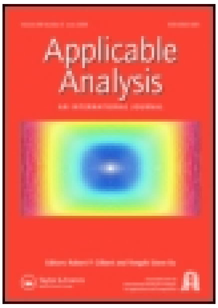


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# Young measures supported on invertible matrices

Barbora Benešová [a b e], Martin Kružík [c d] & Gabriel Pathó [b d]

[a] Department of Ultrasound Methods, Institute of Thermomechanics of the ASCR, Dolejškova 5 CZ-182 08, Praha 8, Czech Republic;

[b] Charles University, Faculty of Mathematics and Physics, Sokolovská 83 CZ-186 75, Praha 8, Czech Republic;

[c] Department of Decision Making, Institute of Information Theory and Automation of the ASCR, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic;

[d] Czech Technical University, Faculty of Civil Engineering, Thákurova 7, CZ-166 Praha 6, Czech Republic

[e] Department of Mathematics I, D-52056 RWTH Aachen, Aachen, Germany

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# Young measures supported on invertible matrices

Barbora Benešová[abe], Martin Kružík[cd]* and Gabriel Pathó[bd]

[a] *Department of Ultrasound Methods, Institute of Thermomechanics of the ASCR, Dolejškova 5, CZ-182 08 Praha 8, Czech Republic;* [b] *Faculty of Mathematics and Physics, Charles University, Sokolovská 83, CZ-186 75 Praha 8, Czech Republic;* [c] *Department of Decision Making, Institute of Information Theory and Automation of the ASCR, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic;* [d] *Faculty of Civil Engineering, Czech Technical University, Thákurova 7, CZ-166 Praha 6, Czech Republic;* [e] *Department of Mathematics I, RWTH Aachen, D-52056 Aachen, Germany*

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Motivated by variational problems in non-linear elasticity, we explicitly characterize the set of Young measures generated by gradients of a uniformly bounded sequence in $W^{1,\infty}(\Omega;\mathbb{R}^n)$ where the inverted gradients are also bounded in $L^\infty(\Omega;\mathbb{R}^{n\times n})$. This extends the original results due to the studies of Kinderlehrer and Pedregal. Besides, we completely describe Young measures generated by a sequence of matrix-valued mappings $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$, such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is bounded, too, and the generating sequence satisfies the constraint $\det Y_k > 0$.

**Keywords:** orientation-preserving mappings; relaxation; Young measures

**AMS Subject Classifications:** 49J45; 35B05

## 1. Introduction

In this paper, we investigate a new tool to study minimization problems for integral functionals defined over matrix-valued mappings that take values only in the set of invertible matrices. Typical examples are found, e.g. in non-linear elasticity where static equilibria are minimizers of the elastic energy

$$J(y) := \int_\Omega W(\nabla y(x))\,\mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ denotes the reference configuration of the material, $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ is the deformation, $1 < p \leq +\infty$, $y = y_0$ on $\partial\Omega$ and $W\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is the stored energy density, i.e. the potential of the first Piola–Kirchhoff stress tensor. Further, usually in elasticity, one demands either $\det \nabla y \neq 0$ to assure local invertibility of $\nabla y$ or even $\det \nabla y > 0$ in order to preserve orientation of $y$.

If $W$ is polyconvex, i.e. $A \mapsto W(A)$ can be written as a convex function of all minors of $A$, then the existence of minimizers to (1.1) was proved by J.M. Ball in his pioneering

*Corresponding author. Email: kruzik@utia.cas.cz

paper [1]. We refer, for example, to [2,3] for various further results in this direction. Namely, the existence theory for polyconvex materials can even cope with the important physical assumption

$$W(A) \to +\infty \text{whenever } \det A \to 0_+. \tag{1.2}$$

On the other hand, there are many materials that cannot be modelled by polyconvex stored energies, prominent examples are materials with microstructure, like shape-memory materials [4,5]. If we give up (1.2) and suppose that $W$ has polynomial growth at infinity, i.e. there exist $c, \tilde{c} > 0$ such that

$$c(-1 + |A|^p) \le W(A) \le \tilde{c}(1 + |A|^p), \tag{1.3}$$

the existence of a solution to (1.1) is guaranteed if $W$ is quasiconvex [6], which means that for all $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ and all $A \in \mathbb{R}^{n\times n}$ it holds that

$$|\Omega| W(A) \le \int_\Omega W(A + \nabla\varphi(x))\, \mathrm{d}x. \tag{1.4}$$

However, stored energy densities of materials with microstructure do not possess this property either. As a result, solutions to (1.1) might not exist. Various relaxation techniques were developed [3,5,7] and used in numerical approximations [8] to overcome this drawback for integrands satisfying (1.3). Some relaxation results for the case $W(A) \to +\infty$ if $\det A \to 0$ but $W(A) < +\infty$ even if $\det A < 0$ were recently stated in [9]. In both situations, one replaces the integrand by its quasiconvex envelope (the pointwise supremum of all quasiconvex functions not greater than $W$).

Another approach is to extend the notion of solutions from Sobolev mappings to parameterized measures called Young measures [7,10–14,30]. The idea is to describe the limit behaviour of $\{J(y_k)\}_{k\in\mathbb{N}}$ along a minimizing sequence $\{y_k\}_{k\in\mathbb{N}}$. Nevertheless, the growth condition (1.3) is still a key ingredient in these considerations.

Our goal is to tailor the Young-measure relaxation to functions satisfying (1.2). In order to reach this, we allow $W$ to depend on the inverse of its argument, more precisely, we suppose that $W$ is continuous on invertible matrices and that there exist positive constants $c, \tilde{c} > 0$ such that

$$c(-1 + |A|^p + |A^{-1}|^p) \le W(A) \le \tilde{c}(1 + |A|^p + |A^{-1}|^p). \tag{1.5}$$

Notice that (1.5) implies (1.2) and that $W$ has polynomial growth in $|A|$ and $|A^{-1}|$ at infinity. In the context of non-linear elasticity, $A$ plays the role of a deformation gradient measuring deformation strain and $A^{-1}$ is just another strain measure. We refer, for example e.g. to [15,16] for the so-called Seth–Hill family of strain measures or to [17] where the physical meaning of the Piola tensor and of the Finger tensor depending on $A^{-1}A^{-\top}$ and on $A^{-\top}A^{-1}$, respectively, is discussed in great detail.

To justify (1.5), we notice that if $y : \Omega \to \mathbb{R}^n$ is a deformation of the reference domain $\Omega \subset \mathbb{R}^n$ and $y^{-1} : y(\Omega) \to \Omega$ is its differentiable inverse then, for $x \in \Omega$ $(\nabla y(x))^{-1} = \nabla y^{-1}(z)$, $z := y(x)$. Hence, if we exchange the role of the reference and deformed configurations, our model requires the same integrability for the original deformation gradient as well as for the deformation gradient of the inverse deformation. Also, if we consider $n = 3$, $p \ge 2$, $q \ge 1$ satisfying $r := pq/(p + 2q) \ge 1$, and a polyconvex stored energy density of the form

$$W(F) := \begin{cases} |F|^p + 1/(\det F)^q & \text{if } \det F > 0 \\ +\infty \text{ otherwise} \end{cases}$$

then we see that every minimizing sequence $\{y_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^n)$ of $J$ from (1.1) is such that $\{\|(\nabla y_k)^{-1}\|_{L^r(\Omega;\mathbb{R}^{n\times n})}\}_{k\in\mathbb{N}}$ is bounded by a constant independent of $k \in \mathbb{N}$. Hence, boundedness of the sequences of "inverted gradients" $\{(\nabla y_k)^{-1}\}_{k\in\mathbb{N}}$ in some Lebesgue space may appear as a necessary condition on minimizing sequences in non-linear elasticity. On the other hand, if $W$ satisfies (1.5) then $\{1/\det \nabla y_k\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ for any minimizing sequence $\{y_k\}_{k\in\mathbb{N}}$ of $J$.

This all motivates the idea to perform relaxation in terms of gradient Young measures generated by gradients of functions $\{y_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^n)$ such that $\{(\nabla y_k)^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is bounded, too. In order to do so, an explicit characterization of this specific set of measures is essential. In this work, however, we concentrate merely on parameterized measures generated by gradients of Lipschitz maps. Therefore, these results should be understood as a first step in the analysis of more realistic hyperelastic models where $p$ is finite.

In particular, we completely and explicitly describe Young measures generated by gradients of functions $\{y_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ with $\{(\nabla y_k)^{-1}\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ also bounded. The main characterization is exposed in Theorem 2.5 following the work [18], only additional constraints on the support of the measure and a restricted set of test functions for the Jensen inequality needs to be introduced for which the envelope from (1.10) does not need to be quasiconvex anymore.

Moreover, we also characterize Young measures generated by matrix-valued mappings $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ with $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ bounded. Namely, we show that, in this case, the Young measures are necessarily supported on invertible matrices and satisfy a certain integral condition, cf. (2.1), Theorem 2.2 and Proposition 2.4. Contrary to the general theory of Young measures generated by $L^p$-maps [7,19], where only the behaviour of test functions at infinity is important, Young measures supported on invertible matrices are also sensitive to the asymptotics of test functions as the argument approaches a singular matrix. The constraint $\det Y_k > 0$ almost everywhere in $\Omega$ can be incorporated, too.

