# The Quantum Entropy Cone of Stabiliser States

**Noah Linden[1], František Matúš[2], Mary Beth Ruskai[3,4], and Andreas Winter[5,1,6]**

1 School of Mathematics, University of Bristol
Bristol BS8 1TW, United Kingdom
n.linden@bristol.ac.uk

2 Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Prague, Czech Republic
matus@utia.cas.cz

3 Institute for Quantum Computing, University of Waterloo
Waterloo, Ontario, Canada
mbruskai@gmail.com

4 Tufts University, Medford, MA 02155 USA
marybeth.ruskai@tufts.edu

5 ICREA & Física Teòrica: Informació i Fenomens Quàntics
Universitat Autònoma de Barcelona
ES-08193 Bellaterra (Barcelona), Spain
andreas.winter@uab.cat

6 Centre for Quantum Technologies, National University of Singapore
2 Science Drive 3, Singapore 117542, Singapore

**Abstract**

We investigate the universal linear inequalities that hold for the von Neumann entropies in a multi-party system, prepared in a stabiliser state. We demonstrate here that entropy vectors for stabiliser states satisfy, in addition to the classic inequalities, a type of linear rank inequalities associated with the combinatorial structure of normal subgroups of certain matrix groups.

In the 4-party case, there is only one such inequality, the so-called Ingleton inequality. For these systems we show that strong subadditivity, weak monotonicity and Ingleton inequality exactly characterize the entropy cone for stabiliser states.

**1998 ACM Subject Classification** 94A17, 81P45,54C70

**Keywords and phrases** Entropy inequalities, Stabiliser states, Ingleton inequality

**Digital Object Identifier** 10.4230/LIPIcs.TQC.2013.270

## 1 Introduction

### 1.1 Background

Undoubtedly, the single most important quantity in (classical) information theory is the Shannon entropy, and its properties play a key role: for a discrete probability distribution $p$ on $\mathcal{T}$

$$H(p) = -\sum_{t\in\mathcal{T}} p(t)\log p(t) \ . \tag{1}$$

The quantum (von Neumann) entropy is understood to be of equal importance to quantum information: for a quantum state (density operator) $\rho \geq 0$, $\operatorname{Tr}\rho = 1$

$$S(\rho) = -\operatorname{Tr}\rho\log\rho \tag{2}$$


8th Conference on Theory of Quantum Computation, Communication and Cryptography, Guelph, Canada.
Editors: Simone Severini and Fernando Brandao; Vol. **22**; pp. 270–284
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

which reduces to (1) when $\rho$ is diagonal.

For $N$-party systems, one can apply these definitions to obtain the entropy of all marginal probability distributions (in the classical case) and reduced density operators (aka quantum marginals) in the quantum case. The collection of these entropies can be regarded as a vector in $\mathbf{R}_{2^N}$, and the collection of all such vectors forms a set whose closure is a convex cone. It is an interesting open question to determine the inequalities which characterize this cone. As discussed in Section 1.3, it is now known that in the classical setting the Shannon inequalities given below do not suffice; they describe a strictly larger cone.

This work has motivated us to consider analogous questions for the von Neumann entropy in $N$-party quantum systems. Although we are unable to answer this question, we can fully characterize the cone associated with a subset of quantum states known as stabiliser states in the 4-party case. Moreover, we can show that for any number of parties, entropy vectors for stabiliser states satisfy additional inequalities in the class known as linear rank inequalities discussed in Section 3. In the classical setting, distributions whose entropies satisfy this subclass of stronger inequalities, suffice to achieve maximum throughput in certain network coding problems [28].

## 1.2 Classic inequalities and Definitions

It is well-known that the classical Shannon entropy for an $N$-party classical probability distribution $p$ on a discrete space $\mathcal{T}_1 \times \cdots \times \mathcal{T}_N$, has the following properties, commonly known as the *Shannon inequalities*:

1. It is non-negative, *i.e.* $H(A) \geq 0$; $H(\emptyset) = 0$. (+)
2. It is strongly subadditive (aka submodular), *i.e.*

$$H(A) + H(B) - H(A \cap B) - H(A \cup B) \geq 0. \tag{SSA}$$

3. It is monotone non-decreasing, *i.e.*

$$A \subset B \quad \Longrightarrow \quad H(A) \leq H(B). \tag{MO}$$

where $H(A)$ denotes the entropy $H(p_A)$ of the marginal distribution $p_A$ on $\mathcal{T}_A = \bigotimes_{j \in A} \mathcal{T}_j$.

The monotonicity property (MO) implies that if $H(A) = 0$ then $H(B) = 0$ for all $B \subset A$ and, thus, $p_A = \bigotimes_{j \in A} \delta_{t_j}$ is a product of point masses. Some of the most remarkable features of quantum systems arise when (MO) is violated. Indeed, for a pure entangled state $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ for which $S(\rho_{AB}) = 0$, but the entropy of the reduced states $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ and $\rho_B = \mathrm{Tr}_A\, \rho_{AB}$ can be (and usually is) strictly positive. In fact, $S(\rho_{AB}) - S(\rho_A)$ can be as large as $-\log d$, where $d$ is the Hilbert space dimension of the smaller of $A$ and $B$.

