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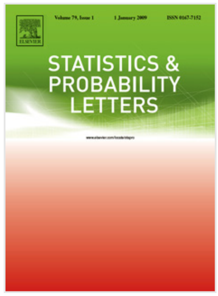

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On a strong law of large numbers for monotone measures

Hamzeh Agahi[a,b*] Adel Mohammadpour[a†] Radko Mesiar[c,d ‡] Yao Ouyang[e§]

[a]*Department of Statistics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran*

[b]*Statistical Research and Training Center (SRTC), Tehran, Iran*

[c]*Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, SK-81368 Bratislava, Slovakia*

[d]*Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodarenskou vezi 4, 182 08 Praha 8, Czech Republic*

[e]*Faculty of Science, Huzhou Teacher's College, Huzhou, Zhejiang 313000, People's Republic of China*

Abstract

Recently, Maccheroni *et al.* [Ann. Probab. 33, (2005) 1171–1178] provided an extension of the strong laws of large numbers of iid random variables for capacities. In this paper, we formulate new versions of strong laws of large number based on a submodular continuous monotone measure without the independence condition.

Keywords: Probability theory; The Borel-Cantelli lemma; Strong law of large numbers; Choquet expectation.

1 Introduction

The importance of the strong laws of large numbers in probability theory is well recognized and requires no discussion (see Durrett (2004)). Many papers have been published on the topic of the strong laws of large numbers, in several different research communities; see Chen (2012), Gadidov (1998), Jajte (2003), Korchevsky (2011), Maccheroni et al. (2005), Latała et al. (2000), Li et al. (2011), Rébillé (2009). We focus on capacities, an important class of nonadditive probabilities.

*h_agahi@aut.ac.ir (H. Agahi)

†Corresponding author. Tel: +98 21 66499064. Fax: +98 21 66497930. e-mail: adel@aut.ac.ir (A. Mohammadpour)

‡mesiar@math.sk (R. Mesiar)

§oyy@hutc.zj.cn (Y. Ouyang)

Recently, Maccheroni et al. (2005) established for capacities a strong law of large numbers for iid random variables. The main results subsequently proved relative to their strong law are still based on iid random variables. However, in probability theory many attempts have been made to weaken the independence condition in the strong law of large numbers; see Korchevsky (2011), Korchevsky et al. (2010), Luzia (2012). For this reason in this paper we address the question of how to prove strong laws of large numbers for capacities without the independence condition. Recently, Luzia (2012) proved an interesting version of the strong law of large numbers without assuming the random variables are pairwise independent, which is derived from the following theorem.

Theorem 1.1 *Let $X_1, X_2, ...$ be non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$ and $S_n = \sum_{i=1}^{n} X_i$. If $\sup_i \mathbb{E}[X_i^2] < \infty$, $\mathbb{E}[S_n] \to \infty$ and there exists $\gamma > 1$ such that*

$$\operatorname{var}[S_n] = O\left(\frac{(\mathbb{E}[S_n])^2}{(\log(\mathbb{E}[S_n]))(\log\log(\mathbb{E}[S_n]))^{\gamma}}\right),$$

then $\frac{S_n}{\mathbb{E}[S_n]} \overset{a.e.}{\to} 1$.

Corollary 1.2 *Let $X_1, X_2, ...$ be identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}X_i^2 < \infty$, $\mathbb{E}X_i = m$ and $S_n = \sum_{i=1}^{n} X_i$. If $X_i \geq -M$, for some constant $M > 0$, and there exists $\gamma > 1$ such that*

$$\sum_{1 \leq i < j \leq n} \left(\mathbb{E}(X_i X_j) - m^2\right) = O\left(\frac{n^2}{(\log n)(\log\log n)^{\gamma}}\right),$$

then $\frac{S_n}{n} \overset{a.e.}{\to} m$.

The aim of this paper is to generalize the strong laws of large number for some monotone measures. In particular, the previous results of Luzia (2012) based on the strong law of large numbers are extended for Choquet (-like) expectation based on a submodular continuous monotone measure.

The rest of the paper is organized as follows. Some notions and theorems that are useful in this paper are given in Section 2, including some generalizations of Borel-Cantelli lemmas. Generalizations of strong laws of large number are given in Section 3. Finally, some concluding remarks are added.