Let us mention that while the matrix case is an extension of the results due to Freddi and Paroni [20] who considered a related case for vector-valued maps, the results concerning the curl constraint are the main novelty of the paper. We refer also to [21] for another refinement of Young measures involving discontinuous integrands.

The plan of the paper is as follows. After introducing Young measures we state our main results – Theorems 2.1, 2.2 and 2.5 in Section 2. The proofs of our statements are left, however, to Section 3 for the $L^p$-case and Section 4 for the $W^{1,\infty}$-case. In particular, Propositions 3.1 and 3.2 are of special interest as they form an $L^\infty$-version of our main Theorems 2.1 and 2.2.

### 1.1. *Notation*

Throughout the paper, we use standard notation for Lebesgue $L^p$, Sobolev $W^{1,p}$ spaces and the space $C(S)$ of continuous functions on $S \subset \mathbb{R}^n$. If not said otherwise, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. For $p \geq 0$, we define the following subspace of $C(\mathbb{R}^{n\times n})$:

$$C_p(\mathbb{R}^{n\times n}) := \left\{ v \in C(\mathbb{R}^{n\times n});\ \lim_{|s|\to\infty} \frac{v(s)}{|s|^p} = 0 \right\}.$$

$\mathbb{R}^{n\times n}_{\rm inv}$ shall denote the set of invertible matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times n}_{\rm inv+}$ denotes the set of matrices in $\mathbb{R}^{n\times n}$ with positive determinant. Further, we define the following subsets of the set of invertible matrices:

$$R^{n\times n}_\varrho := \{A \in \mathbb{R}^{n\times n}_{\rm inv};\ \max(|A|, |A^{-1}|) \leq \varrho\}, \tag{1.6}$$

$$R^{n\times n}_{\varrho+} := \{A \in R^{n\times n}_\varrho;\ \det A > 0\} \tag{1.7}$$

for $0 < \varrho < \infty$, while $R^{n\times n}_{+\infty} := \mathbb{R}^{n\times n}_{\rm inv}$. Note that both $R^{n\times n}_\varrho$ and $R^{n\times n}_{\varrho+}$ are compact for every $1 \leq \varrho < \infty$ and empty for $\varrho < 1$.

When analysing the $W^{1,\infty}$-case, we shall need, for $\varrho \in [1; +\infty]$, the following set

$$\mathcal{O}(\varrho) := \{v : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\};\ v \in C(R^{n\times n}_\varrho),\ v(s) = +\infty \text{ if } s \in \mathbb{R}^{n\times n} \setminus R^{n\times n}_\varrho\}. \tag{1.8}$$

In the $L^p$-case, we will work with the following subspace of $C(\mathbb{R}^{n\times n}_{\rm inv})$

$$C_{p,-p}(\mathbb{R}^{n\times n}_{\rm inv}) := \left\{ v \in C(\mathbb{R}^{n\times n}_{\rm inv});\ \lim_{|s|+|s^{-1}|\to\infty} \frac{v(s)}{|s|^p + |s^{-1}|^p} = 0 \right\} \tag{1.9}$$

and $C^{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$ is defined as

$$C^{p,-p}(\mathbb{R}^{n\times n}_{\rm inv}) := \{f \in C(\mathbb{R}^{n\times n}_{\rm inv});\ |f(s)| \leq C(1 + |s|^p + |s^{-1}|^p)\ \forall s \in \mathbb{R}^{n\times n}_{\rm inv}\};$$

here and in the sequel $|A|$ is the spectral norm of the matrix $A$, i.e. the largest singular value of $A$ (the largest eigenvalue of $\sqrt{AA^{\rm T}}$).

If $v : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$ is bounded from below and Borel measurable we define

$$Z^\infty v(A) := \inf_{\varphi \in W^{1,\infty}_A(\Omega;\mathbb{R}^n)} |\Omega|^{-1} \int_\Omega v(\nabla\varphi(x))\,{\rm d}x, \tag{1.10}$$

where $W^{1,\infty}_A(\Omega;\mathbb{R}^n) := \{\psi \in W^{1,\infty}(\Omega;\mathbb{R}^n);\ \psi(x) = Ax \text{ for } x \in \partial\Omega\}$.

It is well known that the right-hand side of (1.10) is the same if we replace $\Omega$ by any other bounded Lipschitz domain in $\mathbb{R}^n$.

Note that $v$ is quasiconvex if $v = Z^\infty v$. The quasiconvex envelope of $v$, $Qv$ is defined as:

$$Qv(A) := \sup\{g(A);\ g : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\};\ g \leq v,\ g \text{ is quasiconvex}\}.$$

We say that $\{u_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ is equi-integrable if we can extract a subsequence weakly converging in $L^1(\Omega)$. We refer, e.g. to [11,22] for details about equi-integrability and relative weak compactness in $L^1(\Omega)$. Finally, $C$ denotes a generic positive constant which may change from place to place.

### 1.2. *Young measures*

Young measures on a bounded domain $\Omega \subset \mathbb{R}^n$ are weakly* measurable mappings $x \mapsto \nu_x : \Omega \to \text{rca}(\mathbb{R}^{n\times n})$ with values in probability measures; and the adjective "weakly* measurable" means that, for any $v \in C_0(\mathbb{R}^{n\times n})$, the mapping $\Omega \to \mathbb{R} : x \mapsto \langle \nu_x, v\rangle = \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $\mathrm{rca}(\mathbb{R}^{n\times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{n\times n})^*$, where $C_0(\mathbb{R}^{n\times n})$ stands for the space of all continuous functions $\mathbb{R}^{n\times n} \to \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega; \mathbb{R}^{n\times n})$. It is known that $\mathcal{Y}(\Omega; \mathbb{R}^{n\times n})$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega; \mathrm{rca}(\mathbb{R}^{n\times n})) \cong L^1(\Omega; C_0(\mathbb{R}^{n\times n}))^*$, where the subscript "w" indicates the aforementioned property of weak* measurability. Let $S \subset \mathbb{R}^{n\times n}$ be a compact set. A classical result [12,23] is that for every sequence $\{Y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega; \mathbb{R}^{n\times n})$ such that $Y_k(x) \in S$, there exists a subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega; \mathbb{R}^{n\times n})$ satisfying

$$\forall v \in C(S) : \lim_{k\to\infty} v(Y_k) = \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s) \text{ weakly* in } L^\infty(\Omega). \tag{1.11}$$

Moreover, $\nu_x$ is supported on $S$ for almost all $x \in \Omega$. On the other hand, if $\mu = \{\mu_x\}_{x\in\Omega}$, $\mu_x$ is supported on $S$ for almost all $x \in \Omega$ and $x \mapsto \mu_x$ is weakly* measurable, then there exists a sequence $\{Z_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega; \mathbb{R}^{n\times n})$, $Z_k(x) \in S$ and (1.11) holds with $\mu$ and $Z_k$ instead of $\nu$ and $Y_k$, respectively.

Let us denote by $\mathcal{Y}^\infty(\Omega; \mathbb{R}^{n\times n})$ the set of all Young measures that are created in this way, i.e. by taking all bounded sequences in $L^\infty(\Omega; \mathbb{R}^{n\times n})$. Moreover, we denote by $\mathcal{GY}^\infty(\Omega; \mathbb{R}^{n\times n})$ the subset of $\mathcal{Y}^\infty(\Omega; \mathbb{R}^{n\times n})$ consisting of measures generated by gradients of $\{y_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega; \mathbb{R}^n)$, i.e. $Y_k := \nabla y_k$ in (1.11). It is due to Kinderlehrer and Pedregal [18] that $\nu \in \mathcal{Y}^\infty(\Omega; \mathbb{R}^{n\times n})$ is in $\mathcal{GY}^\infty(\Omega; \mathbb{R}^{n\times n})$ if and only if

(1) there exists $z \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ such that $\nabla z(x) = \int_{\mathbb{R}^{n\times n}} A\nu_x(\mathrm{d}A)$ for a.e. $x \in \Omega$,
(2) $\psi(\nabla z(x)) \le \int_{\mathbb{R}^{n\times n}} \psi(A)\nu_x(\mathrm{d}A)$ for a.e. $x \in \Omega$ and for all $\psi$ quasiconvex, continuous and bounded from below,
(3) $\mathrm{supp}\,\nu_x \subset K$ for some compact set $K \subset \mathbb{R}^{n\times n}$ for a.e. $x \in \Omega$.

A generalization of the $L^\infty$-result (1.11) was formulated by Schonbek [19] (cf. also [10]): if $1 \le p < +\infty$ then for every sequence $\{Y_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega; \mathbb{R}^{n\times n})$ there exists a subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega; \mathbb{R}^{n\times n})$ such that

$$\forall v \in C_p(\mathbb{R}^{n\times n}) : \lim_{k\to\infty} v(Y_k) = \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s) \text{ weakly in } L^1(\Omega)\,. \tag{1.12}$$

We say that $\{Y_k\}_{k\in\mathbb{N}}$ generates $\nu$ if (1.12) holds. Let us denote by $\mathcal{Y}^p(\Omega; \mathbb{R}^{n\times n})$ the set of all Young measures that are obtained through the latter procedure, i.e. by taking all bounded sequences in $L^p(\Omega; \mathbb{R}^{n\times n})$. It was shown in [24] that if $\nu \in \mathcal{Y}(\Omega; \mathbb{R}^{n\times n})$ satisfies the bound

$$\int_\Omega \int_{\mathbb{R}^{n\times n}} |s|^p \nu_x(\mathrm{d}s)\,\mathrm{d}x < +\infty \tag{1.13}$$

then $\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^{n\times n})$.