For multi-party quantum systems, (+) and (SSA) are still valid [29], but (MO) has to be replaced by the third property below – in analogy to the classical case, we call them Shannon inequalities:

1. Non-negativity: $S(A) \geq 0$; $S(\emptyset) = 0$. (+)
2. Strong subadditivity:

$$S(A) + S(B) - S(A \cap B) - S(A \cup B) \geq 0. \tag{SSA}$$

3. Weak monotonicity:

$$S(A) + S(B) - S(A \setminus B) - S(B \setminus A) \geq 0. \tag{WMO}$$

However, in contrast to the classical setting, this weaker version of monotonicity is not completely independent of strong subadditivity (SSA). In fact, it can be obtained from the latter by the (non-linear) process known as *purification* described in Section 2.2. Using a slight abuse of notation, we use $I(A:B)$ and $I(A:B|C)$ to denote, respectively, the mutual information and conditional mutual information for both classical and quantum systems, defined explicitly in the latter case as

$$\begin{aligned} I(A:B) &= S(A) + S(B) - S(AB), \\ I(A:B|C) &= S(AC) + S(BC) - S(ABC) - S(C), \end{aligned}$$

for pairwise disjoint sets $A$, $B$, $C$. Note that SSA can then be written as $I(A:B|C) \geq 0$.

## 1.3 Entropy cones and non-Shannon inequalities

The first non-Shannon entropy inequality was obtained in 1997-98 by Yeung and Zhang [43, 44, 45] for 4-party systems. Their work established that the classical entropy cone is strictly smaller than the polyhedral cone defined by the Shannon inequalities. This was the only non-Shannon inequality known until 2006, when Dougherty, Freiling and Zeger [12, 13] used a computer search to generate new inequalities. Then Matúš [34] found two infinite families, one of which can be written as

$$t\,\mathrm{Ing}(AB:CD) + I(A:B|D) + \frac{t(t+1)}{2}\big[I(B:D|C) + I(C:D|B))\big] \quad \geq \quad 0 \tag{3}$$

where $t$ is a non-negative integer, and $\mathrm{Ing}(AB:CD)$ is defined in (ING) below. The case $t=1$ in (3) yields the inequality in [45]. Moreover, either of the Matúš families can be used to show that the 4-party entropy cone is not polyhedral. In [15] additional non-Shannon inequalities were found.

In the quantum setting, Lieb [30] considered the question of additional inequalities in a form that could be regarded as extending SSA to more parties, but found none. Much later Pippenger [39] rediscovered one of Lieb's results and used it to show constructively that there are no additional inequalities for 3-party systems. He also explicitly raised the question of whether or not additional inequalities hold for more parties. Despite the fact that (SSA) is still the only known inequality, it has been shown that for 4-party systems there are constrained inequalities [4, 31] that do not follow from SSA. (Numerical evidence for additional inequalities is given in the thesis of Ibinson [21].)

## 1.4 Structure of the paper

This paper is organized as follows. In Section 2 we give some basic notation and review some well-known facts. In Section 3 we discuss what is known about linear rank inequalities beginning with the Ingleton inequality in Section 3.1 and concluding with a discussion of their connection to the notion of common information in Section 3.3. In Section 4 we discuss stabiliser states, beginning with some basic definitions in Section 4.1. In Section 4.2 we consider the entropies of stabiliser states, showing half of our main result that pure stabiliser states generate entropy vectors which satisfy the Ingleton inequality and a large class of other linear rank inequalities. In Section 5 we prove the other half, *i.e.*, that all extremal rays of the 4-party Ingleton cone can be achieved using 5-party stabiliser states. We conclude with some open questions and challenges.

# 2 Preliminaries

## 2.1 Notation

We now introduce some notation needed to make precise the notion of entropy vectors and entropy cones. We will let $\mathcal{X} = \{A, B, C, \dots\}$ denote an index set of finite size $|\mathcal{X}| = N$ so that in many cases we could just assume that $\mathcal{X} = \{1, 2, \dots N\}$. However, it will occasionally be useful to consider the partition of some the index set into smaller groups, e.g, by grouping $A$ and $B$ as well as $D$ and $E$, $\mathcal{X}_5 = \{A, B, C, D, E\}$ gives rise to a 3-element $\mathcal{X}_3 = \{AB, C, DE\}$. When the size of $\mathcal{X}$ is important, we write $\mathcal{X}_N$.

An arbitrary $N$-partite quantum system is associated with a Hilbert space $\mathcal{H} = \bigotimes_{x\in\mathcal{X}} \mathcal{H}_x$ (with no restrictions on the dimension of the Hilbert spaces $\mathcal{H}_x$) with $|\mathcal{X}| = N$. The reduced states (properly called reduced density operators, but more often referred to as reduced density matrices (RDM) and also known as quantum marginals) are given by $\rho_J = \mathrm{Tr}_{J^c}\rho$, where $J^c = \mathcal{X} \setminus J$. This gives rise to a function $S : J \longmapsto S(J) = S(\rho_J)$ on the subsets $J \subset \mathcal{X}$. An element of the output of $S$ can be viewed as a vector in $\mathbf{R}^{2^{\mathcal{X}}}$, whose coordinates are indexed by the power set $2^{\mathcal{X}}$ of $\mathcal{X}$. We study the question of which such vectors arise from classical or quantum states, *i.e.*, when their elements are given by the entropies $S(\rho_J)$ of the reduced states of some $N$-party quantum state.

Classical probability distributions can be embedded into the quantum framework by restricting density matrices to those which are diagonal in a fixed product basis. A function $H : 2^{\mathcal{X}} \to \mathbf{R}$, associating real numbers to the subsets of a finite set $\mathcal{X}$, which satisfies the Shannon inequalities, eqs. (+), (SSA) and (MO), is called *poly-matroid*. By analogy with poly-matroids, we propose to call a function $S : 2^{\mathcal{X}} \to \mathbf{R}$ a *poly-quantoid*, if it satisfies (+), (SSA) and (WMO) [36].