2 Preliminaries

In order to derive our main results, we have to recall here the following results and notations.

Let $(\Omega, \mathcal{F})$ be a fixed measurable space. An $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$ will be called a random variable. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see Agahi et al. (2012)).

2.1 Definition and Fundamental Properties

A set function $\mu : \mathcal{F} \to [0, \infty]$ is called a monotone measure whenever $\mu(\emptyset) = 0$, $\mu(\Omega) > 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. The triple $(\Omega, \mathcal{F}, \mu)$ is also called a *monotone measure space*. If $\mu(B_n) \downarrow \mu(B)$ for all sequences of measurable sets such that $B_n \downarrow B$ and $\mu(B_n) \uparrow \mu(B)$ for all sequences of measurable sets such that $B_n \uparrow B$, the monotone set function μ is called *continuous*. If, for a set $B \subset \Omega$, there exist $A \in \mathcal{F}$ such that $B \subset A$ and $\mu(A) = 0$, then B is called a *μ-nullset*. Note that a monotone set function μ is also *submodular (2-alternating)* whenever $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ *for all* $A, B \in \mathcal{F}$. If $\upsilon : \mathcal{F} \to [0, \infty]$ is a submodular continuous measure, then the triple $(\Omega, \mathcal{F}, \upsilon)$ is also called a *submodular continuous (SC-) measure space*. A monotone set function $\lambda : \mathcal{F} \to [0, 1]$ is said to be a *capacity*, if $\lambda(\Omega) = 1$ (see Choquet (1954)). In capacities, most work on upper probabilities has focused on the 2-alternating case; see Huber et al. (1973), Huber et al. (1974), Shapley (1971). For example, Huber et al. (1973) obtained that the 2-alternating property is necessary and sufficient for generalizing the Neyman-Pearson lemma to sets of probabilities.

Definition 2.1 (*Maccheroni et al. 2005*). *Let* $(\Omega, \mathcal{F}, \mu)$ *be a monotone measure space. Random variables* $Y_1, \ldots, Y_n : \Omega \to \mathbb{R}$ *are independent if* $\mu\{Y_1 \in B_1, \ldots, Y_n \in B_n\} = \prod_{i=1}^n \mu\{Y_i \in B_i\}$ *for all Borel sets* $B_1, \ldots, B_n$. *We also say they are identically distributed if, for each* $n, m \geq 1$ *and each Borel set* B, $\mu\{Y_n \in B\} = \mu\{Y_m \in B\}$.

Given a monotone measure space $(\Omega, \mathcal{F}, \mu)$, we shall denote by ω any element of Ω and we put $\{X > t\} = \{\omega : X(\omega) > t\}$ for any $t > 0$. The Choquet expectation of X with respect to (w.r.t.) the finite monotone measure μ is defined by

$$\mathbb{E}_C^{\mu}(X) = \int_{\Omega} X d\mu = \int_0^{\infty} \mu(\{X > t\})\, dt - \int_{-\infty}^{0} [\mu(\Omega) - \mu(\{X > t\})]\, dt. \tag{2.1}$$

We also denote $\mathbb{V}_C^{\mu}(X) = \mathbb{E}_C^{\mu}\left[(X - \mathbb{E}_C^{\mu}[X])^2\right]$. Throughout this paper, we always consider the existence of $\mathbb{E}_C^{\mu}(X)$ and of $\mathbb{V}_C^{\mu}(X)$. Notice that if μ is probability measure, $\mu = P$, then $\mathbb{E}_C^{\mu}(X) = \mathbb{E}_C^{P}(X) = \mathbb{E}(X)$ and $\mathbb{V}_C^{\mu}(X) = \mathbb{V}_C^{P}(X) = \mathrm{var}(X)$.

In order to consider the convergence analysis of random variables defined on a monotone measure space, we need the following definition, which is mainly due to Agahi et al. (2012). The authors proved some theorems in the convergence of a sequence of random variables on monotone measure spaces and also discussed relationships among forms of convergence.

