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Calculus of Variations

Sequential weak continuity of null Lagrangians at the boundary

Agnieszka Kałamajska · Stefan Krömer · Martin Kružík

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Abstract We show weak* in measures on $\bar{\Omega}$/ weak-L^1 sequential continuity of $u \mapsto f(x, \nabla u) : W^{1,p}(\Omega; \mathbb{R}^m) \to L^1(\Omega)$, where $f(x, \cdot)$ is a null Lagrangian for $x \in \Omega$, it is a null Lagrangian at the boundary for $x \in \partial\Omega$ and $|f(x, A)| \leq C(1 + |A|^p)$. We also give a precise characterization of null Lagrangians at the boundary in arbitrary dimensions. Our results explain, for instance, why $u \mapsto \det \nabla u : W^{1,n}(\Omega; \mathbb{R}^n) \to L^1(\Omega)$ fails to be weakly continuous. Further, we state a new weak lower semicontinuity theorem for integrands depending on null Lagrangians at the boundary. The paper closes with an example indicating that a well-known result on higher integrability of determinant by Müller (Bull. Am. Math. Soc. New Ser. 21(2): 245–248, 1989) need not necessarily extend to our setting. The notion of quasiconvexity at the boundary due to J.M. Ball and J. Marsden is central to our analysis.

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A. Kałamajska
Institute of Mathematics, Warsaw University, ul. Banacha 2, PL-02-097 Warsaw, Poland

S. Krömer
Mathematisches Institut, Universität zu Köln, 50923 Köln, Germany

M. Kružík (✉)
Institute of Information Theory and Automation of the ASCR,
Pod vodárenskou věží 4,
182 08 Praha 8, Czech Republic
e-mail: kruzik@utia.cas.cz

M. Kružík
Faculty of Civil Engineering, Czech Technical University,
Thákurova 7, 166 29 Praha 6, Czech Republic

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1 Introduction

This paper is inspired by the well-known example [3, Example 7.3] or [7, Example 8.6] showing that if $\Omega \subset \mathbb{R}^2$ is bounded and Lipschitz and $\{u_k\} \subset W^{1,2}(\Omega;\mathbb{R}^2)$ weakly converges to the origin then, in general, $\int_\Omega \det \nabla u_k(x)\,\mathrm{d}x \not\to 0$ which means that $\det \nabla u_k \not\rightharpoonup 0$ in $L^1(\Omega)$, neither $\det \nabla u_k \overset{*}{\rightharpoonup} 0$ in $\mathrm{rca}(\bar{\Omega})$ (Radon measures on $\bar{\Omega}$). Contrary to that, if the sequence were bounded in $W^{1,p}(\Omega;\mathbb{R}^2)$ for $p > 2$ then $\{\det \nabla u_k\}_{k\in\mathbb{N}}$ would weakly tend to zero in $L^1(\Omega)$. Therefore, a natural question arises which functions $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, $|f(A)| \le C(1+|A|^p)$, have the property that $u \mapsto f(\nabla u)$ is (weakly,weakly*) sequentially continuous as maps from $W^{1,p}(\Omega;\mathbb{R}^m)$ to $\mathrm{rca}(\bar{\Omega})$, $p > 1$. It is obvious that such functions must be quasiaffine, i.e., f is an affine function of all subdeterminants of its argument [7], however, as the above mentioned example shows, it is far from being sufficient. It turns out that this question is intimately related to concentrations of $\{|\nabla u_k|^p\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ at the boundary of Ω and that, for a general domain Ω, f must also depend on $x \in \Omega$. We also show that the notion of quasiconvexity at the boundary, introduced in [4] to study necessary conditions for local minimizers of variational integral functionals plays a key role in our analysis.

The plan of the paper is as follows. After introducing necessary notation we recall the notions of quasiconvexity and quasiconvexity at the boundary. Then we explicitly characterize all functions which, together with their negative multiple, are quasiconvex at the boundary. These are here called null Lagrangians at the boundary. Our characterization is a slight adaptation of the result of P. Sprenger [34] which does not seem to be well-known to the calculus-of-variations community. We state our main result Theorem 3.1 using a recently discovered characterization of DiPerna-Majda measures generated by gradients [21] and get a new weak lower semicontinuity result for integral functionals depending on null Lagrangians at the boundary. Finally, we construct an example indicating that a result analogous to higher integrability of determinants due to Müller [28] may not hold for null Lagrangians at the boundary.

2 Basic notation

Let us start with a few definitions and with the explanation of our notation. Having a bounded domain $\Omega \subset \mathbb{R}^n$ we denote by $C(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$. Then $C_0(\Omega)$ consists of functions from $C(\Omega)$ whose support is contained in Ω. More generally, for any topological space S, by $C(S)$ we denote all continuous functions on S. In what follows "rca(S)" denotes the set of regular countably additive set functions on the Borel σ-algebra on a metrizable set S (cf. [9]), its subset, $\mathrm{rca}_1^+(S)$, denotes regular probability measures on a set S. We write "γ-almost all" or "γ-a.e." if we mean "up to a set with the γ-measure zero". If γ is the n-dimensional Lebesgue measure and $M \subset \mathbb{R}^n$ we omit writing γ in the notation. Further, $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \le p < +\infty$ denotes the usual space of measurable mappings which are together with their first (distributional) derivatives integrable with the p-th power. The support of a measure $\sigma \in \mathrm{rca}(\Omega)$ is the smallest closed set S such that $\sigma(A) = 0$ if $S \cap A = \emptyset$. Finally, if $\sigma \in \mathrm{rca}(S)$ we write σ_s and d_σ for the singular part and density of σ defined by the Lebesgue decomposition, respectively. We denote by 'w-lim' the weak limit and by $B(x_0, r)$ an open ball in $\mathbb{R}^n$ centered at x_0 and the radius $r > 0$. The scalar product on $\mathbb{R}^n$ is standardly defined as $a \cdot b := \sum_{i=1}^n a_i b_i$ and analogously on $\mathbb{R}^{m\times n}$. Finally, if $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ then $a \otimes b \in \mathbb{R}^{m\times n}$ with $(a\otimes b)_{ij} = a_i b_j$, and $\mathbb{I}$ denotes the identity matrix.

2.1 Quasiconvex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex [27] if for any $F \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$

$$v(F)|\Omega| \le \int_\Omega v(F+\nabla\varphi(x))\,\mathrm{d}x\ . \tag{2.1}$$

If $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is not quasiconvex we define its quasiconvex envelope $Qv : \mathbb{R}^{m\times n} \to \mathbb{R}$ as

$$Qv = \sup\left\{h \le v;\ \ h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex}\right\}$$

and if the set on the right-hand side is empty we put $Qv = -\infty$. If v is locally bounded and Borel measurable then for any $F \in \mathbb{R}^{m\times n}$ (see [7])

$$Qv(F) = \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega v(F+\nabla\varphi(x))\,\mathrm{d}x\ . \tag{2.2}$$

Following [4,33,34] we define the notion of quasiconvexity at the boundary. In order to proceed, we first define the so-called *standard boundary domain*.