We will let $\Gamma^C_{\mathcal{X}}$ and $\Gamma^Q_{\mathcal{X}}$ denote, respectively, the convex cone of vectors in a poly-matroid or poly-quantoid. The existence of non-Shannon entropy inequalities implies that there are vectors in $\Gamma^C_{\mathcal{X}}$ which can not be achieved by any classical state. Neither the classical nor quantum set of true entropy vectors is convex, because their boundaries have a complicated structure [4, 31, 35, 39]. However, the closure of the set of classical or quantum entropy vectors, which we denote $\overline{\Sigma}^C_{\mathcal{X}}$ or $\overline{\Sigma}^Q_{\mathcal{X}}$, respectively, is a closed convex cone. The inclusion $\overline{\Sigma}^C_{\mathcal{X}} \subset \Gamma^C_{\mathcal{X}}$ is strict for $N \geq 4$ [45]. It is an important open question whether or not this also holds in the quantum setting, *i.e.*, is the inclusion $\overline{\Sigma}^Q_{\mathcal{X}} \subseteq \Gamma^Q_{\mathcal{X}}$ also strict?

In this paper, we consider entropy vectors which satisfy additional inequalities known as linear rank inequalities, i.e. those satisfied by the dimensions of subspaces of a vector space and their intersections. A poly-matroid $H$ is called *linearly represented* if $H(J) = \dim \sum_{j\in J} V_j$ for subspaces $V_j$ of a common vector space $V$.

The simplest linear rank inequality is the 4-party Ingleton inequality (see section 3 below). Poly-matroids and poly-quantoids which also satisfy these additional inequalities will be denoted $\Lambda^C_{\mathcal{X}}$ and $\Lambda^Q_{\mathcal{X}}$ respectively.

## 2.2 Purification and complementarity

For statements about $J$ and $J^c = \mathcal{X} \setminus J$, it suffices to consider a bipartite quantum system with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. It is well-known that any pure state $|\psi_{AB}\rangle$ can be written in the form

$$|\psi_{AB}\rangle = \sum_k \mu_k |\phi^A_k\rangle \otimes |\phi^B_k\rangle \tag{4}$$

with $\mu_k > 0$ and $\{\phi_k^A\}$ and $\{\phi_k^B\}$ orthonormal. Indeed, this is an immediate consequence of the isomorphism between $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$, the set of linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$, and the singular value decomposition. It then follows that both $\rho_A$ and $\rho_B$ have the same non-zero eigenvalues $\mu_k^2$, and hence $S(\rho_A) = S(\rho_B)$.

This motivates the process known as purification. Given a density matrix $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, one can define the bipartite state

$$|\psi\rangle = \sum_k \sqrt{\lambda_k}\, |\phi_k\rangle \otimes |\phi_k\rangle$$

whose reduced density matrix $\mathrm{Tr}_B\, |\psi\rangle\langle\psi|$ is $\rho$.

Therefore, every vector in an $N$-party quantum entropy cone $\Sigma_N^Q$ can be obtained from the entropies of some reduced state of a $(N+1)$-party pure state $|\Psi\rangle$. In that case, we say that the entropy vector is realized by $|\Psi\rangle$.

In an abstract setting, we could define a cone $\widetilde{\Gamma}_{\mathcal{X}}^Q$ whose elements satisfy (+), (SSA) and the complementarity property $S(J) = S(J^c)$, and let $\Gamma_N^Q$ be the cone of vectors which arise as subvectors of $\widetilde{\Gamma}_{N+1}^Q$. Although we will not need this level of abstraction, this correspondence is used in Section 5.

## 2.3 Group inequalities

Consider a (finite) group $G$ and a family of subgroups $G_x < G$, $x \in \mathcal{X}$. Then, $H(J) = \log |G/G_J|$, with $G_J = \bigcap_{j \in J} G_j$ is a poly-matroid. In fact, Chan and Yeung [9] show that it is entropic because it can be realised by the random variables $X_j = gG_j \in G/G_j$ for a uniformly distributed $g \in G$. The fact that for two subgroups $G_1, G_2 < G$, the mappings

$$G/(G_1 \cap G_2) \longrightarrow G/G_1 \times G/G_2 \qquad \text{and} \qquad g(G_1 \cap G_2) \longmapsto (gG_1, gG_2),$$

are one-to-one [42], guarantees that indeed $H(X_J) = H(J)$.

Thus, the inequalities satisfied by poly-matroids, and more specifically entropic poly-matroids give rise to relations between the cardinalities of subgroups and their intersections in a generic group. Conversely, Chan and Yeung [9] have shown that every such relation for groups, is valid for all entropic poly-matroids. This result motivates the search for a similar, combinatorial or group theoretical, characterization of the von Neumann entropic poly-quantoids, and our interest in stabiliser states originally grew out of it.

However, it must be noted that if some subgroups of $G$ are not general, but, e..g, normal subgroups as in Theorem 6 below, then the Chan-Yeung correspondence breaks down. In this case further inequalities hold for the group poly-matroid that are not satisfied by entropic poly-matroids.

# 3 Linear rank inequalities

## 3.1 The Ingleton inequality

The classic *Ingleton inequality*, when stated in information theoretical terms, and as manifestly balanced, reads

$$\mathrm{Ing}(AB : CD) \equiv I(A:B|C) + I(A:B|D) + I(C:D) - I(A:B) \geq 0, \tag{ING}$$

where $A$, $B$, $C$ and $D$ are elements (more generally pairwise disjoint subsets) of $\mathcal{X}$. It was discovered by Ingleton [22] as a constraint on linearly represented matroids.