Definition 2.2 *Let* X *be a random variable and let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of random variables defined on an SC-measure space* $(\Omega, \mathcal{F}, \upsilon)$.
(I) *We say that* X_n *converges in* υ *to* X *and write* $X_n \xrightarrow{\upsilon} X$ *if*

$$\forall \epsilon > 0 : \quad \upsilon[|X_n - X| > \epsilon] \to 0 \quad \text{as } n \to \infty.$$

(II) *We say that* X_n *is almost everywhere convergent to* X *and write* $X_n \xrightarrow{a.e.} X$ *if there exists a* υ*-nullset* $N \in \mathcal{F}$ *such that*

$$\forall \omega \in N^c : |X_n(\omega) - X(\omega)| \to 0 \quad \text{as } n \to \infty.$$

A Choquet-like expectation is based on pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$; see Mesiar (1995). Therefore, at first we state their definitions.

Let $[a,b]$ be a closed subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus\colon [a,b]^2 \to [a,b]$ is pseudo-addition if it is commutative with a zero (neutral) element denoted by $\mathbf{0}$, non-decreasing (with respect to $\preceq$), associative and if $r_n \to r$ and $s_n \to s$, then $r_n \oplus s_n \to r \oplus s$. Let $[a,b]_+ = \{x \mid x \in [a,b], \mathbf{0} \preceq x\}$. A binary operation $\otimes$ on $[a,b]$ is said to be a pseudo-multiplication corresponding to $\oplus$ if it is commutative with a unit element $e \in [a,b]$, i.e., for each $x \in [a,b], e \otimes x = x$, positively non-decreasing, i.e., $x \preceq y$ implies $x \otimes z \preceq y \otimes z$ for all $z \in [a,b]_+$, associative and if $r_n \to r \in (\mathbf{0},b)$ and $x_n \to x$, then $(r_n \otimes x_n) \to (r \otimes x)$ and $b \otimes x = \lim_{r \to b}(r \otimes x)$. We assume also $\mathbf{0} \otimes x = \mathbf{0}$ and that $\otimes$ is distributive over $\oplus$, i.e., $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.

Mesiar (1995) showed that if $\otimes$ is a pseudo-multiplication corresponding to a given pseudo-addition $\oplus$ fulfilling the aforesaid axioms (considering $a = \mathbf{0}$) and if its identity element e is not an idempotent of $\oplus$, then there is a continuous strictly increasing function $g\colon [a,b] \to [0,\infty]$ with $g(a) = 0$ and $g(b) = \infty$, such that $g(e) = 1$ and

$$\begin{aligned} c \oplus d &= g^{-1}(g(c) + g(d)) \quad \oplus \text{ is called a } g\text{-addition}, \\ c \otimes d &= g^{-1}(g(c) \cdot g(d)) \quad \otimes \text{ is called a } g\text{-multiplication}. \end{aligned}$$

Clearly, $\bigoplus_{i=1}^{n} x_i = g^{-1}\left(\sum_{i=1}^{n} g(x_i)\right)$ for any $x_i \in [a,b], i = 1, \ldots n$. We define the pseudo-power $x_{\otimes}^{(n)}$ as $x_{\otimes}^{(n)} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n-\text{times}}$, where $x \in [a,b]$ and n is a natural number. Evidently, if $x \otimes y = g^{-1}(g(x) \cdot g(y))$, then $x_{\otimes}^{(n)} = g^{-1}\left(g^n(x)\right)$.

Definition 2.3 (*Mesiar 1995*) *Let $(\Omega, \mathcal{F}, \mu)$ be a monotone measure space. Let $\oplus$ and $\otimes$ be generated by a generator g. The Choquet-like expectation of a measurable function $X : \Omega \to [a,b]$ w.r.t. the monotone measure μ can be represented as*

$$\mathbb{E}_{Cl,g}^{\mu}[X] = g^{-1}\left(\mathbb{E}_{C}^{g(\mu)}[g(X)]\right). \tag{2.2}$$

During the context, the Choquet-like expectation defined on an SC-measure space $(\Omega, \mathcal{F}, \upsilon)$ is defined by $\mathbb{E}_{Cl,g}^{\upsilon}[X] = g^{-1}\left(\mathbb{E}_{C}^{g(\upsilon)}[g(X)]\right)$. We also denote $\mathbb{V}_{Cl,g}^{\upsilon}[X] = g^{-1}\left(\mathbb{V}_{C}^{g(\upsilon)}[g(X)]\right)$.

Before stating our main result, we need the following theorems.