Definition 2.1 Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $\Omega_\varrho \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that Ω_ϱ is a standard boundary domain with the normal ϱ if there is $a \in \mathbb{R}$ such that $\Omega_\varrho \subset H_{a,\varrho} := \{x \in \mathbb{R}^n;\ \varrho\cdot x < a\}$ and the $(n-1)$- dimensional interior Γ_ϱ of $\partial\Omega_\varrho \cap \partial H_{a,\varrho}$ is not empty.

For $1 \le p \le +\infty$, and any bounded Lipschitz domain Ω, we define

$$W^{1,p}_{\partial\Omega\setminus\Gamma}(\Omega;\mathbb{R}^m) := \{u \in W^{1,p}(\Omega;\mathbb{R}^m);\quad u \equiv 0 \text{ on } \partial\Omega\setminus\Gamma\}, \tag{2.3}$$

where the condition $u \equiv 0$ is understood in the sense of operator of trace, in particular the equality holds $\mathcal{H}^{n-1}$-almost everywhere with respect to the $n-1$-dimensional Hausdorff measure on $\partial\Omega$.

Definition 2.2 ([4]) Let $\varrho \in \mathbb{R}^n$ be a unit vector, and let $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given function.

(i) v is called quasiconvex at the boundary at $F \in \mathbb{R}^{m\times n}$ (where $F \in \mathbb{R}^{m\times n}$ is given), with respect to ϱ (shortly v is qcb at (F,ϱ)), if there is $q \in \mathbb{R}^m$ such that for every standard boundary domain Ω_ϱ with the normal ϱ and for every $u \in W^{1,\infty}_{\partial\Omega_\varrho\setminus\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$, we have

$$\int_{\Gamma_\varrho} q\cdot u(x)\,\mathrm{d}S + v(F)|\Omega_\varrho| \le \int_{\Omega_\varrho} v(F+\nabla u(x))\,\mathrm{d}x\ . \tag{2.4}$$

(ii) v is called quasiconvex at the boundary if it is quasiconvex at the boundary at every $F \in \mathbb{R}^{m\times n}$ and every $\varrho \in \mathbb{R}^n$.

An immediate generalization of the above definition is the following one.

Definition 2.3 Let $\varrho \in \mathbb{R}^n$ be a unit vector, $F \in \mathbb{R}^{m\times n}$, $1 \le p < +\infty$, $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is such that $|v| \le C(1+|\cdot|^p)$ for some $C > 0$.

(i) A function v is called $W^{1,p}$-quasiconvex at the boundary at given $F \in \mathbb{R}^{m\times n}$ with respect to ϱ (shortly v is p-qcb at (F,ϱ)), if there is $q \in \mathbb{R}^m$ such that for every standard boundary domain Ω_ϱ with the normal ϱ and for every $u \in W^{1,p}_{\partial\Omega_\varrho\setminus\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$, we have

$$\int_{\Gamma_\varrho} q\cdot u(x)\,\mathrm{d}S + v(F)|\Omega_\varrho| \le \int_{\Omega_\varrho} v(F+\nabla u(x))\,\mathrm{d}x\ . \tag{2.5}$$

(ii) A function v is called $W^{1,p}$-quasiconvex at the boundary if it is $W^{1,p}$-quasiconvex at the boundary at every $F \in \mathbb{R}^{m\times n}$ and every $\varrho \in \mathbb{R}^n$.

Let us formulate several remarks, concerning the notion of quasiconvexity at the boundary.

Remark 2.4 (i) If v is differentiable at F then vector q satisfying (2.4) is uniquely defined and $q = \nabla v(F)\varrho$, cf. [34].

(ii) It is clear that if v is qcb at (F, ϱ) it is also quasiconvex at F, i.e., (2.1) holds.

(iii) If (2.4) holds for one standard boundary domain it holds for other standard boundary domains with the normal ρ, too, [4].

(iv) If $p > 1$, $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is positively p-homogeneous, i.e. $v(\lambda F) = \lambda^p v(F)$ for all $F \in \mathbb{R}^{m\times n}$, continuous, and p-qcb at $(0, \varrho)$ then $q = 0$ in (2.4). Indeed, as $v(t\theta) = t^p v(\theta)$, then $\partial_\theta v = \lim_{t\to 0} \frac{v(t\theta)-v(0)}{t} = \lim_{t\to 0} t^{p-1} v(\theta) = 0$, whenever $\theta \in \mathbb{R}^{m\times n}$. Therefore v is differentiable at 0 with $\nabla v(0) = 0$ and according to our Remark (i), $q = \nabla v(0)\rho = 0$. Moreover, let us note that (2.4) implies that $\int_{\Omega_\varrho} v(\nabla u(x))\,\mathrm{d}x \geq 0$.

(v) Under the growth assumption $|v| \leq C(1 + |\cdot|^p)$ for some $1 \leq p < +\infty$ and $C > 0$, $W^{1,p}$-quasiconvexity at the boundary is equivalent to the quasiconvexity at the boundary [21].

(vi) We refer an interested reader to [14,15,26] for other applications of quasiconvexity at the boundary in variational context.

It will be convenient to define the following notion of quasiconvex at the boundary envelope of v at zero. Note that we integrate only over a standard boundary domain with a given normal.

Definition 2.5 Let $\Omega_\varrho \subset \mathbb{R}^n$ be the standard boundary domain with the normal $\varrho \in \mathbb{R}^n$ of the unit length and let Γ_ϱ be as in Definition 2.1. Let $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ be continuous and positively p-homogeneous. By the $W^{1,p}$-quasiconvex envelop at the boundary at 0, we define the quantity:

$$Q_{b,\varrho} v(0) := \inf_{u \in W^{1,p}_{\Omega_\varrho \setminus \Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)} \frac{1}{|\Omega_\varrho|} \int_{\Omega_\varrho} v(\nabla u(x))\,\mathrm{d}x\ . \tag{2.6}$$

Below we state an example of a function which is quasiconvex at the boundary.

Example 2.6 It is shown in [33, Prop. 17.2.4] that the function $v : \mathbb{R}^{3\times 3} \to \mathbb{R}$ given by

$$v(F) := a \cdot [\mathrm{Cof}\, F]\varrho$$

is quasiconvex at the boundary with the unit normal $\varrho \in \mathbb{R}^3$. Here $a \in \mathbb{R}^3$ is an arbitrary constant and "Cof" is the cofactor matrix, i.e., $[\mathrm{Cof}\, F]_{ij} = (-1)^{i+j} \det F'_{ij}$, where $F'_{ij} \in \mathbb{R}^{2\times 2}$ is the submatrix of F obtained from F by removing the i-th row and the j-th column. Hence, v is positively 2-homogeneous. This particular function is also called an interface null Lagrangian in [32].