Although this inequality does not hold universally, it is of considerable importance, and continues to be studied [32, 37, 41, 36], particularly when reformulated as an inequality for subgroup ranks. In Theorem 11 we show that (ING) always holds for a special class of states. Before doing that, we give some basic properties first. Observe that (ING) is symmetric with respect to the interchanges $A \leftrightarrow B$ and $C \leftrightarrow D$, so that it suffices to consider special properties only for $A$ and $D$.

Because it is not always easy to see if a 4-party state $\rho_{ABCD}$ is the reduction of a pure stabiliser state, it is worth listing some easily checked conditions under which (ING) holds.

▶ **Proposition 1.** *The Ingleton inequality* (ING) *holds if any one of the following conditions holds.*

*a)* $\rho_{ABCD} = |\psi_{ABCD}\rangle\langle\psi_{ABCD}|$ *is any pure 4-party state.*

*b)* $\rho_{ABCD} = \rho_{ABC} \otimes \rho_D$ *or* $\rho_A \otimes \rho_{BCD}$

*c) The two-party component of the entropy vector for* $(\rho_{ABCD})$ *is symmetric under a partial exchange between* $(A, B)$ *and* $(C, D)$, i.e. *under any* one *(but not two) of the exchanges* $A \leftrightarrow C$, $B \leftrightarrow D$, $A \leftrightarrow D$ *or* $B \leftrightarrow C$.

**Proof.** To prove (a) it suffices to observe that

$$\begin{aligned}
\mathrm{Ing}(AB:CD) &= I(A:B|C) + S(AD) + S(BD) - S(D) - S(ABD) \\
&\quad + S(C) + S(D) - S(CD) - S(A) - S(B) + S(AB) \\
&= I(A:B|C) + S(AD) + S(AC) - S(A) - S(ACD) \\
&= I(A:B|C) + I(C:D|A) \geq 0.
\end{aligned}$$

To prove (b) observe that when $\rho_{ABCD} = \rho_{ABC} \otimes \rho_D$ then $I(A:B|D) = I(A:B)$ and $I(C:D) = 0$ so that (ING) follows immediately from (SSA). For $\rho_{ABCD} = \rho_A \otimes \rho_{BCD}$ the first, second and last terms in (ING) are zero so that it becomes simply $I(C:D) \geq 0$.

For (c) we observe that (ING) is equivalent to

$$I(B:C|A) + I(A:D|B) + R \geq 0 \quad \text{with} \quad R = S(BC) + S(AD) - S(CD) - S(AB). \tag{5}$$

The exchange $A \leftrightarrow C$ takes $R$ to $-R$. Thus, if $\rho_{ABCD}$ is symmetric under this exchange, then $R = 0$ and (ING) holds. The sufficiency of the other exchanges can be shown similarly. ◀

If (ING) holds, then all of the Matúš inequalities (3) also hold, since they add only conditional mutual informations $I(X:Y|Z) \geq 0$ to it. However, it is well-known that entropies do not universally obey the Ingleton inequality. A simple, well-known counterexample is given by independent and uniform binary variables $C$ and $D$, and $A = C \vee D$, $B = C \wedge D$. Then the first three terms in (ING) vanish, so that $\mathrm{Ing}(AB:CD) = -I(A:B) < 0$.

To obtain a quantum state which violates Ingleton, let $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0000\rangle + |1111\rangle\big)$ and

$$\rho_{ABCD} = \tfrac{1}{2}|\psi\rangle\langle\psi| + \tfrac{1}{4}|1010\rangle\langle 1010| + \tfrac{1}{4}|1001\rangle\langle 1001|. \tag{6}$$

All the reduced states $\rho_{ABC}, \rho_{BD}$, etc. are separable and identical to those of the state

$$\rho_{ABCD} \;=\; \tfrac{1}{4}|0000\rangle\langle 0000| + \tfrac{1}{4}|1111\rangle\langle 1111| + \tfrac{1}{4}|1010\rangle\langle 1010| + \tfrac{1}{4}|1001\rangle\langle 1001|.$$

corresponding to the classical example above. Therefore (6) violates the Ingleton inequality, but still satisfies all of the Matúš inequalities. Note that the state $|\psi\rangle$ is maximally entangled wrt the splitting $A$ and $BCD$. Additional quantum states with the same entropy vectors as classical states which violate Ingleton [32, 33] can be similarly constructed. However, we do not seem to know "genuinely quantum" counterexamples to the Ingleton inequality.

▶ **Question 2.** *Do there exist quantum states which violate Ingleton and are neither separable nor have the same entropy vectors as some classical state?*

## 3.2 Families of inequalities

When the subsystem $C$ or $D$ is trivial, the Ingleton inequality reduces to the 3-party SSA inequality, $I(A:B|C) \geq 0$; when subsystem $A$ or $B$ is trivial, it reduces to the 2-party subadditivity inequality $I(C:D) \geq 0$. This suggests that the Ingleton inequality is a member of a more general family of $N$-party inequalities. In 2011, Kinser [23] found the first such family, which can be written (for $N \geq 4$) as

$$K[N] = I(1:N|3) + H(1N) - H(12) - H(3N) + H(23) + \sum_{k=4}^{N} I(2:k-1|k) \geq 0. \qquad (7)$$

This is equivalent to the Ingleton inequality when $N = 4$.

▶ Remark. As in the proof of Proposition 1(c), it can be shown that Kinser's inequalities hold if $\rho$ is symmetric with respect to the interchange $1 \leftrightarrow 3$ or $2 \leftrightarrow N$. They also hold if $\rho_{1,2,\ldots N} = \rho_2 \otimes \rho_{1,3,\ldots N}$ One can ask if part (a) of Theorem 1 can be extended to the new inequalities, *i.e.*, do they hold for hold for $N$-party pure quantum states?