Theorem 2.4 (Borel-Cantelli lemma) *Let $(\Omega, \mathcal{F}, \upsilon)$ be an SC-measure space and let $\{A_i\}_{i=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$. If $\sum_{n=1}^{\infty} \upsilon(A_n) < \infty$, then $\upsilon\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = 0$.*

Proof. If $\sum_{n=1}^{\infty} \upsilon(A_n) < \infty$, then the convergence of this sum implies that $\inf_{m \geq 1} \sum_{n=m}^{\infty} \upsilon(A_n) = 0$. Since υ is an SC-measure, we have

$$0 \leq \upsilon\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) \leq \inf_{m \geq 1} \upsilon\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \inf_{m \geq 1} \sum_{n=m}^{\infty} \upsilon(A_n) = 0,$$

where the last inequality follows from the subadditivity and the continuity of υ. This completes the proof. □

The next result extends the Theorem 2.4.

Theorem 2.5 *Let $(\Omega, \mathcal{F}, \upsilon)$ be an SC-measure space, $\oplus, \otimes : [\mathbf{0}, \infty]^2 \to [\mathbf{0}, \infty]$ be generated by a generator $g : [\mathbf{0}, \infty] \to [0, \infty]$ and $\{A_i\}_{i=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$. If $\bigoplus_{n=1}^{\infty} \upsilon(A_n) < \infty$, then $\upsilon\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = \mathbf{0}$.*

Proof. If $\bigoplus_{n=1}^{\infty} \upsilon(A_n) < \infty$, then $\sum_{n=1}^{\infty} g(\upsilon(A_n)) < g(\infty) = \infty$. Now, by the Theorem 2.4, $g\left(\upsilon\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right)\right) = 0$. Therefore, $\upsilon\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = g^{-1}(0) = \mathbf{0}$. □

Now, our results can be stated as follows.

3 The strong law of large numbers

Theorem 3.1 *Let $X_1, X_2, ...$ be non-negative random variables defined on an SC- measure space $(\Omega, \mathcal{F}, \upsilon)$ and $S_n = \sum_{i=1}^{n} X_i$. Let $\sup_i \mathbb{E}_C^{\upsilon}[X_i^2] < \infty$ and $\mathbb{E}_C^{\upsilon}[S_n] \to \infty$. If there exist $a > 1$ and $k > 0$ such that*

$$\mathbb{V}_C^{\upsilon}[S_n] \leq \frac{k\left(\mathbb{E}_C^{\upsilon}[S_n]\right)^2}{\left(\log\left(\mathbb{E}_C^{\upsilon}[S_n]\right)\right)\left(\log\log\left(\mathbb{E}_C^{\upsilon}[S_n]\right)\right)^a}, \tag{3.1}$$

then $\frac{S_n}{\mathbb{E}_C^{\upsilon}[S_n]} \xrightarrow{a.e.} 1$.

Proof. Using the Chebyshev type inequality for Choquet expectation (see Sheng et al. (2011)), then for each $\epsilon > 0$, we have

$$\upsilon\left(|S_n - \mathbb{E}_C^{\upsilon}[S_n]| > \left(\epsilon \mathbb{E}_C^{\upsilon}[S_n]\right)\right) \leq \frac{\mathbb{V}_C^{\upsilon}[S_n]}{\epsilon^2\left(\mathbb{E}_C^{\upsilon}[S_n]\right)^2} \leq \frac{k}{\epsilon^2\left(\log\left(\mathbb{E}_C^{\upsilon}[S_n]\right)\right)\left(\log\log\left(\mathbb{E}_C^{\upsilon}[S_n]\right)\right)^a} \tag{3.2}$$

for some constant $k > 0$. So, $\frac{S_n}{\mathbb{E}_C^{\upsilon}[S_n]} \xrightarrow{\upsilon} 1$. Let $0 < b < a - 1$ and $n_r = \inf\left\{n : \log\left(\mathbb{E}_C^{\upsilon}[S_n]\right) \geq \frac{r}{(\log r)^b}\right\}$ be subsequences of n. Since $\mathbb{E}_C^{\upsilon}[X_i] \leq C < \infty$, then we have $\exp\left(\frac{r}{(\log r)^b}\right) \leq \mathbb{E}_C^{\upsilon}[S_{n_r}] \leq \exp\left(\frac{r}{(\log r)^b}\right) + C$. Thus, Ineq. (3.2) implies that