2.2 Null Lagrangians at the boundary

Definition 2.7 Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given function.

(i) v is called a *null Lagrangian at the boundary* at given $F \in \mathbb{R}^{m\times n}$; cf. [33], if both v and $-v$ are quasiconvex at the boundary at F, i.e., there exists $q \in \mathbb{R}^m$ such that for every standard boundary domain Ω_ϱ with the normal ϱ and for all $u \in W^{1,p}_{\partial\Omega_\varrho \setminus \Gamma_\varrho}(\Omega_\varrho; \mathbb{R}^m)$, we have

$$\int_{\Gamma_\varrho} q \cdot u(x)\,\mathrm{d}S + v(F)|\Omega_\varrho| = \int_{\Omega_\varrho} v(F + \nabla u(x))\,\mathrm{d}x\,. \tag{2.7}$$

(ii) If v is a *null Lagrangian at the boundary* at every $F \in \mathbb{R}^{m\times n}$, we call it a null Lagrangian at the boundary.

Definition 2.8 Let $\varrho \in \mathbb{R}^n$ be a unit vector. A mapping $\mathcal{N} : \mathbb{R}^{m\times n} \to \mathbb{R}$ will be called a special null Lagrangian at the boundary at $F \in \mathbb{R}^{m\times n}$, with respect to the normal ϱ if for every $W^{1,\infty}_{\partial\Omega_\varrho\setminus\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ it holds that

$$\int_{\Omega_\varrho} \mathcal{N}(F + \nabla u(x))\,\mathrm{d}x = \mathcal{N}(F)|\Omega_\varrho|\,. \tag{2.8}$$

In particular, Eq. (2.7) holds with $q = 0$.

Remark 2.9 Given a fixed $F_0 \in \mathbb{R}^{m\times n}$, every null Lagrangian at the boundary at F_0 can be transformed into a special null Lagrangian at the boundary at F_0 by adding a linear term. More precisely, we have the following result: If $\mathcal{N}$ is a null Lagrangian at the boundary at F_0 with normal ϱ, then $\mathcal{N}$ is differentiable (in fact, it is a null Lagrangian and thus a polynomial), and according to Remark 2.4 (i), the vector q in (2.7) is given by $q = \frac{\partial}{\partial F}\mathcal{N}(F_0)\varrho$. If we define $\tilde{\mathcal{N}} : \mathbb{R}^{m\times n} \to \mathbb{R}$ by

$$\tilde{\mathcal{N}}(F) := \mathcal{N}(F) - \frac{\partial}{\partial F}\mathcal{N}(F_0)\cdot F, \quad F \in \mathbb{R}^{m\times n},$$

we have that $\frac{\partial}{\partial F}\tilde{\mathcal{N}}(F_0) = 0$, and consequently, $\tilde{\mathcal{N}}$ is a special null Lagrangian at the boundary at F_0.

In view of the previous remark, the following theorem explicitly characterizes all possible null Lagrangians at the boundary. It was first proved by P. Sprenger in his thesis [34, Satz 1.27] written in German. We give here his original proof with some minor simplifications. Before stating the result we recall that $\mathrm{SO}(n) := \{R \in \mathbb{R}^{n\times n};\ R^\top R = RR^\top = \mathbb{I}\,,\ \det R = 1\}$ denotes the set of orientation-preserving rotations and if we write $A = (B|\varrho)$ for some $B \in \mathbb{R}^{n\times(n-1)}$ and $\varrho \in \mathbb{R}^n$ then $A \in \mathbb{R}^{n\times n}$, its last column is ϱ and $A_{ij} = B_{ij}$ for $1 \le i \le n$ and $1 \le j \le n-1$.

Theorem 2.10 *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $\mathcal{N} : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given continuous function. Then the following four statements are equivalent.*

(i) $\mathcal{N}$ satisfies (2.8) for every $F \in \mathbb{R}^{m\times n}$;
(ii) $\mathcal{N}$ satisfies (2.8) for $F = 0$, i.e., $\mathcal{N}$ is a special null Lagrangian at 0;
(iii) There are constants $\tilde{\beta}_s \in \mathbb{R}^{\binom{m}{s}\times\binom{n-1}{s}}$, $1 \le s \le \min(m, n-1)$, such that for all $H \in \mathbb{R}^{m\times n}$,

$$\mathcal{N}(H) = \mathcal{N}(0) + \sum_{s=1}^{\min(m,n-1)} \tilde{\beta}_s \cdot \mathrm{ad}_s(H\tilde{R}), \tag{2.9}$$

where $\tilde{R} \in \mathbb{R}^{n\times(n-1)}$ is a matrix such that $R = (\tilde{R}|\varrho)$ belongs to $\mathrm{SO}(n)$;
(iv) $\mathcal{N}(F + a\otimes\varrho) = \mathcal{N}(F)$ for every $F \in \mathbb{R}^{m\times n}$ and every $a \in \mathbb{R}^m$.

Remark 2.11 Condition (ii) is not a part of the statement of [34, Satz 1.27], but it follows from the proof. On the other hand, we omitted a simple variant of (iv) in terms of the derivative of $\mathcal{N}$ that was given by Sprenger.

Proof of Theorem 2.10 At first we note that the proof can be reduced to the case when $\varrho = e_n$ in the formulation of the statements. Indeed, let $e_1, \dots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$ and observe that $Re_n = \varrho$, with R defined in (iii).

Moreover, let $\Omega_{e_n} := R^\top \Omega_\varrho$, $\Gamma_{e_n} := R^\top \Gamma_\varrho$ and $\hat{\mathcal{N}}(F) := \mathcal{N}(FR^\top)$ for $F \in \mathbb{R}^{m\times n}$, and let $\tilde{F} := FR$. Note that $\hat{\mathcal{N}}$ also is a null Lagrangian, and Ω_{e_n} is a standard boundary domain to the normal vector e_n. By a change of variables, (2.7) is equivalent to

$$\int_{\Omega_{e_n}} \hat{\mathcal{N}}(\tilde{F} + \nabla u(x))\,\mathrm{d}x = \hat{\mathcal{N}}(\tilde{F})|\Omega_{e_n}| \text{ for every } u \in W^{1,\infty}_{\partial\Omega_{e_n}\setminus\Gamma_{e_n}}(\Omega_{e_n}; \mathbb{R}^m). \tag{2.10}$$

An easy verification shows that (i)–(iv) defined for $\mathcal{N}$ are equivalent to the analogous counterparts for $\hat{\mathcal{N}}$.

As a consequence, it suffices to prove the assertion for the case $\mathcal{N} = \hat{\mathcal{N}}$, $\varrho = e_n$ and $\tilde{R} = (e_1|\dots|e_{n-1}) \in \mathbb{R}^{n\times(n-1)}$. Moreover, as (i)–(iv) clearly remain unchanged if we add a constant to $\mathcal{N}$, therefore we may assume that $\mathcal{N}(0) = 0$.