## 3.3 Inequalities from common information

Soon after Kinser's work, another group [14] found new families of linear rank inequalities for poly-matroids for $N > 4$ that are independent of both Ingleton's inequality and Kinser's family. In the 5-party case, they found a set of 24 inequalities which generate all linear rank inequalities for poly-matroids. Moreover, they give an algorithm which allows one to generate many more families of linear rank inequalities based on the notion of common information, considered much earlier in [1, 2, 16] and used below. However, it was shown in [8] that there are $N$-party linear rank inequalities that cannot be obtained from the process described in [14].

▶ **Definition 3.** In a poly-matroid $H$ on $\mathcal{X}$, two subsets $A$ and $B$ are said to *have a common information*, if there exists an extension of $H$ to a poly-matroid on the larger set $\mathcal{X} \dot\cup \{\zeta\}$, such that $H(\{\zeta\} \cup A) - H(A) =: H(\zeta|A) = 0$, $H(\{\zeta\} \cup B) - H(B) =: H(\zeta|B) = 0$ and $H(\zeta) = I(A:B)$.

Here we used $H(Z|A) = H(AZ) - H(A)$ to denote the conditional entropy. For completeness we include a proof (courtesy of a Banff talk by Dougherty) of the following result, as well as a proof of Lemma 5 below, which appear in [14].

▶ **Proposition 4.** *Let $h$ be a poly-matroid on $\mathcal{X}$, and $A, B, C, D \subset \mathcal{X}$ such that $A$ and $B$ have a common information. Then the Ingleton inequality (ING) holds for $A$, $B$, $C$ and $D$.*

**Proof.** Let $\zeta$ be a common information of $A$ and $B$. Then, using $H(F|A) \geq H(F|AC)$ in Lemma 5 below, and letting $F = \zeta$, gives

$$I(A:B|C) + H(\zeta|A) \geq I(\zeta:B|C).$$

Using this a total of six times, we obtain

$$\begin{aligned} I(A:B|C) &+ I(A:B|D) + I(C:D) + 2H(\zeta|A) + 2H(\zeta|B) \\ &\geq I(A:\zeta|C) + I(A:\zeta|D) + I(C:D) + 2H(\zeta|A) \\ &\geq I(\zeta:\zeta|C) + I(\zeta:\zeta|D) + I(C:D) \\ &= H(\zeta|C) + H(\zeta|D) + I(C:D) \geq H(\zeta|C) + I(\zeta:D) \geq I(\zeta:\zeta) = H(\zeta). \end{aligned}$$

Inserting the conditions for $\zeta$ being a common information, completes the proof. ◀

▶ **Lemma 5.** *In a poly-matroid $H$ on a set $\mathcal{X}$ with subsets $A, B, C, F \subset \mathcal{X}$.*

$$I(A:B|C) + H(F|AC) \geq I(F:B|C) \tag{8}$$

**Proof.** By a direct application of the poly-matroid axioms:

$$\begin{aligned} I(A:B|C) + H(F|AC) - I(F:B|C) &= H(B|FC) - H(B|AC) + H(F|AC) \\ &= H(BCF) + H(ACF) - H(CF) - H(ABC) \qquad (9) \\ &\geq H(BCF) + H(ACF) - H(CF) - H(ABCF) \\ &= I(A:B|CF) \geq 0, \qquad (10) \end{aligned}$$

where we used only algebraic identities, SSA and monotonicity. ◀

In a linearly represented poly-matroid, (ING) is universally true: There, $H(J) = \dim V_J$, with $V_J = \sum_{j\in J} V_j$ for a family of linear subspaces $V_j \subset V$ of a vector space. The common information of any $A, B \subset \mathcal{X}$ is constructed by defining $V_\zeta = V_A \cap V_B$.

▶ **Theorem 6.** *Any linear rank inequality for a poly-matroid obtained using common information and the poly-matroid inequalities, also holds for a group poly-matroid when its defining subgroups are normal.*

**Proof.** It suffices to show that when $G_A, G_B \triangleleft G$ are normal subgroups for $A, B \subset \mathcal{X}$, then $A$ and $B$ have a common information given by $G_\zeta = G_A G_B \triangleleft G$ (the latter from the normality of $G_A$ and $G_B$). The first two conditions for a common information are clearly satisfied, as $G_A, G_B \subset G_A G_B$, and the third follows from the well-known natural isomorphisms [42]

$$G/(G_A G_B) \approx {(G/G_A)}\big/{((G_A G_B)/G_A)} \quad \text{and} \quad (G_A G_B)/G_A \approx G_B/(G_A \cap G_B),$$

which imply

$$\begin{aligned} H(\zeta) = \log|G/(G_A G_B)| &= \log|G/G_A| - \log|(G_A G_B)/G_A| \\ &= \log|G/G_A| + \log|G/G_B| - \log|G/(G_A \cap G_B)| = I(A:B). \end{aligned}$$ ◀

# 4 Entropies of stabiliser states

## 4.1 Stabiliser groups and stabiliser states

Motivated by the stabiliser states encountered in the extremal rays of $\Sigma_2$, $\Sigma_3$ and $\Sigma_4$, we now focus on (pure) stabiliser states, *i.e.* 1-dimensional quantum codes. Stabilizer codes have emerged in successively more general forms. We use the formulation described by Klappenecker and Rötteler [24, 25] (following Knill [26]) which relies on the notion of *abstract error group*: This is a finite subgroup $W < \mathcal{U}(\mathcal{H})$ of the unitary group of a (finite dimensional) Hilbert space $\mathcal{H}$, which satisfies the following axioms:

1. The center $C$ of $W$ consists only of multiples of the identity matrix ("scalars"): $C \subset \mathbf{C}\mathbb{1}$.
2. $\widehat{W} \equiv W/C$ is an abelian group of order $|\mathcal{H}|^2$, called the *abelian part* of $W$.
3. For all $g \in W \setminus C$, $\operatorname{Tr} g = 0$.