## 2. Main results

Let us, at this point, summarize the main results of the paper.

### 2.1. $L^p$-case

We define the following subsets of $\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$:

$$\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n}) := \Big\{\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n});\ \int_\Omega\int_{\mathbb{R}^{n\times n}_{\rm inv}}(|s|^p+|s^{-1}|^p)\nu_x(\mathrm{d}s)\,\mathrm{d}x<+\infty,\ \nu_x(\mathbb{R}^{n\times n}_{\rm inv})=1 \text{ for a.a. } x\in\Omega\Big\}, \tag{2.1}$$

$$\mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n}) := \Big\{\nu\in\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n});\ \nu_x(\mathbb{R}^{n\times n}_{\rm inv+})=1 \text{ for a.a. } x\in\Omega\Big\}. \tag{2.2}$$

Our results concerning the $L^p$-case are then summarized in the following theorems.

Theorem 2.1 *Let $+\infty>p\geq 1$, let $\Omega\subset\mathbb{R}^n$ be open and bounded, and let $\{Y_k\}_{k\in\mathbb{N}}$, $\{Y_k^{-1}\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{n\times n})$ be bounded. Then there is a subsequence of $\{Y_k\}_{k\in\mathbb{N}}$ (not relabelled) and $\nu\in\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ such that for every $g\in L^\infty(\Omega)$ and every $v\in C_{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$ it holds that*

$$\lim_{k\to\infty}\int_\Omega v(Y_k(x))g(x)\,\mathrm{d}x=\int_\Omega\int_{\mathbb{R}^{n\times n}_{\rm inv}}v(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x, \tag{2.3}$$

*Conversely, if $\nu\in\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ then there is a bounded sequence $\{Y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{n\times n})$ is also bounded and* (2.3) *holds for all $g$ and $v$ defined above.*

Theorem 2.2 *Let $+\infty>p\geq 1$, let $\Omega\subset\mathbb{R}^n$ be open and bounded, and let $\{Y_k\}_{k\in\mathbb{N}}$, $\{Y_k^{-1}\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{n\times n})$ be bounded and for every $k\in\mathbb{N}$ $\det Y_k>0$ almost everywhere in $\Omega$. Then there is a subsequence of $\{Y_k\}_{k\in\mathbb{N}}$ (not relabelled) and $\nu\in\mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n})$ such that for every $g\in L^\infty(\Omega)$ and every $v\in C_{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$* (2.3) *holds.*

*Conversely, if $\nu\in\mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n})$ then there is a bounded sequence $\{Y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{n\times n})$ is also bounded, for every $k\in\mathbb{N}$ $\det Y_k>0$ almost everywhere in $\Omega$, and* (2.3) *holds for all $g$ and $v$ defined above.*

*Remark 2.3* We could also define the sets

$$\mathcal{Y}^{p,f(\cdot)}(\Omega;\mathbb{R}^{n\times n}) := \Big\{\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n});\ \int_\Omega\int_{\mathbb{R}^{n\times n}_{\rm inv}}(|s|^p+f(s^{-1}))\nu_x(\mathrm{d}s)\,\mathrm{d}x<+\infty\ ,\ \nu_x(\mathbb{R}^{n\times n}_{\rm inv})=1 \text{ for a.a. } x\in\Omega\Big\}, \tag{2.4}$$

$$\mathcal{Y}^{p,f(\cdot)}_+(\Omega;\mathbb{R}^{n\times n}) := \Big\{\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n});\ \int_\Omega\int_{\mathbb{R}^{n\times n}_{\rm inv+}}(|s|^p+f(s^{-1}))\nu_x(\mathrm{d}s)\,\mathrm{d}x<+\infty\ ,\ \nu_x(\mathbb{R}^{n\times n}_{\rm inv+})=1 \text{ for a.a. } x\in\Omega\Big\}, \tag{2.5}$$

with $f(\cdot)\geq|\det(\cdot)|^q$ for some $q>0$. Obvious modifications of the proofs below give that $\nu$ is in $\mathcal{Y}^{p,f(\cdot)}(\Omega;\mathbb{R}^{n\times n})(\mathcal{Y}^{p,f(\cdot)}_+(\Omega;\mathbb{R}^{n\times n}))$ if and only if it can be generated by a sequence of invertible matrices with inverses $\left\{Y_k^{-1}\right\}_{k\in\mathbb{N}}$ bounded in $L^q(\Omega;\mathbb{R}^{n\times n})$ (and $\det Y_k(x)>0$ for all $k\in\mathbb{N}$ and a.a. $x\in\Omega$). Defining these sets allows us to relax even a larger class of functions than $C_{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$.

The next result shows that the weak limit of a sequence of gradients with positive determinant inherits this property if we control the behaviour of the inverse.

PROPOSITION 2.4 *Let $p > n$. If $y_k \rightharpoonup y$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ is such that* $\det \nabla y_k > 0$ *a.e. in $\Omega$ for all $k \in \mathbb{N}$ and $\{(\nabla y_k)^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is bounded then* $\det \nabla y > 0$ *a.e. in $\Omega$. Moreover, every Young measure generated by a subsequence of $\{\nabla y_k\}_{k\in\mathbb{N}}$ is supported on $\mathbb{R}^{n\times n}_{\mathrm{inv}+}$.*

## 2.2. $W^{1,\infty}$-case

We now turn to a characterization of gradient Young measures supported on invertible matrices. We shall see that the characterization is similar to the one obtained by Kinderlehrer and Pedregal for gradient Young measures [18,25], however, the set of test functions for the Jensen inequality is restricted to $\mathcal{O}(\varrho)$ from (1.8), it is not known if $Z^\infty v$ is still quasiconvex in this case.

Let us define the following sets of Young measures generated by bounded and invertible gradients of $W^{1,\infty}(\Omega;\mathbb{R}^n)$ maps:

$$\mathcal{GY}^{+\infty,-\infty}_{\varrho}(\Omega;\mathbb{R}^{n\times n}) := \Big\{ \nu \in \mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n});\ \exists \{y_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)\ , \\ \text{for a.a. } x \in \Omega\ \{\nabla y_k(x)\}_{k\in\mathbb{N}} \subset R^{n\times n}_{\varrho} \text{ and } \{\nabla y_k\}_{k\in\mathbb{N}} \text{ generates } \nu \Big\} \tag{2.6}$$

and $\mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n}) := \bigcup_{\varrho>0} \mathcal{GY}^{+\infty,-\infty}_{\varrho}(\Omega;\mathbb{R}^{n\times n})$.

THEOREM 2.5 *Let $\nu \in \mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n})$. Then $\nu \in \mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$ if and only if the following three conditions hold:*

$$\operatorname{supp} \nu_x \subset R^{n\times n}_{\varrho} \textit{ for a.a. } x \in \Omega \textit{ and some } \varrho \ge 1, \tag{2.7}$$

$$\exists\, u \in W^{1,\infty}(\Omega;\mathbb{R}^n)\ :\ \nabla u(x) = \int_{\mathbb{R}^{n\times n}_{\mathrm{inv}}} s\nu_x(\mathrm{d}s), \tag{2.8}$$

*for a.a. $x \in \Omega$, all $\tilde\varrho \in (\varrho; +\infty]$, and all $v \in \mathcal{O}(\tilde\varrho)$ the following inequality is valid*

$$Z^\infty v(\nabla u(x)) \le \int_{\mathbb{R}^{n\times n}_{\mathrm{inv}}} v(s)\nu_x(\mathrm{d}s). \tag{2.9}$$

*Remark 2.6* It will follow from the proof of Theorem 2.5 that if $|\nabla u(x)| \le \varrho - \epsilon$ for some $\epsilon < \varrho$ and almost all $x \in \Omega$ then we can take $\tilde\varrho \ge \varrho$ in (2.9). Otherwise $\tilde\varrho > \varrho$ seems to be necessary, similarly as in [18].

If there is a convex compact $K \subset R^{n\times n}_{\varrho}$ such that $\operatorname{supp} \nu_x \subset K$ for almost all $x \in \Omega$ in Theorem 2.5 then it is sufficient to consider (2.9) only for all $v : \mathbb{R}^{n\times n} \to \mathbb{R}$ quasiconvex and bounded from below; cf. [26, Cor. 3]. In particular, either $K \subset R^{n\times n}_{\varrho+}$ or $\det A < 0$ for all $A \in K$. Assume that $K \subset R^{n\times n}_{\varrho+}$, i.e. $\det A > 0$ for all $A \in K$. Following [26, Cor. 3] we find a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $\nu$ and $\|\mathrm{dist}(\nabla u_k, K)\|_{L^\infty} \to 0$ as $k \to \infty$. Hence, $\det \nabla u_k > 0$ for almost all $x \in \Omega$ if $k \ge k_0 \in \mathbb{N}$ is large enough. This observation can be used in approximating minimizers of $\nu \mapsto \bar J(\nu) := \int_\Omega \int_{\mathbb{R}^{n\times n}_{\mathrm{inv}+}} W(A)\nu_x(\mathrm{d}A)\,\mathrm{d}x$.

## 3. Proofs in the $L^p$-case

This section is devoted to prove Theorems 2.1 and 2.2 when proving the necessity part of Theorem 2.1 by a combination of Propositions 3.3 and 3.5; the sufficiency part heavily relies on Proposition 3.6.