Since (i) obviously implies (ii), we only have to show that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(ii) implies (iii):

Since $\mathcal{N}$ is a null Lagrangian with $\mathcal{N}(0) = 0$, there are constants $\beta_s \in \mathbb{R}^{\binom{m}{s}\times\binom{n}{s}}$, $1 \le s \le \min(m, n)$ such that

$$\mathcal{N}(H) = \sum_{s=1}^{\min(m,n)} \beta_s \cdot \mathrm{ad}_s H. \tag{2.11}$$

(see e.g. [7]). By (ii), for every $\varphi \in W^{1,\infty}_{\partial\Omega_{e_n}\setminus\Gamma_{e_n}}(\Omega_{e_n}; \mathbb{R}^m)$ we have that

$$0 = \int_{\Omega_{e_n}} \mathcal{N}(\nabla\varphi(y))\,\mathrm{d}y,$$

and using (2.11), we infer that

$$0 = \sum_{s=1}^{\min(m,n)} \beta_s \cdot \int_{\Omega_{e_n}} \mathrm{ad}_s \nabla\varphi(y)\,\mathrm{d}y.$$

As this must hold for all admissible mappings φ and ad_s is positively homogeneous of degree s, by rescaling φ it is easy to see that in fact,

$$0 = \beta_s \cdot \int_{\Omega_{e_n}} \mathrm{ad}_s \nabla\varphi(y)\,\mathrm{d}y \quad \text{for each } s = 1, \dots, \min(m, n). \tag{2.12}$$

Let I^r_s denote the set of all $(\alpha) := \{\alpha_1, \dots, \alpha_s\} \subset \mathbb{N}$ with $1 \le \alpha_1 < \alpha_2 < \dots < \alpha_s \le r$. For $(p) \in I^m_s$ and $(q) \in I^n_s$ we write $\nabla_{(q)}\varphi_{(p)}(x) = \left(\frac{\partial}{\partial x_{q_j}}\varphi_{p_k}(x)\right)_{jk} \in \mathbb{R}^{s\times s}$. With this notation and the divergence structure of determinants, the entries of ad_s are defined[1] as

$$\mathrm{ad}_s^{(p)(q)} \nabla\varphi = \det \nabla_{(q)}\varphi_{(p)}.$$

If $s = 1$, $(p) = \{p_1\}$ and $(q) = \{q_1\}$ for some integers p_1, q_1, and integration by parts in (2.12) gives

[1] The standard definition requires an additional factor $(-1)^{p+q}$, where p and q denote positions of (p) and (q) in an appropriate ordering of the elements of I^m_s and I^n_s, respectively. However, as this factor plays no role in the proof (and could be absorbed into the corresponding constant, anyway), we omit it here.

$$0=\sum_{p_1=1}^{m}\sum_{q_1=1}^{n}\beta_1^{(p)(q)}\int\limits_{\Gamma_{en}}\varrho_{q_1}\varphi_{p_1}\,\mathrm{d}S, \tag{2.13}$$

where ϱ is the outer normal to Γ_{e_n} and $\varrho_{q_1}=\varrho\cdot e_{q_1}$. In our case $\varrho\equiv e_n$ on Γ_{e_n}, the flat part of the boundary of Ω_{e_n}, where φ is not subject to a Dirichlet boundary condition. Hence, all terms below the integral in (2.13) vanish unless $q_1=n$, and since φ is arbitrary, we get that

$$\beta_1^{(p)(q)}=0 \qquad \text{for every}(p)\in I_1^m\text{and}(q)=\{n\}. \tag{2.14}$$

In case $s\geq 2$, we can use the divergence structure of determinants as follows:

$$\mathrm{ad}_s^{(p)(q)}\nabla\varphi=\det\nabla_{(q)}\varphi_{(p)}=\sum_{i=1}^{s}\frac{d}{dx_{q_i}}\Big[(-1)^{q_i+p_s}\varphi_{p_s}\det\big(\nabla_{(q)\setminus\{q_i\}}\varphi_{(p)\setminus\{p_s\}}\big)\Big].$$

Integrating by parts in (2.12) yields that

$$\beta_s\cdot\int_{\Omega_{e_n}}\mathrm{ad}_s\nabla\varphi(y)\,\mathrm{d}y=\sum_{(p)\in I_s^m}\sum_{(q)\in I_s^n}\beta_s^{(p)(q)}$$
$$\times\int\limits_{\Gamma_{en}}\sum_{i=1}^{s}(-1)^{q_i+p_s}\varrho_{q_i}\varphi_{p_s}\det\big(\nabla_{(q)\setminus\{q_i\}}\varphi_{(p)\setminus\{p_s\}}\big)\,\mathrm{d}S. \tag{2.15}$$

As $\varrho=e_n$ on Γ_{e_n}, the inner sum in (2.15) only contributes if $i=s$ and $q_s=n$, because otherwise, $q_i<n$ and thus $\varrho_{q_i}=0$. Denoting

$$\bar{I}_s^n:=\{(\alpha)=\{\alpha_1,\ldots,\alpha_s\}\subset\mathbb{N}\mid 1\leq\alpha_1<\ldots<\alpha_s=n\},$$

we can combine (2.12) and (2.15) to get

$$0=\sum_{(p)\in I_s^m}\sum_{(q)\in\bar{I}_s^n}\beta_s^{(p)(q)}(-1)^{n+p_s}\int\limits_{\Gamma_{en}}\varphi_{p_s}\det\big(\nabla_{(q)\setminus\{n\}}\varphi_{(p)\setminus\{p_s\}}\big)\,\mathrm{d}S.$$

As we are free to choose the vector-valued function φ with arbitrary components vanishing, this implies that for every $(p)\in I_s^m$,

$$0=\sum_{(q)\in\bar{I}_s^n}\beta_s^{(p)(q)}\int\limits_{\Gamma_{en}}\psi\det\big(\nabla_{(q)\setminus\{n\}}\eta\big)\,\mathrm{d}S,$$

for every $\psi\in W^{1,\infty}_{\partial\Omega_{en}\setminus\Gamma_{en}}(\Omega_{e_n})$ and every $\eta\in W^{1,\infty}_{\partial\Omega_{en}\setminus\Gamma_{en}}(\Omega_{e_n};\mathbb{R}^{s-1})$, and since ψ is arbitrary on Γ_{e_n}, we get that

$$0=\sum_{(q)\in\bar{I}_s^n}\beta_s^{(p)(q)}\det\nabla_{(q)\setminus\{n\}}\eta(y)\quad\text{for a.e. } y\in\Gamma_{e_n}(\text{with respect to the surface measure}).$$

For any given $(q)\in\bar{I}_s^n$, it is not difficult to find an admissible function $\eta(y)$ which, on some neighborhood of a point in Γ_{e_n}, only depends on $y_{q_1},\ldots,y_{q_{s-1}}$, such that $\det\nabla_{(q)\setminus\{n\}}\eta\not\equiv 0$ on Γ_{e_n}. Together with (2.14), we conclude that for $s=1,\ldots,\min(m,n-1)$,

$$\beta_s^{(p)(q)}=0\qquad\text{for every}(p)\in I_s^m\text{and every}(q)\in\bar{I}_s^n.$$

Plugging this into (2.11), we obtain (2.9) for $\tilde{R}=(e_1|\ldots|e_{n-1})$, with $\tilde{\beta}_s^{(p)(q)}:=\beta_s^{(p)(q)}$ for every $(p)\in I_s^m$ and every $(q)\in I_s^{n-1}=I_s^n\setminus\bar{I}_s^n$.