Note that conditions 1 and 2 imply that $W$ is non-abelian; whereas condition 2 says that the non-commutativity is played out only on the level of complex phases: for $g, h \in W$,

$$gh = \omega(g,h)hg, \ \text{with } \omega(g,h) \in \mathbf{C}.$$

Finally, condition 3 means that $g, h \in W$ in different cosets modulo $C$ are orthogonal with respect to the Frobenius (or Hilbert-Schmidt) inner product: $\operatorname{Tr} g^\dagger h = 0$. It is known that $\widehat{W}$ is a direct product of an abelian group $T$ with itself, such that $|T| = |\mathcal{H}|$.

▶ **Example 7** (Discrete Weyl-Heisenberg group)**.** Let $\mathcal{H}$ be a $d$-dimensional Hilbert space, with a computational orthonormal basis $\{|j\rangle\}_{j=0}^{d}$. Define discrete Weyl operators

$$X|j\rangle = |j+1\rangle \mod d, \qquad Z|j\rangle = e^{j\frac{2\pi i}{d}}|j\rangle.$$

They are clearly both of order $d$, and congruent via the discrete Fourier transform. The fundamental commutation relation, $XZ = e^{2\pi i/d}ZX$ ensures that the group $W$ generated by $X$ and $Z$ is finite, and indeed an abstract error group with center $C = \left\{e^{j\frac{2\pi i}{d}} : j = 0, \dots, d-1\right\}$.

Note that the tensor product of abstract error groups is again an abstract error group. Now, assume that each party $x \in \mathcal{X}$ of the composite quantum system can be associated with an abstract error group $W_x < \mathcal{U}(\mathcal{H}_x)$ of unitaries with center $C_x$, which satisfy $W_x \supset C_x \subset \mathbf{C}\mathbb{1}$, such that $\widehat{W_x} = W_x/C_x$ is abelian and has cardinality $d_x^2 \equiv |\mathcal{H}_x|^2$. Let $W \equiv \bigotimes_{x\in\mathcal{X}} W_x$ be the tensor product abstract error group, acting on $\mathcal{H} = \bigotimes_{x\in\mathcal{X}} \mathcal{H}_x$. For any subgroup $\Gamma < W$, we let $\widehat{\Gamma} = (C\Gamma)/C \simeq \Gamma/(\Gamma \cap C)$ denote the quotient of $\Gamma$ by the center of $W$.

Stabiliser codes [17, 5] are subspaces of $\mathcal{H}$ which are simultaneous eigenspaces of abelian subgroups of $W$.

Consider a maximal abelian subgroup $G < W$, which contains the center $C = \bigotimes_{x\in\mathcal{X}} C_x < \mathbf{C}\mathbb{1}$ of $W$, so that $\widehat{G} = G/C$ has cardinality $\sqrt{|\widehat{W}|} = |\mathcal{H}| = \prod_{j=1}^{N} |\mathcal{H}_j|$. Since $G$ is abelian it has a common eigenbasis, each state of which is called a stabiliser state $|\psi\rangle$.

More generally, let $G < W$ be any abelian subgroup of an abstract error group $W < \mathcal{U}(\mathcal{H})$. Since all $g \in G$ commute, they are jointly diagonalisable: let $P$ be one of the maximal joint eigenspace projections. Then for $g \in G$, $gP = \chi(g)P$, for a complex number $\chi(g)$. Thus $\chi : G \longrightarrow \mathbf{C}$ is necessarily the character of a 1-dimensional group representation, which gives rise to the following expression for $P$:

$$P = \frac{1}{|G|} \sum_{g\in G} \overline{\chi(g)}\, g. \tag{11}$$

If $\chi(g_0) = 1$ and $g = c\,g_0$ is in the coset $g_0\,C$, then $c = \chi(g)$ and $\overline{\chi(g)}\,g = g_0$. Thus, $G_0 = \{g \in G : \chi(g) = 1\}$ is a subgroup of $G$ isomorphic to $\widehat{G} = G/C$ and (11) can be rewritten as

$$P = \frac{1}{|G_0|} \sum_{g\in G_0} g. \tag{12}$$

Since $g^{-1} = g^{\dagger}$ this sum is self-adjoint, and

$$P^2 = \frac{1}{|G_0|^2} \sum_{g,h\in G_0} gh = \frac{1}{|G_0|} \sum_{g\in G_0} g = P,$$

so that (12) is indeed a projection.

**Note:** The above reasoning is true because we assumed that $\chi(g)$ records the eigenvalues of $g$ on the eigenspace with projector $P$; as such, it has the property $\chi(t\mathbb{1}) = t$ for $t \in \mathbf{C}$. For a general character $\chi$, however, only $G_0 < \chi^{-1}(1)$ holds.

Because of the importance of the case of rank one projections, we summarize the results above in the case of maximal abelian subgroups.