PROPOSITION 3.1 *Let $\nu \in \mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$ and suppose that there is $\varrho > 0$ such that for almost all $x \in \Omega$ supp $\nu_x \subset R_\varrho^{n\times n}$. Then there exists $\{Y_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k(x)\}_{k\in\mathbb{N}} \subset R_\varrho^{n\times n}$ for almost all $x \in \Omega$ and $\{Y_k\}_{k\in\mathbb{N}}$ generates $\nu$. The same result holds if we replace $R_\varrho^{n\times n}$ with $R_{\varrho+}^{n\times n}$.*

*Proof* This is a classical result mentioned in (1.11). See e.g. [12, Th. 1] for details. □

PROPOSITION 3.2 *Let $\varrho > 0$ and let $\{Y_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$, $\{Y_k\}_{k\in\mathbb{N}} \subset R_\varrho^{n\times n}$ for almost all $x \in \Omega$ and all $k \in \mathbb{N}$. If $\{Y_k\}_{k\in\mathbb{N}}$ generates $\nu \in \mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n})$ and if $\{Y_k^{-1}\}_{k\in\mathbb{N}}$ generates $\mu \in \mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n})$ then for almost all $x \in \Omega$ and every continuous $f : R_\varrho^{n\times n} \to \mathbb{R}$ it holds*

$$\int_{R_\varrho^{n\times n}} f(s)\mu_x(\mathrm{d}s) = \int_{R_\varrho^{n\times n}} f(s^{-1})\nu_x(\mathrm{d}s). \tag{3.1}$$

*Moreover,* supp $\nu_x \subset R_\varrho^{n\times n}$ *for almost all $x \in \Omega$. The same result holds for $R_{\varrho+}^{n\times n}$ instead of $R_\varrho^{n\times n}$.*

*Proof* First of all, recall that [27,28] for almost all $x \in \Omega$ $\nu_x$ is supported on the set $\cap_{l=1}^\infty \overline{\{Y_k(x);\ k \geq l\}}$, i.e. $\nu_x$ is supported on $R_\varrho^{n\times n}$. Further, notice that $\{Y_k^{-1}(x)\}_{k\in\mathbb{N}} \subset R_\varrho^{n\times n}$ for a.a. $x \in \Omega$. If $f : R_\varrho^{n\times n} \to \mathbb{R}$ is continuous, so is $F : R_\varrho^{n\times n} \to \mathbb{R}$, $F(s) := f(s^{-1})$. Then we have for any $g \in L^1(\Omega)$

$$\lim_{k\to\infty} \int_\Omega f(Y_k^{-1}(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{R_\varrho^{n\times n}} f(s)\mu_x(\mathrm{d}s)g(x)\,\mathrm{d}x.$$

At the same time,

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega F(Y_k(x))g(x)\,\mathrm{d}x &= \int_\Omega \int_{R_\varrho^{n\times n}} F(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x \\ &= \int_\Omega \int_{R_\varrho^{n\times n}} f(s^{-1})\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x.\end{aligned}$$

Note that the above procedure stays valid for $R_{\varrho+}^{n\times n}$ instead of $R_\varrho^{n\times n}$. □

PROPOSITION 3.3 *Let $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ generate $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$ and let $\int_\Omega |\det\ Y_k^{-1}|^q \mathrm{d}x \leq C$ for some $C > 0$ and some $q > 0$. Then*

$$\nu_x(\mathbb{R}^{n\times n} \setminus \mathbb{R}_{\mathrm{inv}}^{n\times n}) = 0 \quad \textit{for almost all} \quad x \in \Omega. \tag{3.2}$$

*Moreover, if* $\det Y_k > 0$ *a.e. in $\Omega$ then $\nu_x(\mathbb{R}^{n\times n} \setminus \mathbb{R}_{\mathrm{inv}+}^{n\times n}) = 0$.*

*Proof* Define $v : \mathbb{R}^{n\times n} \to [0;+\infty]$

$$v(Y) := \begin{cases} |\det\ Y^{-1}|^q & \text{if } Y \in \mathbb{R}_{\mathrm{inv}}^{n\times n}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $v$ is lower semicontinuous and by a fundamental result on Young measures (see e.g. [11, Th. 8.61]) we have that

$$\int_\Omega \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega v(Y_k(x))\,\mathrm{d}x = \liminf_{k\to\infty} \int_\Omega |\det\, Y_k^{-1}|^q\,\mathrm{d}x \le C,$$

which means that $\nu_x(\mathbb{R}^{n\times n}\setminus \mathbb{R}^{n\times n}_{\rm inv}) = 0$ for almost all $x \in \Omega$. To prove the second claim, we argue the same way with a re-defined function $v : \mathbb{R}^{n\times n} \to [0; +\infty]$

$$v(Y) := \begin{cases} |\det\, Y^{-1}|^q & \text{if } \det Y > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

□

Lemma 3.4 *Let* $\nu \in \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n})$, $\mu \in \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n})$. *Let* $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$. *Let also,*

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} f^\varrho(s)\mu_x(\mathrm{d}s)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} f^\varrho(s^{-1})\nu_x(\mathrm{d}s)\mathrm{d}x \tag{3.3}$$

*for all* $f^\varrho \in \{h \in C_0(\mathbb{R}^{n\times n});\ \mathrm{supp}\, h\ \subset R^{n\times n}_\varrho\}$, *for any* $\varrho > 0$. *Then*

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} f(s)\mu_x(\mathrm{d}s)\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} f(s^{-1})\nu_x(\mathrm{d}s)\mathrm{d}x\ . \tag{3.4}$$

*Proof* The proof is a simple application of suitable cut-off functions and of Lebesgue's dominated convergence theorem. □

Proposition 3.5 *Let* $p \in [1,\infty)$ *and* $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{n\times n})$, $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega; R^{n\times n})$ *be bounded. Then there is a (not relabelled) subsequence of* $\{Y_k\}_{k\in\mathbb{N}}$ *generating a Young measure* $\nu \in \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n})$.

*Moreover, if we denoted* $\mu$ *the Young measure generated by (a further subsequence of)* $\{Y_k^{-1}\}_{k\in\mathbb{N}}$ *then* (3.4) *holds for all* $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$.

*Proof* It follows from (1.12) that a (not relabelled) subsequence of $\{Y_k\}_{k\in\mathbb{N}}$ generates a Young measure $\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^{n\times n})$ and $\{Y_k^{-1}\}_{k\in\mathbb{N}}$ generates a Young measure $\mu \in \mathcal{Y}^p(\Omega; \mathbb{R}^{n\times n})$. As $\int_\Omega |\det\,(Y_k^{-1})|^{p/n}\mathrm{d}x \le C \int_\Omega |Y_k^{-1}|^p\mathrm{d}x < +\infty$, we know from Proposition 3.3 that $\nu_x$ and $\mu_x$ are both supported on $\mathbb{R}^{n\times n}_{\rm inv}$ for almost all $x \in \Omega$. We have for all $g \in L^\infty(\Omega)$ and all $v \in C_0(\mathbb{R}^{n\times n})$

$$\lim_{k\to\infty} \int_\Omega v(Y_k^{-1}(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s^{-1})\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x,$$

$$\lim_{k\to\infty} \int_\Omega v(Y_k^{-1}(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\mu_x(\mathrm{d}s)g(x)\,\mathrm{d}x.$$

This means that for all $g \in L^\infty(\Omega)$ and all $v \in C_0(\mathbb{R}^{n\times n})$

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s^{-1})\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\mu_x(\mathrm{d}s)g(x)\,\mathrm{d}x; \tag{3.5}$$

in particular, the equality holds for all $v$ supported on $R^{n\times n}_\varrho$.

It remains only to prove that $\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} (|s|^p + |s^{-1}|^p)\nu_x(\mathrm{d}s)\mathrm{d}x$ is bounded. This can be shown by an application of [11, Th. 8.61], similarly as in the proof of Proposition 3.3, when setting $s^{-1} = +\infty$ in singular matrices. Note that, since $\nu_x$ is supported on invertible matrices, this extension will not play a role.

Therefore, by Lemma 3.4, (3.4) holds for all $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$. $\square$

**Proposition 3.6** *Let $\nu \in \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n})$. Then there is a generating sequence $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{n\times n})$ is bounded. Moreover, $\{|Y_k^{-1}|^p\}_{k\in\mathbb{N}}$ as well as $\{|Y_k|^p\}_{k\in\mathbb{N}}$ are equi-integrable.*

*Proof* Notice that inevitably supp $\nu_x \subset \mathbb{R}^{n\times n}_{\rm inv}$ for a.a. $x \in \Omega$ (cf. (2.1)). Therefore, we define smooth cut-off functions $\Phi_\varrho$ which are 1 on $R^{n\times n}_\varrho$ and 0 on $\mathbb{R}^{n\times n}_{\rm inv} \setminus R^{n\times n}_{\varrho+1}$; note that $\Phi_\varrho$ can be found as follows: construct $\Theta_\varrho$ a smooth function which is 1 inside the ball $B(0,\varrho) \subset \mathbb{R}^{n\times n}$ and equals 0 on $\mathbb{R}^{n\times n} \setminus B(0,\varrho+1)$. Now, we may set $\Phi_\varrho(s) := \Theta_\varrho(s)\Theta_\varrho(s^{-1})$. Then, we define

$$\nu_x^\varrho := \Phi_\varrho \nu_x + \left(\int_{\mathbb{R}^{n\times n}_{\rm inv}} \left(1 - \Phi_\varrho(s)\right) \nu_x(\mathrm{d}s)\right) \delta_I, \tag{3.6}$$

where $\delta_I$ denotes the Dirac measure supported at the identity matrix. It is a simple observation that $\{\nu_x^\varrho\}_{x\in\Omega} =: \nu^\varrho \in \mathcal{Y}(\Omega; \mathbb{R}^{n\times n})$ such that supp $\nu_x^\varrho \subset R^{n\times n}_{\varrho+1}$ for a.e. $x \in \Omega$.