(iii) implies (iv):

This is a simple consequence of the fact that $(F + a \otimes \varrho)\tilde{R} = F\tilde{R} + a \otimes (\tilde{R}^\top \varrho) = F\tilde{R}$, where we used that $(\tilde{R}|\varrho) \in O(n)$ and thus $\tilde{R}^\top \varrho = 0$.

(iv) implies (i):

For given $F \in \mathbb{R}^{m\times n}$ and $\varphi \in W^{1,\infty}_{\partial\Omega_\varrho \setminus \Gamma_\varrho}(\Omega_\varrho; \mathbb{R}^m)$, we define

$$g(t) := \int_{\Omega_\varrho} \mathcal{N}(F + t\nabla\varphi)\, \mathrm{d}x, \quad t \in \mathbb{R}.$$

Integrating by parts, we obtain that

$$\begin{aligned} g'(t) &= \int_{\Omega_\varrho} \nabla_F \mathcal{N}(F + t\nabla\varphi) \cdot \nabla\varphi\, \mathrm{d}x \\ &= -\int_{\Omega_\varrho} [\operatorname{div} \nabla_F \mathcal{N}(F + t\nabla\varphi)] \cdot \varphi\, \mathrm{d}x + \int_{\partial\Omega_\varrho} \nabla_F \mathcal{N}(F + t\nabla\varphi) \cdot (\varphi \otimes \varrho)\, \mathrm{d}S \end{aligned} \tag{2.16}$$

Since $\mathcal{N}$ is null Lagrange, we know that $\operatorname{div} \nabla_F \mathcal{N}(F + \nabla\psi) = 0$ a.e. in Ω_ϱ for every $\psi \in C^2(\Omega_\varrho)$. Hence, the first term in (2.16) vanishes, and since $\varphi = 0$ on $\partial\Omega_\varrho \setminus \Gamma_\varrho$, we see that

$$g'(t) = \int_{\Gamma_\varrho} \nabla_F \mathcal{N}(F + t\nabla\varphi) \cdot (\varphi \otimes \varrho)\, \mathrm{d}S$$

On the other hand, (iv) implies that $\nabla\mathcal{N}(H) \cdot (a \otimes \varrho) = 0$ for every $a \in \mathbb{R}^N$ and every $H \in \mathbb{R}^{m\times n}$. As a consequence, $g'(t) = 0$ for every $t \in \mathbb{R}$, whence $g(0) = g(1)$. □

2.3 DiPerna-Majda measures

While Young measures [36] successfully capture oscillatory behavior of sequences, and proved to be very useful in the calculus of variations see e.g. [22], they completely miss concentrations. There are several tools how to deal with concentrations. They can be considered as generalization of Young measures, see for example Alibert's and Bouchitté's approach [1], DiPerna's and Majda's treatment of concentrations [8], or Fonseca's method described in [11]. An overview can be found in [31,35]. Moreover, in many cases, we are interested in oscillation/concentration effects generated by sequences of gradients. A characterization of Young measures generated by gradients was completely given by Kinderlehrer and Pedregal [19,18], cf. also [30]. The first attempt to characterize both oscillations and concentrations in sequences of gradients is due to Fonseca, Müller, and Pedregal [13]. They dealt with a special situation of $\{gv(\nabla u_k)\}_{k\in\mathbb{N}}$ where v is essentially positively p-homogeneous, $u_k \in W^{1,p}(\Omega; \mathbb{R}^m)$, $p > 1$, with g continuous and vanishing on $\partial\Omega$. Later on, a characterization of oscillation/concentration effects in terms of DiPerna's and Majda's generalization of Young measures was given in [17] for arbitrary integrands and in [12] for sequences living in the kernel of a first-order differential operator. Recently Kristensen and Rindler [20] characterized oscillation/concentration effects in the case $p = 1$.

Let us take a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable ring $\mathcal{R}$ of continuous bounded functions $\mathbb{R}^{m\times n} \to \mathbb{R}$. It is known [10, Sect. 3.12.21] that there is a one-to-one correspondence $\mathcal{R} \mapsto \beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ between such rings and metrizable compactifications of $\mathbb{R}^{m\times n}$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$, into which $\mathbb{R}^{m\times n}$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^{m\times n}$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$. Similarly, we will not distinguish between elements of $\mathcal{R}$ and their unique continuous extensions on $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$.

Let $\sigma \in \mathrm{rca}(\bar{\Omega})$ be a positive Radon measure on a bounded domain $\Omega \subset \mathbb{R}^n$. A mapping $\hat{\nu} : x \mapsto \hat{\nu}_x$ belongs to the space $L^\infty_{\mathrm{w}}(\bar{\Omega}, \sigma; \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ if it is weakly* σ-measurable (i.e., for any $v_0 \in C_0(\mathbb{R}^{m\times n})$, the mapping $\bar{\Omega} \to \mathbb{R} : x \mapsto \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)$ is σ-measurable in the usual sense). If additionally $\hat{\nu}_x \in \mathrm{rca}^+_1(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ for σ-a.a. $x \in \bar{\Omega}$ the collection $\{\hat{\nu}_x\}_{x\in\bar{\Omega}}$ is the so-called Young measure on $(\bar{\Omega}, \sigma)$ [36], see also [2,31,35].