▶ **Theorem 8.** *Let $G$ be a maximal abelian subgroup of an abstract error group $W$ with center $C$. Any simultaneous eigenstate of $G$ can be associated with a subgroup $G_0 \simeq G/C$ for which* $|\psi\rangle\langle\psi| = \frac{1}{|G_0|} \sum_{g\in G_0} g.$ ◀

▶ Remark. The use of the trivial representation is not essential in the expression above. It was used only to define $G_0$. Once this has been done, one can use the (1-dim) irreducible representations of $G_0$ to describe the orthonormal basis of stabiliser states associated with $G$. Let $\chi_k(g)$ denote the $d = |G_0|$ irreducible representations of $G_0$ and define

$$|\psi_k\rangle\langle\psi_k| = \tfrac{1}{|G_0|} \sum_{g \in G_0} \chi_k(g) g. \tag{13}$$

Then the orthogonality property of group characters implies that $\operatorname{Tr} |\psi_j\rangle\langle\psi_j||\psi_k\rangle\langle\psi_k| = |\langle\psi_j|\psi_k\rangle|^2 = \delta_{jk}$.

## 4.2 Entropies of stabiliser states

The next result seems to have been obtained independently by several groups [20, 10, 11].

▶ **Proposition 9.** *For a pure stabiliser state $\rho = |\psi\rangle\langle\psi|$ with associated error group $G < W$, and any $J \subset \mathcal{X}$, the entropy*

$$S(J) = S(\rho_J) = \log \frac{d_J}{|\widehat{G_J}|}. \tag{14}$$

*Here, $d_J = \prod_{x \in J} d_x$ and*

$$G_J \equiv \{\otimes_{x \in \mathcal{X}} g_x \in G : \forall x \notin J \; g_x = \mathbb{1}\} \subset G,$$

*and $\widehat{G_J} = G_J / C_J$ is the quotient of $G_J$ with respect to the center $C_J = G_J \cap C$.*

**Proof.** It is enough to consider a bipartite system with local error groups $W_A$ and $W_B$, by considering party $A$ all systems in $J$, and $B$ all systems in $\mathcal{X} \setminus J$. Then,

$$|\psi\rangle\langle\psi| = \frac{1}{|\widehat{G}|} \sum_{(g_A, g_B) \in \widehat{G}} g_A \otimes g_B.$$

Since $\operatorname{Tr} g_B = 0$ unless $g_B = \mathbb{1}$ and $|\widehat{G}| = d_A d_B$, this implies

$$\begin{aligned}
\rho_A = \operatorname{Tr}_B |\psi\rangle\langle\psi| &= \frac{1}{|\widehat{G}|} \sum_{(g_A, g_B) \in \widehat{G}} (\operatorname{Tr} g_B)\, g_A \\
&= \frac{1}{|\widehat{G}|} \sum_{g_A \in \widehat{G_A}} |\mathcal{H}_B|\, g_A \\
&= \frac{1}{|\mathcal{H}_A|} \sum_{g_A \in \widehat{G_A}} g_A = \frac{|\widehat{G_A}|}{d_A} \Bigg( \frac{1}{|\widehat{G_A}|} \sum_{g_A \in \widehat{G_A}} g_A \Bigg).
\end{aligned}$$

Since, $\operatorname{Tr} \rho_A = 1$ , the last line implies that $\rho_A$ is proportional to a projector of rank $\frac{d_A}{|\widehat{G_A}|}$. Thus, its entropy is simply $S(\rho_A) = \log \frac{d_A}{|\widehat{G_A}|}$. ◀

The following corollary is the key to our main result, Theorem 11.

▶ **Corollary 10.** *For a pure stabiliser state as in Proposition 9, the entropy of the reduced state $\rho_J$ satisfies*

$$S(J) = S(\rho_J) = \log \frac{|\widehat{G}|}{|\widehat{G_{J^c}}|} - \log d_J = \log \frac{|\widehat{G}|}{|\widehat{G_J}|} - \log d_{J^c}. \tag{15}$$

**Proof.** As in Proposition 9, it suffices to consider the bipartite case. Since $|\psi\rangle\langle\psi|$ is pure,

$$S(\rho_A) = S(\rho_B) = \log \frac{d_B}{|\widehat{G_B}|} = \log \frac{d_A d_B}{|\widehat{G_B}|} - \log d_A .$$

Since $d_A d_B = |\widehat{G}|$ this gives the desired result. ◀

▶ **Theorem 11.** *Any pure stabiliser state $\rho = |\psi\rangle\langle\psi|$ on an 5-party system gives rise to 4-party reduced states whose entropies satisfy the Ingleton inequality.*

**Proof.** By Corollary 10, we have

$$S(J) = \log \frac{|\widehat{G}|}{|\widehat{G_{J^c}}|} - \sum_{x \in J} \log d_x \, . \tag{16}$$

The first term $H(J) = \log \frac{|\widehat{G}|}{|\widehat{G_{J^c}}|}$ is a Shannon entropy of the type used in [9]. To be precise, observe that $\widehat{G_{J^c}} = \bigcap_{x\in J} \widehat{G_{\mathcal{X}\setminus x}}$. Moreover, since $\widehat{G}$ and its subgroups $\widehat{G_{J^c}}$ are abelian, this implies that the entropy vector for each of the 4-party reduced states satisfies the Ingleton inequality. (This was observed in [9] and also follows from Theorem 6.)

To complete the argument, it suffices to observe that the Ingleton inequality is balanced, so that the Ingleton expression is identically zero for the sum-type "rank function" from the second term in (16), *i.e.* $h_0(J) \equiv \sum_{x\in J} \log d_x$ defines a poly-matroid satisfying (ING) with equality. ◀

Any linear combination of mutual informations and conditional mutual informations is a balanced expression (and vice versa, any balanced expression can be written as such a linear combination). Kinser's family of inequalities is balanced, which can be seen by inspection of (7). It also holds by construction for the inequalities obtained from [14, Thms. 3 and 4] and, more generally, any inequality obtained using a "common information" as in [14]. Therefore, we can conclude using the same argument as above that

▶ **Theorem 12.** *Any pure stabiliser state on an $(N+1)$-party system generate an $N$-party entropy vector which satisfies the Kinser [23] family (7) of inequalities, and more generally those of Dougherty et al. [14].*

A consequence of Theorem 11 is that the Matúš family of inequalities holds for stabiliser states; however, rays generated by the stabiliser state entropy vectors do not span the entropy cone $\overline{\Sigma}_4^Q$. In fact, from the proof of Theorem 11, we see that *every* balanced inequality that holds for the Shannon entropy, holds automatically for stabilizer quantum entropies.[1] Note also that apart from (MO), all other necessary entropy inequalities for the Shannon entropy are balanced [6].