Due to Propositions 3.1 and 3.2, there exists $\{Y_k^\varrho\}_{k\in\mathbb{N}} \subset R^{n\times n}_{\varrho+1}$ with $\{(Y_k^\varrho)^{-1}\}_{k\in\mathbb{N}} \subset R^{n\times n}_{\varrho+1}$ for a.e. $x \in \Omega$ generating $\nu^\varrho$ and $\mu^\varrho$, respectively, that satisfy for all $v \in C_0(\mathbb{R}^{n\times n})$

$$\int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s^{-1})\nu_x^\varrho(\mathrm{d}s) = \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\mu_x^\varrho(\mathrm{d}s). \tag{3.7}$$

Now, for any $g \in L^\infty(\Omega)$ we can write

$$\begin{aligned}
&\lim_{\varrho\to\infty} \int_\Omega g(x) \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\nu_x^\varrho(\mathrm{d}s)\mathrm{d}x \\
&\quad= \lim_{\varrho\to\infty} \int_\Omega g(x) \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\Phi_\varrho(s)\nu_x(\mathrm{d}s)\mathrm{d}x \\
&\quad+ \lim_{\varrho\to\infty} v(I) \int_\Omega g(x) \int_{\mathbb{R}^{n\times n}_{\rm inv}} (1 - \Phi_\varrho(s))\nu_x(\mathrm{d}s)\mathrm{d}x.
\end{aligned}$$

As $v\Phi_\varrho$ converges strongly in the $C_0$-norm to $v$ and $\int_{\mathbb{R}^{n\times n}_{\rm inv}} (1 - \Phi_\varrho(s))\nu_x(\mathrm{d}s)$ converges to 0 for a.e. $x \in \Omega$, thanks to Lebesgue's dominated convergence theorem, we are in the situation that

$$\lim_{\varrho\to\infty} \lim_{k\to\infty} v(Y_k^\varrho) = \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\nu_x(\mathrm{d}s) \text{ weakly in } L^1(\Omega).$$

Further, we verify that $\{Y_k^\varrho\}_{k\in\mathbb{N}}$ as well as $\{(Y_k^\varrho)^{-1}\}_{k\in\mathbb{N}}$ are bounded in $L^p(\Omega; \mathbb{R}^{n\times n})$ independently of $\varrho$. Indeed, for every $\varrho \geq 1$ fixed we have that,

$$\begin{aligned}
\lim_{k\to\infty} \int_\Omega |Y_k^\varrho|^p \,\mathrm{d}x &= \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} |s|^p \,\nu_x^\varrho(\mathrm{d}s)\,\mathrm{d}x \\
&\leq \int_\Omega \int_{B(0,\varrho+1)} |s|^p \nu_x(\mathrm{d}s)\,\mathrm{d}x \leq \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm inv}} |s|^p \,\nu_x(\mathrm{d}s)\,\mathrm{d}x \leq C;
\end{aligned} \tag{3.8}$$

an analogous calculation could be carried out for $\{(Y_k^\varrho)^{-1}\}_{k\in\mathbb{N}}$.

Applying the diagonalization argument (as $L^1(\Omega; C_0(\mathbb{R}^{n\times n}))$ is separable) we get a sequence $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{n\times n})$ generating $\nu$ that is, thanks to (3.8), also equi-integrable; the same holds for the inverse.

Moreover, if we defined $\mu$ as the weak* limit of $\mu_\varrho$, then $\mu$ would be generated by $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{n\times n})$ as, due to its definition,

$$\lim_{\varrho\to\infty}\lim_{k\to\infty} v((Y_k^\varrho)^{-1}) = \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\mu_x(\mathrm{d}s) \text{ weakly in } L^1(\Omega)\,.$$

Also, by applying $\varrho\to\infty$ in (3.7), it holds that

$$\int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s^{-1})\nu_x(\mathrm{d}s) = \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\mu_x(\mathrm{d}s), \tag{3.9}$$

for all $v \in C_0(\mathbb{R}^{n\times n})$ and hence, by Lemma 3.4, also for all $v \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$. □

*Proof of Theorem 2.1.* The necessity part follows from Propositions 3.3 and 3.5 while the sufficiency part follows from Proposition 3.6.

It thus remains to prove relation (2.3), which can be proven analogously to [11, Th. 8.6]; however, we need to show that if $f(x,s) = g(x)v(s)$ for some $g \in L^\infty(\Omega)$ and $v \in C_{p,-p}(\mathbb{R}^{n\times n}_{\rm inv})$ then $\{f(x, Y_k(x))\}_{k\in\mathbb{N}}$ is equi-integrable. To see this, we use [30, Lemma 6.1] and show only that for every $\varepsilon > 0$ there exists $K > 0$ such that $\int_{\{x\in\Omega;\,|v(Y_k(x))|\geq K\}} |v(Y_k(x))|\,\mathrm{d}x \leq \varepsilon$.

Notice that there exists $C > 0$ such that $v_0(s) := |v(s)|/(|s|^p + |s^{-1}|^p) \leq C$ for every $s \in \mathbb{R}^{n\times n}_{\rm inv}$. Moreover, $\lim_{|s|^p+|s^{-1}|^p\to\infty} v_0(s) = 0$. Let $(\|Y_k\|^p_{L^p} + \|Y_k^{-1}\|^p_{L^p}) \leq M$ and take $\varepsilon > 0$ and $K > 0$ large enough so that $|v_0(s)| < \varepsilon/M$ if $|s|^p + |s^{-1}|^p \geq K/C$. Then for all $k$

$$\begin{aligned}
\int_{\{x\in\Omega;\ |v(Y_k(x))|\geq K\}} |v(Y_k(x))|\,\mathrm{d}x &\leq \int_{\{x\in\Omega;\ |Y_k(x)|^p+|(Y_k(x))^{-1}|^p\geq K/C\}} |v(Y_k(x))|\,\mathrm{d}x\\
&\leq \int_{\{x\in\Omega;|Y_k(x)|^p+|Y_k^{-1}(x)|^p\geq K/C\}} |v_0(Y_k(x))|(|Y_k(x)|^p + |Y_k^{-1}(x)|^p)\mathrm{d}x\\
&\leq \varepsilon/M? \int_\Omega |Y_k(x)|^p + |Y_k^{-1}(x)|^p\mathrm{d}x \leq \varepsilon.
\end{aligned}$$

□

*Proof of Theorem 2.2.* It is analogous to the proof of Theorem 2.1. Notice that $\nu$ is supported on matrices with positive determinant due to Proposition 3.3. The converse implication follows from Proposition 3.1. □

*Proof of Proposition 2.4.* By the Mazur lemma $\det\nabla y \geq 0$. Suppose, by contradiction, there existed a set $\omega \subset \Omega$ of non-zero Lebesgue measure such that $\det\nabla y = 0$ on $\omega$. We have by the sequential weak continuity of $y \mapsto \det\nabla y$ from $W^{1,p}(\Omega;\mathbb{R}^n)$ to $L^{p/n}(\Omega)$ [2] that

$$\int_\omega |\det\nabla y_k(x)|\,\mathrm{d}x = \int_\omega \det\nabla y_k(x)\,\mathrm{d}x \to 0 \text{ as } k\to\infty,$$

so, it holds for a subsequence (not relabelled) that $0 < \det \nabla y_k \to 0$ a.e. in $\omega$. By the Fatou lemma, we have

$$\int_\omega \liminf_{k\to\infty} \frac{\mathrm{d}x}{\det \nabla y_k(x)} \le \liminf_{k\to\infty} \int_\omega \frac{\mathrm{d}x}{\det \nabla y_k(x)} \le C \liminf_{k\to\infty} \int_\omega |(\nabla y_k(x))^{-1}|^n \,\mathrm{d}x,$$

however, the left-hand side tends to $+\infty$. This contradicts the boundedness of $\{(\nabla y_k)^{-1}\}_{k\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^{n\times n})$ because $p > n$ and $\Omega$ is bounded. Hence, $\det \nabla y > 0$ a.e. in $\Omega$. The assertion about the support follows from Proposition 3.3. $\square$

## 4. Proofs in the $W^{1,\infty}$-case

We shall heavily rely on the following convex integration result which can be found in [29, p. 199 and Remark 2.4]; recall that $O(n)$ will standardly denote the set of orthogonal matrices in $\mathbb{R}^{n\times n}$, i.e. $O(n) := \{A \in \mathbb{R}^{n\times n};\ A^\top A = AA^\top = I\}$.