$$\upsilon\left(|S_{n_r} - \mathbb{E}_C^{\upsilon}[S_{n_r}]| > \epsilon \mathbb{E}_C^{\upsilon}[S_{n_r}]\right) \leq \frac{k'}{\epsilon^2 r\left(\log(r)\right)^{a-b}},$$

for some constant $k' > 0$. Therefore, $\sum_{r=1}^{\infty} \upsilon\left(|S_{n_r} - \mathbb{E}_C^{\upsilon}[S_{n_r}]| > \epsilon \mathbb{E}_C^{\upsilon}[S_{n_r}]\right) < \infty$. Then the arbitrariness of ϵ and Theorem 2.4 imply that $\frac{S_{n_r}}{\mathbb{E}_C^{\upsilon}[S_{n_r}]} \xrightarrow{a.e.} 1$. Now choose an ω. If $n_r \leq n \leq n_{r+1}$, then

$$\frac{S_{n_r}(\omega)}{\mathbb{E}_C^{\upsilon}\left[S_{n_{r+1}}\right]} \leq \frac{S_n(\omega)}{\mathbb{E}_C^{\upsilon}[S_n]} \leq \frac{S_{n_{r+1}}(\omega)}{\mathbb{E}_C^{\upsilon}[S_{n_r}]}. \tag{3.3}$$

The relation (3.3) can also be written in the form

$$\frac{\mathbb{E}_C^{\upsilon}[S_{n_r}]}{\mathbb{E}_C^{\upsilon}\left[S_{n_{r+1}}\right]} \frac{S_{n_r}(\omega)}{\mathbb{E}_C^{\upsilon}[S_{n_r}]} \leq \frac{S_n(\omega)}{\mathbb{E}_C^{\upsilon}[S_n]} \leq \frac{S_{n_{r+1}}(\omega)}{\mathbb{E}_C^{\upsilon}\left[S_{n_{r+1}}\right]} \frac{\mathbb{E}_C^{\upsilon}\left[S_{n_{r+1}}\right]}{\mathbb{E}_C^{\upsilon}[S_{n_r}]}.$$

So, it is enough to prove that $\frac{\mathbb{E}_C^{\upsilon}\left[S_{n_{r+1}}\right]}{\mathbb{E}_C^{\upsilon}[S_{n_r}]}$ converges to 1. Since

$$\exp\left(\frac{r}{(\log r)^b}\right) \leq \mathbb{E}_C^{\upsilon}[S_{n_r}] \leq \mathbb{E}_C^{\upsilon}\left[S_{n_{r+1}}\right] \leq \exp\left(\frac{r+1}{\left(\log(r+1)\right)^b}\right) + C,$$

then $\frac{\mathbb{E}_C^v\left[S_{n_{r+1}}\right]}{\mathbb{E}_C^v\left[S_{n_r}\right]}$ converges to 1 because $\frac{\frac{r+1}{(\log(r+1))^b}}{\frac{r}{(\log r)^b}}$ converges to 1. This completes the proof. □

Theorem 3.1 plays an important role in obtaining our results.

Corollary 3.2 (Borel-Cantelli lemma II) *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$. If $\{A_n\}_{n=1}^{\infty}$ are pairwise independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then*

$$\frac{\sum_{i=1}^n \mathbb{I}_{A_i}}{\sum_{i=1}^n P(A_i)} \overset{a.e.}{\to} 1.$$

Proof. Let $X_i = \mathbb{I}_{A_i}$. Since $\{A_i\}_{i=1}^n$ are pairwise independent and $\mathbb{E}_C^P[S_n] \to \infty$, we have

$$\mathbb{V}_C^P[S_n] = \operatorname{var}[S_n] = \operatorname{var}\left[\sum_{i=1}^n \mathbb{I}_{A_i}\right] = \sum_{i=1}^n \operatorname{var}[\mathbb{I}_{A_i}] \leq \sum_{i=1}^n \mathbb{E}[\mathbb{I}_{A_i}] = \mathbb{E}[S_n] = \mathbb{E}_C^P[S_n].$$

Then (3.1) holds readily for some large constant $k > 0$. So, by Theorem 3.1, we have

$$\frac{\sum_{i=1}^n \mathbb{I}_{A_i}}{\mathbb{E}_C^P\left[\sum_{i=1}^n \mathbb{I}_{A_i}\right]} = \frac{\sum_{i=1}^n \mathbb{I}_{A_i}}{\sum_{i=1}^n \mathbb{E}[\mathbb{I}_{A_i}]} = \frac{\sum_{i=1}^n \mathbb{I}_{A_i}}{\sum_{i=1}^n P(A_i)} \overset{a.e.}{\to} 1.$$

The main results of this paper are the following theorems.

