq_2$, then, without loss of generality, we can assume that $a < c$ and $b > d$. If we take $\alpha_1 = \beta_2 = 0$ and $\alpha_2 = \beta_1 = 1$, the result follows. □

Remark 3.2. Consider the following toy interval-valued consensus relation:

$$R = \begin{pmatrix} [0.0, 0.8] & [0.0, 0.7] & [0.0, 0.75] \\ [0.2, 0.3] & [0.3, 0.6] & [0.4, 0.6] \\ [0.2, 0.4] & [0.3, 0.8] & [0.25, 0.6] \end{pmatrix}. \tag{6}$$

In this matrix, the element a_{ij} measures the support of expert j for alternative i. So, if we provide the same value to all the experts, a possible way of determining the best alternative is just by considering the arithmetic mean of the supports for each of the alternatives. If we do so, we see that the support for the first alternative A_1 is equal to $[0, 0.75]$; the support for the second alternative A_2 is equal to $[0.3, 0.5]$ and the support for the third alternative A_3 is equal to $[0.25, 0.6]$. In order to determine which is the consensus alternative, we need to order these values. But, if we consider the order generated by (K_0, K_1), our approach leads to the ordering $A_1 < A_3 < A_2$. On the other hand, the order generated by the pair (K_1, K_0) provides the ranking $A_2 < A_3 < A_1$. Finally, if we choose the order generated by $(K_{0.5}, K_{0.6})$ we arrive at $A_1 < A_2 < A_3$. That is, by picking up the appropriate order we can make that any of the alternatives is the chosen one.

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Proposition 3.8. (i) *Let* $\alpha \in [0, 1[$. *Then all admissible orders* $\preceq_{\alpha,\beta}$ *with* $\beta > \alpha$ *coincide. This admissible order will be denoted by* $\preceq_{\alpha+}$.
(ii) *Let* $\alpha \in]0, 1]$. *Then all admissible orders* $\preceq_{\alpha,\beta}$ *with* $\beta < \alpha$ *coincide. This admissible order will be denoted by* $\preceq_{\alpha-}$.

Proof. (i) Take $\alpha < \beta_1 \le \beta_2$ and assume that the orders $\preceq_{\alpha,\beta_1}$ and $\preceq_{\alpha,\beta_2}$ are not equivalent. Thus, there exist $[a, b]$, $[c, d] \in L([0, 1])$ such that $[a, b] \prec_{\alpha,\beta_1} [c, d]$ and $[c, d] \prec_{\alpha,\beta_2} [a, b]$. This means that $K_\alpha(a, b) = K_\alpha(c, d)$, $K_{\beta_1}(a, b) < K_{\beta_1}(c, d)$ and $K_{\beta_2}(a, b) > K_{\beta_2}(c, d)$. From these facts we can derive

$$a - c = \alpha(d - c - b + a),$$

$$a - c < \beta_1(d - c - b + a),$$

$$a - c > \beta_2(d - c - b + a),$$

which implies

$$\alpha(d - c - b + a) < \beta_1(d - c - b + a),$$

$$\alpha(d - c - b + a) > \beta_2(d - c - b + a).$$

As $[a, b] \neq [c, d]$, the previous inequalities yield either $\beta_2 < \alpha < \beta_1$ or $\beta_1 < \alpha < \beta_2$, which is in contradiction with the assumption.

The claim (ii) can be proved in a similar way. $\square$

Remark 3.3. Observe that the Xu and Yager order $\preceq_{YX}$ corresponds to the order $\preceq_{0.5+}$. From the statistical point of view, this order corresponds to the ordering of random variables based on the expected value as the primary criterion, and on the variance as the secondary criterion (in the case of uniform distributions this is a linear order over their supports).

4. Concluding remarks

In this work we have introduced a method for building linear orders between intervals by means of two continuous aggregation functions. These orders include the most used examples of total orders in the literature, such as the lexicographical orders or the order defined by Xu and Yager. However, by means of an example, we have shown that there exist problems in which, by choosing the appropriate order, we can force the conclusion.

This last comment leads to the following problem: Which is the most appropriate order for a given problem, in order to avoid spurious conclusions? An answer to this problem will include at least two questions: firstly, how are these new orders related to aggregation functions, since the latter are the basic tool in most of the problems considered in the literature. And, secondly, how may these orders be related to the data in the different problems that can be considered. In future works, we intend to handle both problems, focusing in particular in how interval-valued Choquet integrals can be defined by means of these families of orders, since these Choquet integral allow to define many of the most usual aggregation functions.

As an interesting theoretical problem we open the question of generating admissible orders by means of functions which are not continuous, in general.

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