LEMMA 4.1 *Let $\omega \subset \mathbb{R}^n$ be open and Lipschitz. Let $\varphi \in W^{1,\infty}(\omega;\mathbb{R}^n)$ be such that there is $\vartheta > 0$, so that $0 \le |\nabla\varphi| \le 1-\vartheta$ a.e. in $\omega$. Then there exist mappings $u \in W^{1,\infty}(\omega;\mathbb{R}^n)$ for which $\nabla u \in O(n)$ a.e. in $\omega$ and $u = \varphi$ on $\partial\omega$. Moreover, the set of such mappings is dense (in the $L^\infty$-norm) in the set $\{\psi := z + \varphi;\ z \in W_0^{1,\infty}(\omega;\mathbb{R}^n)\ ,\ |\nabla\psi| \le 1-\vartheta$ a.e. in $\omega\}$.*

Before proving Theorem 2.5, let us elaborate more on the connection of the envelope $Z^\infty v$ from (1.10) and the standard quasi-convex envelope. If $v \in \mathcal{O}(\varrho)$ with $\varrho < \infty$ it is not clear whether $Z^\infty v$ is quasiconvex. However, this holds in the case when $\rho = +\infty$ as the following proposition shows.

PROPOSITION 4.2 *Let $v : \mathbb{R}^{n\times n} \to \mathbb{R}\cup\{+\infty\}$ be in $\mathcal{O}(+\infty)$. Then $Z^\infty v = Qv$.*

*Proof* Let us establish that $Z^\infty v(A) < +\infty$ for all $A \in \mathbb{R}^{n\times n}$. This is clear if $A \in \mathbb{R}^{n\times n}_{\mathrm{inv}}$, otherwise we use Lemma 4.1 to construct $\psi \in W^{1,\infty}_A(\Omega;\mathbb{R}^n)$ and $\nabla\psi \in (|A|+\varepsilon)O(n)$ for some $\varepsilon > 0$. Thus, $|\Omega| Z^\infty v(A) \le \int_\Omega v(\nabla\psi(A))\,\mathrm{d}x < +\infty$.

Due to the finiteness of $Z^\infty v$, we know by [31, Thm. 2.4] that $Z^\infty v = Qv$, i.e. $Z^\infty v$ is quasiconvex and continuous. $\square$

### 4.1. *Proof of Theorem 2.5 – necessity*

Conditions (2.7) and (2.8) are standard we only need to prove (2.9).

PROPOSITION 4.3 *Let $F \in \mathbb{R}^{n\times n}$, $u_F(x) := Fx$ if $x \in \Omega$, $y_k \overset{*}{\rightharpoonup} u_F$ in $W^{1,\infty}(\Omega;\mathbb{R}^n)$ and let for some $\alpha > 0$ $\nabla y_k(x) \in R^{n\times n}_\alpha$ for all $k > 0$ and almost all $x \in \Omega$. Then for every $\varepsilon > 0$ there is $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that $\nabla u_k(x) \in R^{n\times n}_{\alpha+\varepsilon}$ for all $k > 0$ and almost all $x \in \Omega$, $u_k - u_F \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)$ and $|\nabla y_k - \nabla u_k| \to 0$ in measure. In particular, $\{\nabla y_k\}_{k\in\mathbb{N}}$ and $\{\nabla u_k\}_{k\in\mathbb{N}}$ generate the same Young measure.*

*Proof* Define for $\ell > 0$, sufficiently large, $\Omega_\ell := \{x \in \Omega;\ \mathrm{dist}(x,\partial\Omega) \ge 1/\ell\}_{\ell\in\mathbb{N}}$ and smooth cut-off functions $\eta_\ell : \Omega \to [0,1]$

$$\eta_\ell(x) = \begin{cases} 1 & \text{if } x \in \Omega_\ell \\ 0 & \text{if } x \in \partial\Omega \end{cases}$$

such that $|\nabla\eta_\ell| \leq C\ell$ for some $C > 0$. Set $z_{k\ell} := \eta_\ell y_k + (1-\eta_\ell)u_F$. Then $z_{k\ell} \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ and $z_{k\ell} = y_k$ in $\Omega_\ell$ and $z_{k\ell} = u_F$ on $\partial\Omega$. We see that $\nabla z_{k\ell} = \eta_\ell \nabla y_k + (1-\eta_\ell)F + (y_k - u_F)\otimes\nabla\eta_\ell$. Hence, in view of the facts that $|F| \leq \liminf_{k\to\infty}\|\nabla y_k\|_{L^\infty} \leq \alpha$ and that $y_k \to u_F$ uniformly in $\overline{\Omega}$, we can extract for every $\varepsilon > 0$ a (not relabelled) subsequence $k = k(\ell)$ such that

$$\|\nabla z_{k(\ell)\ell}\|_{L^\infty} < \alpha + \frac{\varepsilon}{2}.$$

Consequently, $\{z_{k(\ell)\ell}\}_{\ell\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}(\Omega;\mathbb{R}^n)$. Moreover,

$$\left|\frac{\nabla z_{k(\ell)\ell}(x)}{\alpha+\varepsilon}\right| \leq \frac{\|\nabla z_{k(\ell)\ell}\|_{L^\infty}}{\alpha+\varepsilon} \leq 1 - \frac{\varepsilon}{2(\alpha+\varepsilon)}.$$

Let us denote $\omega_\ell := \Omega\setminus\Omega_\ell$, then $w_{k(\ell)\ell} := z_{k(\ell)\ell\ \omega_\ell}/(\alpha+\varepsilon)$ is such that $|\nabla w_{k(\ell)\ell}| \leq 1-\vartheta$ for $\vartheta := \varepsilon/2(\alpha+\varepsilon)$. We use Lemma 4.1 for $\omega := \omega_\ell$ and $\varphi := w_{k(\ell)\ell}$ to obtain $\phi_{k(\ell)\ell} \in W^{1,\infty}(\omega_\ell;\mathbb{R}^n)$ such that $\phi_{k(\ell)\ell} = w_{k(\ell)\ell}$ on $\partial\omega_\ell$ and $\nabla\phi_{k(\ell)\ell} \in O(n)$. Define

$$u_{k(\ell)\ell} := \begin{cases} y_k & \text{if } x \in \Omega_\ell \\ (\alpha+\varepsilon)\phi_{k(\ell)\ell} & \text{if } x \in \Omega\setminus\Omega_\ell. \end{cases}$$

Notice that $\{u_{k(\ell)\ell}\}_{\ell\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ and that $u_{k(\ell)\ell}(x) = Fx$ for $x \in \partial\Omega$. Further, $\nabla u_{k(\ell)\ell}(x) \in R^{n\times n}_{\alpha+\varepsilon}$. Moreover, the Lebesgue measure of $\{x \in \Omega;\ \nabla u_{k(\ell)\ell}(x) \neq \nabla y_k(x)\}$ vanishes if $k \to \infty$ and $\ell \to \infty$ sufficiently fast, therefore both sequences generate the same Young measure by [30, Lemma 8.3]. □

*Remark 4.4*
(i) It follows from the above proof that if $|F| < \alpha$ then we can take $\varepsilon = 0$ in Proposition 4.3.
(ii) If $\{u_k\}_{k\in\mathbb{N}}$ defined in the proof of Proposition 4.3 are homeomorphic and $n = 2$ then either $\det\nabla u_k > 0$ or $\det\nabla u_k < 0$ in $\Omega$ for all $k$. The reason is that homeomorphisms in two dimensions are either orientation preserving or reversing.

LEMMA 4.5 *Let $\nu \in \mathcal{GY}^{+\infty,-\infty}_\varrho(\Omega;\mathbb{R}^{n\times n})$. Then $\mu := \{\nu_a\}_{x\in\Omega} \in \mathcal{GY}^{+\infty,-\infty}_\varrho(\Omega;\mathbb{R}^{n\times n})$ for a.e. $a \in \Omega$.*

*Proof* Note that the construction in the proof of [30, Th. 7.2] does not affect invertibility. □

PROPOSITION 4.6 *Let $\nu \in \mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$, supp $\nu \subset R^{n\times n}_\varrho$ be such that for almost all $x \in \Omega$ $\nabla y(x) = \int_{R^{n\times n}_\varrho} s\nu_x(\mathrm{d}s)$, where $y \in W^{1,\infty}(\Omega;\mathbb{R}^n)$. Then for all $\tilde\varrho \in (\varrho;+\infty]$, almost all $x \in \Omega$ and all $v \in \mathcal{O}(\tilde\varrho)$ we have*

$$\int_{\mathbb{R}^{n\times n}_{\mathrm{inv}}} v(s)\nu_x(\mathrm{d}s) \geq Z^\infty v(\nabla y(x)). \tag{4.1}$$

*Proof* We know from Lemma 4.5 that $\mu = \{\nu_a\}_{x\in\Omega} \in \mathcal{GY}^{+\infty,-\infty}_\varrho(\Omega;\mathbb{R}^{n\times n})$ for a.e. $a \in \Omega$, so there exits its generating sequence $\{\nabla u_k\}_{k\in\mathbb{N}}$ such that $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ and for almost all $x \in \Omega$ and all $k \in \mathbb{N}$ $\nabla u_k(x) \in R^{n\times n}_\varrho$. Moreover, $\{u_k\}_{k\in\mathbb{N}}$ weakly* converges to the map $x \mapsto (\nabla y(a))x$.