DiPerna and Majda [8] showed that having a bounded sequence in $L^p(\Omega; \mathbb{R}^{m\times n})$ with $1 \le p < +\infty$ and Ω an open domain in $\mathbb{R}^n$, there exists its subsequence (denoted by the same indices) a positive Radon measure $\sigma \in \mathrm{rca}(\bar{\Omega})$ and a Young measure $\hat{\nu} : x \mapsto \hat{\nu}_x$ on $(\bar{\Omega}, \sigma)$ such that $(\sigma, \hat{\nu})$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{m\times n})$ in the sense that $\forall g \in C(\bar{\Omega})\ \forall v_0 \in \mathcal{R}$:

$$\lim_{k\to\infty} \int_{\Omega} g(x)v(y_k(x))\mathrm{d}x = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} g(x)v_0(s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\,, \tag{2.17}$$

where

$$v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n}) := \{v_0(1 + |\cdot|^p);\ \ v_0 \in \mathcal{R}\}.$$

In particular, putting $v_0 = 1 \in \mathcal{R}$ in (2.17) we can see that

$$\lim_{k\to\infty} (1 + |y_k|^p) \ = \ \sigma \quad \text{ weakly* in } \mathrm{rca}(\bar{\Omega})\,. \tag{2.18}$$

If (2.17) holds, we say that $\{y_k\}_{\in\mathbb{N}}$ generates $(\sigma, \hat{\nu})$. Let us denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$ the set of all pairs $(\sigma, \hat{\nu}) \in \mathrm{rca}(\bar{\Omega}) \times L^\infty_{\mathrm{w}}(\bar{\Omega}, \sigma; \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ attainable by sequences from $L^p(\Omega; \mathbb{R}^{m\times n})$; note that, taking $v_0 = 1$ in (2.17), one can see that these sequences must be inevitably bounded in $L^p(\Omega; \mathbb{R}^{m\times n})$. We also denote by $\mathcal{GDM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$ measures from $\mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$ generated by a sequence of gradients of some bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^m)$. The explicit description of the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$, called DiPerna-Majda measures, for unconstrained sequences was given in [23, Theorem 2]. See also [24] for applications of DiPerna-Majda measures to optimal control problems. In fact, it is easy to see that (2.17) can be also written in the form

$$\lim_{k\to\infty} \int_{\Omega} h(x, y_k(x))\mathrm{d}x = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} h_0(x, s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\,, \tag{2.19}$$

where $h(x, s) := h_0(x, s)(1 + |s|^p)$ and $h_0 \in C(\bar{\Omega} \otimes \beta_{\mathcal{R}}\mathbb{R}^{m\times n})$.

We say that $\{y_k\}$ generates $(\sigma, \hat{\nu})$ if (2.17) holds. Moreover, we denote $d_\sigma \in L^1(\Omega)$ the absolutely continuous (with respect to the Lebesgue measure) part of σ in the Lebesgue decomposition of σ.

We will denote elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$ which are generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$ for some bounded $\{u_k\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ by $\mathcal{GDM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$.

2.3.1 Compactification of $\mathbb{R}^{m\times n}$ by the sphere

In what follows we will work mostly with a particular compactification of $\mathbb{R}^{m\times n}$, namely, with the compactification by the sphere. We will consider the following ring of continuous bounded functions

$$\mathcal{S} := \Bigg\{ v_0 \in C(\mathbb{R}^{m\times n}) \ :\ \text{there exist } v_{0,0} \in C_0(\mathbb{R}^{m\times n}),\ v_{0,1} \in C(S^{(m\times n)-1}), \quad \text{and} \quad c \in \mathbb{R} \text{ s.t.}$$
$$v_0(F) \coloneqq c + v_{0,0}(F) + v_{0,1}\left(\frac{F}{|F|}\right)\frac{|F|^p}{1+|F|^p} \text{ if } F \neq 0 \text{ and } v_0(0) \coloneqq v_{0,0}(0) \Bigg\}, \tag{2.20}$$

where $S^{m\times n-1}$ denotes the $(mn-1)$-dimensional unit sphere in $\mathbb{R}^{m\times n}$. Then $\beta_{\mathcal{S}}\mathbb{R}^{m\times n}$ is homeomorphic to the unit ball $\overline{B(0,1)} \subset \mathbb{R}^{m\times n}$ via the mapping $d : \mathbb{R}^{m\times n} \to B(0,1)$, $d(s) := s/(1+|s|)$ for all $s \in \mathbb{R}^{m\times n}$. Note that $d(\mathbb{R}^{m\times n})$ is dense in $\overline{B(0,1)}$.

For any $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ there exists a continuous and positively p-homogeneous function $v_\infty : \mathbb{R}^{m\times n} \to \mathbb{R}$ (i.e. $v_\infty(\alpha F) = \alpha^p v_\infty(F)$ for all $\alpha \geq 0$ and $F \in \mathbb{R}^{m\times n}$) such that

$$\lim_{|F|\to\infty} \frac{v(F) - v_\infty(F)}{|F|^p} = 0\,. \tag{2.21}$$

Indeed, if v_0 is as in (2.20) and $v = v_0(1+|\cdot|^p)$ then set

$$v_\infty(F) := \left(c + v_{0,1}\left(\frac{F}{|F|}\right)\right)|F|^p \text{ for } F \in \mathbb{R}^{m\times n} \setminus \{0\}.$$

By continuity we define $v_\infty(0) := 0$. It is easy to see that v_∞ satisfies (2.21). Such v_∞ is called the *recession function* of v.

The following lemma and Theorem 2.14 were proven in [21].

Lemma 2.12 *Let $1 \leq p < +\infty$, $0 \leq h_0 \in C(\bar{\Omega} \times \beta_{\mathcal{S}}\mathbb{R}^{m\times n})$, let $h(x,F) := h_0(x,F)(1+|F|^p)$, and let $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be a bounded sequence with $\{\nabla u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{m\times n})$ generating $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Then*

$$\{h(x,\nabla u_k)\}_{k\in\mathbb{N}} \textit{ is weakly relatively compact in } L^1(\Omega)$$

if and only if

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} h_0(x,F)\hat{\nu}_x(\mathrm{d}F)\sigma(\mathrm{d}x) = 0\,. \tag{2.22}$$

Remark 2.13 In Lemma 2.12, we assumed that h_0 and, consequently, h are non-negative, but this assumption can be relaxed. For the asssertion of the lemma to hold true, it actually suffices to have that $h(x,\nabla u_k) \geq 0$ for every k and a.e. $x \in \Omega$. This can easily be seen by applying Lemma 2.12 with h^+ (the positive part of h) instead of h.

Theorem 2.14 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^1, $1 < p < +\infty$, and $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$. Then there is a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat{\nu})$ if and only if the following four conditions hold:*

$$\exists u \in W^{1,p}(\Omega;\mathbb{R}^m) \ \textit{such that for a.a. } x \in \Omega\textit{: } \nabla u(x) = d_\sigma(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{F}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F) \tag{2.23}$$

for almost all $x \in \Omega$ and for all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ the following inequality is fulfilled

$$Qv(\nabla u(x)) \leq d_\sigma(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{v(F)}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F)\,, \tag{2.24}$$

for σ-almost all $x \in \Omega$ and all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Qv_\infty > -\infty$ it holds that

$$0 \leq \int\limits_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(F)}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F)\ , \tag{2.25}$$

and for σ-almost all $x \in \partial\Omega$ with the outer unit normal to the boundary $\varrho(x)$ and all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Q_{b,\varrho(x)}v_\infty(0) > -\infty$ it holds that

$$0 \leq \int\limits_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(F)}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F)\ . \tag{2.26}$$

Remark 2.15 If the traces of $\{u_k\}$ are fixed near some $x \in \partial\Omega$ and coincide with the trace of u, i.e., $u_k = u$ in the sense of trace on $\partial\Omega$, see e.g. [25], then condition (2.26) holds for a bigger class of admissible v, namely, all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Qv > -\infty$. This can be inferred from [17, above Remark 3.9].