Theorem 3.3 (The strong law of large numbers I) *Let $X_1, X_2, \ldots$ be identically distributed non-negative random variables defined on an SC- measure space $(\Omega, \mathcal{F}, \upsilon)$ with $\mathbb{E}_C^v[X_i] = m$, $\mathbb{E}_C^v[X_i^2] < \infty$ and $S_n = \sum_{i=1}^n X_i$. If there exist $a > 1$ and $k > 0$ such that*

$$\mathbb{E}_C^v\left[(S_n - nm)^2\right] \leq \frac{kn^2}{(\log(n))(\log\log(n))^a},$$

then $\frac{S_n}{n} \overset{a.e.}{\to} m$.

Proof. This is similar to the proof of Theorem 3.1.

Theorem 3.4 (The strong law of large numbers II) *Let $X_1, X_2, \ldots$ be identically distributed non-negative random variables defined on an SC- measure space $(\Omega, \mathcal{F}, \upsilon)$ with $\mathbb{E}_C^v[X_i] = m$, $\mathbb{E}_C^v[X_i^2] < \infty$ and $S_n = \sum_{i=1}^n X_i$. If there exist $a > 1$ and $b, k > 0$ such that*

$$\mathbb{V}_C^v[S_n] \leq T^2, \tag{3.4}$$

where $T = \frac{k^{\frac{1}{2}} n}{\sqrt{(\log(n))(\log\log(n))^a}} - 2n\sqrt{b}\upsilon(\Omega)$, then $\frac{S_n}{n} \overset{a.e.}{\to} m$.

Proof. Since υ is submodular, there holds (see Denneberg (1994) and Wang et al. (2008))

$$\mathbb{E}_C^v[S_n] = \mathbb{E}_C^v\left[\sum_{i=1}^n X_i\right] \leq \sum_{i=1}^n \mathbb{E}_C^v[X_i] = nm. \tag{3.5}$$

$\mathbb{E}_C^{\upsilon}\left[X_i^2\right] < \infty$ implies that there exists $b > 0$ such that $\mathbb{E}_C^{\upsilon}\left[X_i^2\right] \leq b$. Applying first Hölder's inequality and then Minkowski's inequality for Choquet expectation (see Mesiar et al. (2010)), we obtain

$$\begin{aligned}\mathbb{E}_C^{\upsilon}\left[X_i\right] + c\upsilon(\Omega) &= \mathbb{E}_C^{\upsilon}\left[X_i + c\right] = \mathbb{E}_C^{\upsilon}\left[(X_i + c) \times 1\right] \leq \left(\mathbb{E}_C^{\upsilon}\left[(X_i + c)^2\right]\right)^{\frac{1}{2}} \upsilon^{\frac{1}{2}}(\Omega) \\ &\leq \left(\left(\mathbb{E}_C^{\upsilon}\left[X_i^2\right]\right)^{\frac{1}{2}} + c\upsilon^{\frac{1}{2}}(\Omega)\right) \upsilon^{\frac{1}{2}}(\Omega) \leq b^{\frac{1}{2}}\upsilon^{\frac{1}{2}}(\Omega) + c\upsilon(\Omega),\end{aligned}$$

where $c > 0$. Since X_i are non-negative, then we have

$$|m| \leq \sqrt{b\upsilon(\Omega)}. \tag{3.6}$$

So, (3.5) and (3.6) imply that

$$\begin{aligned}\left(\mathbb{E}_C^{\upsilon}\left[\left(\mathbb{E}_C^{\upsilon}\left[S_n\right] - nm\right)^2\right]\right)^{\frac{1}{2}} &\leq \left(\mathbb{E}_C^{\upsilon}\left[(2nm)^2\right]\right)^{\frac{1}{2}} = \left((2nm)^2 \upsilon(\Omega)\right)^{\frac{1}{2}} \\ &= 2n\,|m|\,\sqrt{\upsilon(\Omega)} \leq 2n\sqrt{b\upsilon(\Omega)}\sqrt{\upsilon(\Omega)} \leq 2n\sqrt{b}\upsilon\left(\Omega\right).\end{aligned} \tag{3.7}$$