Using Proposition 4.3, we can, without loss of generality, suppose that $\nabla u_k \in R^{n\times n}_{\tilde{\varrho}}$ for all $k \in \mathbb{N}$ and $u_k(x) = \nabla y(a)x$ if $x \in \partial\Omega$. Therefore, we have

$$|\Omega| \int_{\mathbb{R}^{n\times n}_{\rm inv}} v(s)\nu_a(\mathrm{d}s) = \lim_{k\to\infty} \int_\Omega v(\nabla u_k(x))\,\mathrm{d}x \geq |\Omega| Z^\infty v(\nabla y(a)).$$

$\square$

### 4.2. *Proof of Theorem 2.5 – sufficiency*

We need to show that conditions (2.7),(2.8), and (2.9) are also sufficient for $\nu \in \mathrm{rca}(\mathbb{R}^{n\times n}_{\rm inv})$ to be in $\mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$. Put

$$\mathcal{U}^\varrho_A := \{y \in W^{1,\infty}_A(\Omega;\mathbb{R}^n);\ \nabla y \in R^{n\times n}_\varrho\}. \tag{4.2}$$

Consider for $A \in \mathbb{R}^{n\times n}$ the set

$$\mathcal{M}^\varrho_A := \{\overline{\delta_{\nabla y}};\ y \in \mathcal{U}^\varrho_A\}\,, \tag{4.3}$$

where $\overline{\delta_{\nabla y}} \in \mathrm{rca}(\mathbb{R}^{n\times n})$ is defined as $\langle\overline{\delta_{\nabla y}}, v\rangle := |\Omega|^{-1}\int_\Omega v(\nabla y(x))\,\mathrm{d}x$; $\overline{\mathcal{M}^\varrho_A}$ will denote its weak* closure.

We have the following lemma:

LEMMA 4.7 *Let $A \in \mathbb{R}^{n\times n}$ If $\varrho > |A|$ then the set $\mathcal{M}^\varrho_A$ is nonempty and convex.*

*Proof* First we show that $\mathcal{M}^\varrho_A$ is non-empty. This is clear when $A$ is invertible. Otherwise, note that $|A|/\varrho = 1 - (\varrho - |A|)/\varrho$. Thus, we can apply Lemma 4.1 with $\varphi(x) := Ax/\varrho$, $x \in \Omega$ and $\vartheta := (\varrho - |A|)/\varrho$ to get $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that $\nabla u \in O(n)$ a.e. in $\Omega$ and $u(x) = Ax/\varrho$ if $x \in \partial\Omega$. Therefore, $y := \varrho u \in \mathcal{U}^\varrho_A$. Consequently, $\mathcal{M}^\varrho_A \neq \emptyset$.

The rest of proof is analogous to the proof of [30, Lemma 8.5]. We take $y_1, y_2 \in \mathcal{U}^\varrho_A$ and, for a given $\lambda \in (0,1)$, we find a subset $D \subset \Omega$ such that $|D| = \lambda|\Omega|$. There are two countable families of subsets of $D$ and $\Omega \setminus D$ of the form

$$\{a_i + \epsilon_i\Omega;\ a_i \in D,\ \epsilon_i > 0,\ a_i + \epsilon_i\Omega \subset D\}$$

and

$$\{b_i + \epsilon_i\Omega;\ b_i \in \Omega\setminus D,\ \rho_i > 0,\ b_i + \rho_i\Omega \subset \Omega \setminus D\}$$

such that

$$D = \cup_i(a_i + \epsilon_i\Omega) \cup N_0, \qquad \Omega \setminus D = \cup_i(b_i + \rho_i\Omega) \cup N_1,$$

where the Lebesgue measure of $N_0$ and $N_1$ is zero. We define

$$y(x) := \begin{cases} \epsilon_i y_1\left(\frac{x-a_i}{\epsilon_i}\right) + Aa_i & \text{if } x \in a_i + \epsilon_i\Omega, \\ \rho_i y_2\left(\frac{x-b_i}{\rho_i}\right) + Ab_i & \text{if } x \in b_i + \rho_i\Omega, \\ Ax & \text{otherwise,} \end{cases}$$

yielding

$$\nabla y(x) = \begin{cases} \nabla y_1\left(\frac{x-a_i}{\epsilon_i}\right) & \text{if } x \in a_i + \epsilon_i\Omega, \\ \nabla y_2\left(\frac{x-b_i}{\rho_i}\right) & \text{if } x \in b_i + \rho_i\Omega, \\ A & \text{otherwise.} \end{cases}$$

In particular, $y \in \mathcal{U}_A^\varrho$ and $\overline{\delta_{\nabla y}} = \lambda\overline{\delta_{\nabla y_1}} + (1-\lambda)\overline{\delta_{\nabla y_2}}$. □

The following homogenization lemma can be proved the same way as [30, Th. 7.1].

LEMMA 4.8 *Let $\{u_k\}_{k\in\mathbb{N}} \subset W_A^{1,\infty}(\Omega;\mathbb{R}^n)$ be a bounded sequence such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ are invertible and $\{(\nabla u_k)^{-1}\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ bounded. Let $\nu \in \mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$ be generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$. Then there is a bounded sequence $\{w_k\}_{k\in\mathbb{N}} \subset W_A^{1,\infty}(\Omega;\mathbb{R}^n)$ with $\{\nabla w_k\}_{k\in\mathbb{N}}$ invertible and $\{(\nabla w_k)^{-1}\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ bounded such that $\{\nabla w_k\}_{k\in\mathbb{N}}$ generates $\overline{\nu} \in \mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$ defined through*

$$\int_{\mathbb{R}^{n\times n}} v(s)\overline{\nu}_x(\mathrm{d}s) = \frac{1}{|\Omega|}\int_\Omega\int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s)\,\mathrm{d}x, \tag{4.4}$$

*for any $v \in C_0(\mathbb{R}^{n\times n})$ and almost all $x \in \Omega$.*

PROPOSITION 4.9 *Let $\mu$ be a probability measure supported on a compact set $K \subset \mathbb{R}_\alpha^{n\times n}$ for some $\alpha \geq 1$ and let $A := \int_K s\mu(\mathrm{d}s)$. Let $\varrho > \alpha$ and let*

$$Z^\infty v(A) \leq \int_K v(s)\mu(\mathrm{d}s), \tag{4.5}$$

*for all $v \in \mathcal{O}(\varrho)$. Then $\mu \in \mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$ and it is generated by gradients of mappings from $\mathcal{U}_A^\varrho$.*

*Proof* The proof standardly uses the Hahn–Banach theorem and Lemma 4.8 and it is similar to [30, Proposition 8.17]. First, notice that $|A| \leq \alpha < \varrho < +\infty$. Then, since $\mathcal{M}_A^\varrho$ is non-empty and convex due to Lemma 4.7, we can, by the Hahn–Banach theorem, assume that there is $\tilde{v} \in C(R_\varrho^{n\times n})$ such that

$$0 \leq \langle \nu, \tilde{v}\rangle = \int_{R_\varrho^{n\times n}} \tilde{v}(s)\nu(\mathrm{d}s) = |\Omega|^{-1}\int_\Omega \tilde{v}(\nabla y(x))\,\mathrm{d}x,$$

for all $\nu \in \mathcal{M}_A^\varrho$, and hence all $y \in \mathcal{U}_A^\varrho$, and $0 > \langle\tilde{\nu}, \tilde{v}\rangle$ if $\tilde{\nu} \in \mathrm{rca}(\mathbb{R}^{n\times n}) \setminus \overline{\mathcal{M}_A^\varrho}$.

Now, the function

$$\bar{v}(F) := \begin{cases} \tilde{v}(F) & \text{if } F \in R_\varrho^{n\times n}, \\ +\infty & \text{else,} \end{cases}$$

is in $\mathcal{O}(\varrho)$. Notice that it follows from (4.5) that $Z^\infty\bar{v}(A)$ is finite. Thus, $Z^\infty v(A) = \inf_{\mathcal{U}_A^\varrho} |\Omega|^{-1}\int_\Omega v(\nabla y(x))\,\mathrm{d}x$ and hence $Z^\infty v(A) \geq 0$ and, by (4.5), $0 \leq \int_{R_\varrho^{n\times n}} v(s)\mu(\mathrm{d}s)$. Thus, $\mu \in \overline{\mathcal{M}_A^\varrho}$. As $C(R_\varrho^{n\times n})$ is separable, the weak* topology on bounded sets in $\mathrm{rca}(R_\varrho^{n\times n})$ is metrizable. Hence, there is a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}_A^\varrho$ such that for all $v \in C(R_\varrho^{n\times n})$ (and all $v \in \mathcal{O}(\varrho)$)

$$\lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))\,\mathrm{d}x = |\Omega|\int_{R_\varrho^{n\times n}} v(s)\mu(\mathrm{d}s),$$

and $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,\infty}(\Omega;\mathbb{R}^{n\times n})$ due to the Poincaré inequality. As $u_k(x) = Ax$ for $x \in \partial\Omega$ we use the homogenization procedure from Lemma 4.8 to show that $\mu$ is the homogeneous Young measure generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$. □

We will need the following auxiliary result.