The theorem can be extended (with arguments analogous to case of Young measures as presented in [16]) to allow x-dependent test functions (instead of v) in (2.24)–(2.26):

Corollary 2.16 *In the situation of Theorem 2.14, if $(\sigma, \hat{\nu})$ is generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$, with a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, then in addition to (2.23), the following three conditions hold for all functions h of the form $h(x,F) := h_0(x,F)(1+|F|^p)$ with some $h_0 \in C(\overline{\Omega}\times\beta_{\mathcal{S}}\mathbb{R}^{m\times n})$:*

For almost all $x \in \Omega$ and all h, we have that

$$Qh(x,\nabla u(x)) \leq d_\sigma(x) \int\limits_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{h(x,F)}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F)\ , \tag{2.27}$$

for σ-almost all $x \in \Omega$ and all h with $Qh_\infty(x,\cdot) > -\infty$, where h_∞ is the recession function of h with respect to the second variable, it holds that

$$0 \leq \int\limits_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{h(x,F)}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F)\ , \tag{2.28}$$

and for σ-almost all $x \in \partial\Omega$ with the outer unit normal to the boundary $\varrho(x)$ and all h with $[Q_{b,\varrho(x)}h_\infty(x,\cdot)](0) > -\infty$, we have that

$$0 \leq \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{h(x,F)}{1+|F|^p}\hat{\nu}_x(\mathrm{d}F)\ . \tag{2.29}$$

3 Weak continuity up to the boundary

Theorem 3.1 *Let $m, n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary of class C^1, and let $f : \overline{\Omega}\times\mathbb{R}^{m\times n} \to \mathbb{R}$ be a continuous function. In addition, suppose that for every $x \in \Omega$, $f(x,\cdot)$ is a null Lagrangian and for every $x \in \partial\Omega$, $f(x,\cdot)$ is a null Lagrangian at the boundary with respect to $\varrho(x)$, the outer normal to $\partial\Omega$ at x. Hence, by Theorem 2.10, $f(x,\cdot)$ is a polynomial, whose degree we denote by $d_f(x)$. Finally, let $p \in (1,\infty)$ with $p \geq d_f(x)$ for every $x \in \overline{\Omega}$ and let $(u_k) \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence with $u_k \rightharpoonup u$ weakly in $W^{1,p}$. If*

$$f(x,\nabla u_k(x)) \geq 0 \ \text{ for every } \ k \in \mathbb{N} \ \text{ and a.e. } \ x \in \Omega,$$

then $f(\cdot, \nabla u_n) \rightharpoonup f(\cdot, \nabla u)$ *weakly in* $L^1(\Omega)$.

The proof relies on the following auxiliary result, justifying that $h = f$ is admissible as a test function in Corollary 2.16:

Lemma 3.2 *Let* $p \geq 0$ *and suppose that* $f : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *is continuous, and for each* $x \in \overline{\Omega}$, $F \mapsto f(x, F)$ *is a polynomial of degree at most* p. *Then*

$$f_0 : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}, \quad f_0(x, F) := \frac{f(x, F)}{1 + |F|^p}$$

has a continuous extension to $\overline{\Omega} \times \beta_{\mathcal{S}}\mathbb{R}^{m\times n}$.

Proof Let $d := \max_{x\in\overline{\Omega}} d_f(x)$ denote the maximal degree of f. For every x, we split

$$f(x, F) = a_0(x, F) + a_1(x, F) + \ldots + a_d(x, F),$$

where for each i, $a_i(x, \cdot)$ is a positively i-homogeneous polynomial. We first claim that a_i is continuous on $\overline{\Omega} \times \mathbb{R}^{m\times n}$ for each i, which we prove by induction with respect to d. If $d = 0$, the continuity of $a_0 = f$ is trivial. If $d > 1$, since f is continuous, so is

$$g(x, F) := 2^d f(x, F) - f(x, 2F) = \sum_{i=0}^{d-1} (2^d - 2^i) a_i(x, F),$$

whose maximal degree is (at most) $d - 1$. By assumption of the induction, we obtain that $(2^d - 2^i)a_i$ and thus a_i is continuous for each $i = 1, \ldots, d-1$. As a consequence, $a_d = f - \sum_{i=1}^{d-1} a_i$ is continuous as well.

Due to the preceding observation, we may now assume that $f = a_i$ for some i, i.e., f is positively i-homogeneous in its second variable. It is enough to obtain a continuous extension of f_0 for F outside a fixed ball. For any F with $|F| > 0$, we have that

$$f_0(x, F) = \frac{f(x, F)}{1 + |F|^p} = \frac{|F|^i}{1 + |F|^p} f\Big(x, \frac{F}{|F|}\Big).$$

This clearly has a continuous extension to $\overline{\Omega} \times \beta_{\mathcal{S}}\mathbb{R}^{m\times n}$. □

Proof of Theorem 3.1 Let $(\sigma, \hat{\nu})$ be the DiPerna-Majda measure generated by (a subsequence of) u_k. In particular,

$$\int_{\Omega} \varphi(x) f(x, \nabla u_k)\,\mathrm{d}x \underset{k\to\infty}{\longrightarrow} \int_{\bar{\Omega}} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \varphi(x) \frac{f(x, F)}{1 + |F|^p}\, \hat{\nu}_x(\mathrm{d}F)\, \sigma(\mathrm{d}x) \tag{3.1}$$

for every $\varphi \in C(\bar{\Omega})$. By Lemma 3.2, $h := \pm f$ is admissible in the conditions (2.27),(2.28) and (2.29) in Corollary 2.16, which also means that all three inequalities actually are equalities. By (2.28) and (2.29), we obtain that

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{f(x, F)}{1 + |F|^p} \hat{\nu}_x(\mathrm{d}F)\sigma(\mathrm{d}x) = 0. \tag{3.2}$$

Using this together with (2.27), the right hand side in (3.1) can be expressed as

$$\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}} \varphi(x)\frac{f(x,F)}{1+|F|^p}\,\hat\nu_x(\mathrm{d}F)\,\sigma(\mathrm{d}x) = \int\limits_{\bar\Omega}\int\limits_{\mathbb{R}^{m\times n}} \varphi(x)\frac{f(x,F)}{1+|F|^p}\,\hat\nu_x(\mathrm{d}F)\,\sigma(dx)$$
$$= \int\limits_{\bar\Omega}\int\limits_{\mathbb{R}^{m\times n}} \varphi(x)\frac{f(x,F)}{1+|F|^p}\,\hat\nu_x(\mathrm{d}F)\,\mathrm{d}_\sigma(x)\,\mathrm{d}x$$
$$= \int\limits_{\Omega} \varphi(x) f(x,\nabla u)\,\mathrm{d}x\,.$$

Consequently, (3.1) implies that $f(\cdot,\nabla u_k)\to f(\cdot,\nabla u)$ weakly* in $(C(\bar\Omega))'=\mathrm{rca}(\bar\Omega)$. Finally, if $f(x,\nabla u_k(x))\geq 0$ for almost all $x\in\Omega$ and all $k\in\mathbb{N}$, then $f(\cdot,\nabla u_k)\rightharpoonup f(\cdot,\nabla u)$ in $L^1(\Omega)$ by Lemma 2.12 and Remark 2.13, using (3.2). ◻

The following result evokes Müller's generalization [29] of Ball's result [5]. In our setting, however, we can drop nonegativity of the integrand. The condition $f(\cdot,\nabla u)\geq 0$ can be seen as a kind of "orientation-preservation". We refer to [6] for elasticity of shells including a normal-orientation condition.