For any $\epsilon > 0$, by Chebyshev's inequality and then Minkowski's inequality for Choquet expectation, we get

$$\begin{aligned}&\upsilon\left(\left|\frac{S_n}{n} - m\right| > \epsilon\right) = \upsilon\left(|S_n - nm| > \epsilon n\right) \\ &= \upsilon\left(\left|\left(S_n - \mathbb{E}_C^{\upsilon}\left[S_n\right]\right) + \left(\mathbb{E}_C^{\upsilon}\left[S_n\right] - nm\right)\right| > \epsilon n\right) \\ &\leq \frac{\mathbb{E}_C^{\upsilon}\left[\left(\left(S_n - \mathbb{E}_C^{\upsilon}\left[S_n\right]\right) + \left(\mathbb{E}_C^{\upsilon}\left[S_n\right] - nm\right)\right)^2\right]}{\epsilon^2 n^2} \\ &\leq \frac{\left(\left(\mathbb{E}_C^{\upsilon}\left[\left(S_n - \mathbb{E}_C^{\upsilon}\left[S_n\right]\right)^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}_C^{\upsilon}\left[\left(\mathbb{E}_C^{\upsilon}\left[S_n\right] - nm\right)^2\right]\right)^{\frac{1}{2}}\right)^2}{\epsilon^2 n^2},\end{aligned}$$

and hence, by (3.7) and assumption (3.4), we have

$$\begin{aligned}\upsilon\left(\left|\frac{S_n}{n} - m\right| > \epsilon\right) &\leq \frac{\left(\left(\mathbb{V}_C^{\upsilon}\left[S_n\right]\right)^{\frac{1}{2}} + 2n\sqrt{b}\upsilon(\Omega)\right)^2}{\epsilon^2 n^2} \\ &\leq \frac{k}{\epsilon^2\left(\log\left(n\right)\right)\left(\log\log\left(n\right)\right)^a}\end{aligned}$$

for some constant $k > 0$. Then the proof is similar to the proof of Theorem 3.1. □

4 Concluding remarks

We have introduced some generalizations of Borel-Cantelli's lemmas, as well as of the strong law of large numbers, considering continuous submodular monotone measures and Choquet expectations. Up to the result mentioned in Theorem 3.4, we obtain a generalization of this result for Choquet-like expectation. Let $\oplus, \otimes : [0, \infty]^2 \to [0, \infty]$ be generated by a generator $g : [0, \infty]^2 \to [0, \infty]$

and $X_1, X_2, ...$ be identically distributed non-negative random variables defined on an SC- measure space $(\Omega, \mathcal{F}, \upsilon)$ with $\mathbb{E}^{g(\upsilon)}_{Cl,g}[X_i] = m$, $\mathbb{E}^{g(\upsilon)}_{Cl,g}\left[(X_i)^{(2)}_{\otimes}\right] < \infty$ and $B_n = \bigoplus_{i=1}^n X_i$. If there exist $a > 1$ and $b, k > 0$ such that

$$\mathbb{V}^{g(\upsilon)}_{Cl,g}[B_n] \leq g^{-1}\left(T^2\right),$$

where $T = \frac{k^{\frac{1}{2}} n}{\sqrt{(\log(n))(\log\log(n))^a}} - 2n\sqrt{b}\upsilon(\Omega)$, then $g^{-1}\left(\frac{1}{n}\right) \otimes B_n \xrightarrow{a.e.} m$.

We believe that versions of these results for some other types of generalized integrals, such as Sugeno integral (see Sugeno (1974)), Shilkret integral (see Shilkret (1971)) or special kinds of universal integrals based on copulas (see Klement et al. (2010)) are valid, too. In our future research we aim to study and investigate this problem.

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1. New versions of strong laws of large number for monotone measures are given.
2. This paper improves the series of papers on the topic.
3. Some generalizations of Borel-Cantelli's lemmas are given.