LEMMA 4.10 (see [18, Lemma 6.1]) *Let $\Omega \subset \mathbb{R}^n$ be an open domain with $|\partial\Omega| = 0$ and let $N \subset \Omega$ be of the zero Lebesgue measure. For $r_k : \Omega \setminus N \to (0, +\infty)$ and $\{f_j\}_{j\in\mathbb{N}} \subset L^1(\Omega)$ there exists a set of points $\{a_{ik}\}_{i\in\mathbb{N}} \subset \Omega \setminus N$ and positive numbers $\{\epsilon_{ik}\}_{i\in\mathbb{N}}$, $\epsilon_{ik} \le r_k(a_{ik})$ such that $\{a_{ik} + \epsilon_{ik}\bar{\Omega}\}_{i\in\mathbb{N}}$ are pairwise disjoint for each $k \in \mathbb{N}$, $\bar{\Omega} = \cup_i\{a_{ik} + \epsilon_{ik}\bar{\Omega}\} \cup N_k$ with $|N_k| = 0$ and for any $j \in \mathbb{N}$*

$$\lim_{k\to\infty} \sum_i f_j(a_{ik})|\epsilon_{ik}\Omega| = \int_\Omega f_j(x)\,\mathrm{d}x.$$

*Proof of Theorem 2.5 – sufficiency.* Some parts of the proof follow [18, Proof of Th. 6.1]. We are looking for a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ with $\{\nabla u_k\}_{k\in\mathbb{N}}$ invertible and $\{(\nabla u_k)^{-1}\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ satisfying

$$\lim_{k\to\infty} \int_\Omega v(\nabla u_k(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x$$

for all $g \in \Gamma$ and any $v \in S$, where $\Gamma$ and $S$ are countable dense subsets of $C(\bar{\Omega})$ and $C_0(\mathbb{R}^{n\times n})$, respectively.

First of all notice that, as $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ from (2.8) is differentiable in $\Omega$ outside a set of measure zero called $N$, we may find for every $a \in \Omega \setminus N$ and every $k > 0$ a $r_k(a) > 0$ such that for any $0 < \epsilon < r_k(a)$ we have

$$\frac{1}{\epsilon}|u(a + \epsilon y) - u(a) - \epsilon\nabla u(a)y| \le \frac{1}{k}. \tag{4.6}$$

Furthermore, as $g$ is continuous, we choose $r_k(a) > 0$ smaller if necessary to assure that for any $0 < \epsilon < r_k(a)$

$$\left|\int_{a+\epsilon\Omega} g(x)\,\mathrm{d}x - g(a)\epsilon\right| < \frac{1}{k}. \tag{4.7}$$

From Lemma 4.10 we can find $a_{ik} \in \Omega \setminus N$, $\epsilon_{ik} \le r_k(a_{ik})$ such that for all $v \in S$ and all $g \in \Gamma$

$$\lim_{k\to\infty} \sum_i \bar{V}(a_{ik})g(a_{ik})|\epsilon_{ik}\Omega| = \int_\Omega \bar{V}(x)g(x)\,\mathrm{d}x, \tag{4.8}$$

where

$$\bar{V}(x) := \int_{\mathbb{R}^{n\times n}_{\mathrm{inv}}} v(s)\nu_x(\mathrm{d}s).$$

In view of Lemma 4.9, let us assume that $\{\nu_{a_{ik}}\}_{x\in\Omega} \in \mathcal{GY}^{+\infty,-\infty}(\Omega;\mathbb{R}^{n\times n})$ is a homogeneous gradient Young measure and call $\{\nabla u_j^{ik}\}_{j\in\mathbb{N}}$ its generating sequence. We know that we can consider $\{u_j^{ik}\}_{j\in\mathbb{N}} \subset \mathcal{U}^{\tilde{\varrho}}_{\nabla u(a_{ik})}$ for arbitrary $+\infty > \tilde{\varrho} > \varrho$. Hence

$$\lim_{j\to\infty} \int_\Omega v(\nabla u_j^{ik}(x))g(x)\,\mathrm{d}x = \bar{V}(a_{ik}) \int_\Omega g(x)\,\mathrm{d}x \tag{4.9}$$

and, in addition, $u_j^{ik}$ weakly* converges to the map $x \mapsto \nabla u(a_{ik})x$ for $j \to \infty$ in $W^{1,\infty}(\Omega;\mathbb{R}^n)$.

Let $\Omega_\ell := \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) \ge \ell^{-1}\}$. We define a sequence of smooth cut-off functions $\{\eta_\ell\}_{\ell\in\mathbb{N}}$

$$\eta_\ell(x) := \begin{cases} 0 & \text{in } \Omega_\ell, \\ 1 & \text{on } \partial\Omega \end{cases}$$

such that $|\nabla\eta_\ell| \le C\ell$ for some $C > 0$. Further, take a sequence $\{u_k^\ell\}_{k,\ell\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ defined by

$$u_k^\ell(x) := \begin{cases} \left[u(a_{ik}) + \epsilon_{ik} u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right]\left(1-\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right) & \\ +u(x)\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) & \text{if } x \in a_{ik}+\epsilon_{ik}\Omega, \\ u(x) & \text{otherwise,} \end{cases}$$

where $j = j(i,k,\ell)$ will be chosen later. Note that for every $k$ we have $u_k^\ell - u \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$.

We calculate for $x \in a_{ik} + \epsilon_{ik}\Omega$

$$\begin{aligned} \nabla u_k^\ell(x) &= \nabla u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\left(1-\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right) \\ &\quad + \nabla u(x)\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) \\ &\quad + \frac{1}{\epsilon_{ik}}\left[u(x)-u(a_{ik})-\epsilon_{ik}\nabla u(a_{ik})\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right]\otimes\nabla\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) \\ &\quad + \left[\nabla u(a_{ik})\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) - u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right]\otimes\nabla\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right). \end{aligned} \tag{4.10}$$

Notice that the moduli of all four terms can be made together uniformly bounded by $\tilde{\varrho} > \varrho$. Namely, notice that the sum of the first two terms is not greater then $\varrho$ and the other two terms can be made arbitrarily small if $k$ is sufficiently large compared to $\ell$ by exploiting (4.6) and the strong convergence in $L^\infty(a_{ik}+\epsilon_{ik}\Omega;\mathbb{R}^n)$ of $u_j^{ik}(x)$ to the map $x \mapsto \nabla u(a_{ik})x$ for $j \to \infty$.

Take the set $(a_{ik}+\varepsilon_{ik}\Omega)\setminus(a_{ik}+\varepsilon_{ik}\Omega_\ell)$ and solve the inclusion $\nabla\tilde{u}_k^\ell \in O(n)$ with the boundary conditions $\tilde{u}_k^\ell = u_k^\ell/\tilde{\varrho}$ if $x \in \partial((a_{ik}+\varepsilon_{ik}\Omega_k)\setminus(a_{ik}+\varepsilon_{ik}\Omega_\ell))$. This inclusion has a solution due to Lemma 4.1. Set

$$z_k^\ell(x) := \begin{cases} u_k^\ell(x) & \text{if } x \in a_{ik}+\epsilon_{ik}\Omega_\ell, \\ \tilde{u}_k^\ell(x) & \text{if } x \in (a_{ik}+\epsilon_{ik}\Omega)\setminus(a_{ik}+\epsilon_{ik}\Omega_\ell), \\ u(x) & \text{otherwise.} \end{cases}$$

Observe, that the Lebesgue measure of the set $\{x \in \Omega;\ \nabla(u_k^\ell(x) - z_k^\ell(x)) \neq 0\}$ vanishes as $\ell \to \infty$. Further, $\{z_k^\ell\}_{k,\ell\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^n)$ is a bounded sequence as well as $\{\nabla z_k^\ell\}^{-1}_{k,\ell\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$.

Let us fix $k, i, \ell$ (with $k$ sufficiently large such that $|\nabla z_k^\ell|$ is uniformly bounded by $\tilde{\varrho}$) and consider the sets $\{E_k\}_{k\in\mathbb{N}}$, $E_k \subset E_{k+1}$ such that $\Gamma \times S = \bigcup_k E_k$ . We can eventually enlarge each $j = j(i,k,\ell)$ so that additionally for any $(g, v_0) \in E_k$

$$\left|\epsilon_{ik}^n \int_\Omega g(a_{ik}+\epsilon_{ik}y)v(\nabla u_j^{ik}(y))\,\mathrm{d}y - \bar{V}(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega} g(x)\,\mathrm{d}x\right| \le \frac{1}{2^i k}. \tag{4.11}$$

We have, by the smallness of $|\Omega\setminus\Omega_\ell|$ and boundedness of $g$ and $v$, that for some $C > 0$

$$\int_\Omega g(x)v(\nabla u_k^\ell(x))\,\mathrm{d}x = \sum_i \epsilon_{ik}^n \int_\Omega g(a_{ik}+\epsilon_{ik}y)v(\nabla u_j^{ik}(y))\,\mathrm{d}y + \frac{C}{\ell}.$$

Consequently, in view of (4.8), (4.7) and (4.11) for all $(g, v) \in \Gamma \times S$

$$\lim_{\ell\to\infty}\lim_{k\to\infty}\int_\Omega g(x)v(\nabla u_k^\ell(x))\,\mathrm{d}x = \int_\Omega\int_{\mathbb{R}^{\times n}} v(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x.$$

Hence, we can pick a subsequence $\{\nabla u^\ell_{k(\ell)}\}_{\ell\in\mathbb{N}}$ generating $\nu$. The measure $\nu$ is also generated by $\{\nabla z^\ell_{k(\ell)}\}_{\ell\in\mathbb{N}}$ because the difference of both sequences vanishes in measure. Finally, we see from the construction that $\{z^\ell_{k(\ell)}\}_{\ell\in\mathbb{N}}$ can be chosen to have the same boundary conditions as $u$. □

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