Theorem 3.3 *Let $h:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ be such that $h(\cdot,s)$ is measurable for all $s\in\mathbb{R}$ and $h(x,\cdot)$ is convex for almost all $x\in\Omega$. Let f and d_f be as in Theorem 3.1. Then $I(u):=\int_\Omega h(x,f(x,\nabla u(x)))\,\mathrm{d}x$ is weakly lower semicontinuous on the set $\{u\in W^{1,p}(\Omega;\mathbb{R}^m);\ f(\cdot,\nabla u)\geq 0$ in $\Omega\}$.*

Proof The proof is standard. ◻

4 Higher integrability

By a result of S. Müller [28,29], for any bounded sequence $(u_k)\subset W^{1,n}(\Omega;\mathbb{R}^N)$ with $\det\nabla u_k\geq 0$ a.e. in Ω, $\det\nabla u_k$ is locally bounded in the class $L\log L$, i.e.,

$$\sup_k\int\limits_K \gamma(\det\nabla u_k)\,\mathrm{d}x<\infty,\quad \text{with}\quad \gamma(s):=s\ln^+ s,$$

for every $K\subset\subset\Omega$, with $\ln^+$ denoting the positive part of the logarithm. It is natural to ask whether an analogous result holds for a null Lagrangian at the boundary in place of the determinant, up to the boundary (i.e., with Ω instead of K). The example closest to Müller's original result for the determinant is the function $\det'$, given by

$$\det{}'\xi:=\det(\xi_{ij})_{i,j=1,\ldots,n-1},\quad\text{for}\quad \xi\in\mathbb{R}^{(n-1)\times n}.$$

This is a null Lagrangian at the boundary, at every boundary point with the normal $\varrho=e_n$. A strict analogue for the estimates in [28,29] in this case would be an inequality as follows:

$$\sup_k\int\limits_K \gamma(\det{}'\nabla u_k)\,\mathrm{d}x\leq C\Big(K,\|u_k\|_{W^{1,n-1}(\Omega;\mathbb{R}^{n-1})}\Big)\quad\text{with}\quad \gamma(s):=s\ln^+ s, \tag{4.1}$$

with a continuous function $C(K,\cdot)$, for every compact $K\subset\bar\Omega$ having a positive distance to the set $\{x\in\partial\Omega;\ \rho(x)\neq e_n\}$ where $\rho(x)$ denotes the outer unit normal to $\partial\Omega$ at x .

However, it seems that it is not possible to extend Müller's proof to this case, at least not in a straightforward way, and the validity of (4.1) remains an open problem.

On the other hand, $\det{}'\nabla u_k$ only depends on the derivatives of u_k with respect to first $n-1$ variables. The anisotropic space $L^{n-1}((0,1); W^{1,n-1}((0,1)^{n-1}; \mathbb{R}^{n-1}))$ suffices to ensure integrability of $\det{}'\nabla u_k$, which makes it a natural alternative to the isotropic space $W^{1,n-1}(\Omega; \mathbb{R}^{n-1})$ used above. It turns out that the analogue of (4.1) with the anisotropic norm on the right hand side fails to hold even in the interior, as illustrated by the example below, an extension of Counterexample 7.2 in [29]. More precisely, we show that one cannot expect an inequality of the following form:

$$\sup_k \int_K \gamma(\det{}'\nabla u_k)\,\mathrm{d}x \le C\Big(K, \|u\|_{L^{n-1}((-1,1); W^{1,n-1}((0,1)^{n-1}; \mathbb{R}^{n-1}))}\Big) \text{ with} \gamma(s) := s\ln^+ s, \tag{4.2}$$

with a continuous function $C(K,\cdot)$.

Example 4.1 For $n \ge 2$ consider

$$\det{}' : \mathbb{R}^{(n-1)\times n} \to \mathbb{R}, \quad \det{}'\xi := \det(\xi_{ij})_{i,j=1,\dots,n-1},$$

which is a boundary Null Lagrangian with respect to the normal $\varrho = \pm e_n$ (the n-th unit vector), together with the sequence (u_k) defined as

$$u_k(x', x_n) := g(x_n) h_k(|x'|) \frac{x'}{|x'|}, \quad (x', x_n) \in Q := (0,1)^{n-1} \times (-1,1),$$

where

$$g(t) := \left[|t| \ln^2(|t|)\right]^{-\frac{1}{n-1}},$$
$$h_k(r) := \begin{cases} k(\ln k)^{-\frac{1}{n-1}} r & \text{if } r < \frac{1}{k}, \\ (\ln k)^{-\frac{1}{n-1}} & \text{else.} \end{cases}$$

In this case, one can check (cf. [29]) that $\det{}'(\nabla u_k) = g(x_n)^{n-1} k^{n-1} (\ln k)^{-1}$ for $|x'| < \frac{1}{k}$ and $\det{}'(\nabla u_k) = 0$ elsewhere. In particular $\det{}'(\nabla u_k) \ge 0$ a.e. in Q, for every k. In addition, $u_k \subset L^{n-1}((0,1); W^{1,n-1}((0,1)^{n-1}; \mathbb{R}^{n-1}))$ is bounded, i.e.,

$$\sup_k \int_Q |\nabla' u_k|^{n-1}\,\mathrm{d}x < \infty,$$

where ∇' denotes the gradient with respect to the first $n-1$ variables. But for every k, the leading term in $\int_{(0,1)^{n-1}} \gamma(\det{}'\nabla u_k(x', x_n))\,\mathrm{d}x'$ for x_n near zero is of the form

$$\frac{-1}{|x_n| \ln(|x_n|) \ln k},$$

which is not integrable near $x_n = 0$, and consequently,

$$\int_{K_\varepsilon} \gamma(\det{}'\nabla u_k)\,\mathrm{d}x = +\infty \quad \text{with } \gamma(s) := s\ln^+ s \text{ and } K_\varepsilon := [\varepsilon, 1-\varepsilon]^{n-1} \times [0, \varepsilon].$$

Hence inequality (4.2) cannot hold.

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