## 5 The 4-party quantum entropy cone

By direct calculation using symbolic software, we can compute the extreme rays of 4-party poly-quantoids plus Ingleton inequalities. The results are given (up to permutation) as rays 0 to 6 in Table 1 below, as elements of the 5-party cone $\widetilde{\Gamma}_{4+1}^Q$ (on subsets of $\{a,b,c,d,e\}$) of vectors which satisfy (+) (SSA) and the complementarity property $S(J) = S(J^c)$ as described at the end of Section 2.2.

[1] We are grateful to D. Gross and M. Walter, whose paper [18] made us aware of this observation.

| subset\ray | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 1 | 1 | 2 | 1 | 1 |
| b | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| c | 0 | 1 | 1 | 1 | 1 | 2 | 1 |
| d | 0 | 1 | 1 | 1 | 1 | 2 | 2 |
| e ($\widehat{=}$ abcd) | 0 | 0 | 0 | 1 | 1 | 2 | 2 |
| ab | 0 | 1 | 2 | 1 | 3 | 2 | 2 |
| ac | 1 | 1 | 2 | 1 | 3 | 3 | 2 |
| ad | 1 | 1 | 2 | 1 | 3 | 3 | 2 |
| ae ($\widehat{=}$ bcd) | 1 | 1 | 1 | 1 | 3 | 3 | 2 |
| bc | 1 | 1 | 2 | 1 | 2 | 3 | 2 |
| bd | 1 | 1 | 2 | 1 | 2 | 3 | 2 |
| be ($\widehat{=}$ acd) | 1 | 1 | 1 | 1 | 2 | 3 | 2 |
| cd | 0 | 1 | 2 | 1 | 2 | 2 | 2 |
| ce ($\widehat{=}$ abd) | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| de ($\widehat{=}$ abc) | 0 | 1 | 1 | 1 | 2 | 2 | 2 |

**Table 1** Extreme rays of the 4-party quantum Ingleton cone

The following stabiliser states found by Ibinson [21] (some of which were known earlier) realise entropy vectors on the rays 1 through 6 shown in Table 1.

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)_{ab}|000\rangle_{cde}, \tag{R1}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\big(|0000\rangle + |1111\rangle\big)_{abcd}|0\rangle_{e}, \tag{R2}$$

$$|\psi_3\rangle = \frac{1}{3}\sum_{i,j=0,1,2} |i\rangle_a|j\rangle_b|i\oplus j\rangle_c|i\oplus 2j\rangle_d|0\rangle_e, \tag{R3}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}\big(|00000\rangle + |11111\rangle\big)_{abcde}, \tag{R4}$$

$$|\psi_5\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_{a'}|0_L\rangle_{a''bcde} + |1\rangle_{a'}|1_L\rangle_{a''bcde}\big), \tag{R5}$$

$$|\psi_6\rangle = \frac{1}{\sqrt{27}}\sum_{i,j,k=0,1,2} |i\rangle_a|j\rangle_b|i\oplus j\rangle_{c'}|k\rangle_{c''}|i\oplus j\rangle_{d'}|k\rangle_{d''}|i\oplus j\rangle_{e'}|k\rangle_{e''}, \tag{R6}$$

where in eq. (R5), $|0_L\rangle$ and $|1_L\rangle$ are the logical 0 and 1 on the famous 5-qubit code [27, 3]. These are also extremal rays of the quantum entropy cone $\Sigma_4^Q$. In addition, ray 0 in Table 1 is realised by the (stabiliser!) state

$$|\psi_0\rangle = \frac{1}{2}\sum_{i,j=0,1} |i\rangle_A|j\rangle_B|i\oplus j\rangle_C|ij\rangle_D|ij\rangle_E. \tag{R0}$$

on $1+1+1+2+2$ qubits.

Let us call an $N$-party poly-quantoid *stabiliser-represented*, if it is in the closure of the cone generated by the entropy vectors of $(N+1)$-party stabiliser states in the sense used above. Then the above reasoning proves the following analogue of a theorem by Hammer, Romashchenko, Shen and Vereshchagin [19]:

▶ **Theorem 13.** *A 4-party poly-quantoid is stabiliser-represented if and only if it satisfies the Ingleton inequality (and all its permutations).* ◀

It seems reasonable to conjecture that the closure of the cone generated by the entropy vectors of stabiliser states is identical to that obtained when inequalities obtained from common information as in [14] are added to the classical ones. However, it is not even clear if stabiliser states satisfy the additional linear rank inequalities shown to exist in [8].

## 6 Conclusion

The difficult question of whether or not the quantum entropy satisfies inequalities beyond positivity and SSA remains open for four or more parties.

Do quantum states which do not satisfy Ingleton always lie within the classical part of the quantum entropy cone? We know that the quantum entropy cone $\overline{\Sigma}_{\mathcal{X}}^{Q}$ is strictly larger than the classical one $\overline{\Sigma}_{\mathcal{X}}^{C}$. Recall that $\Lambda_4^{C,Q}$ denotes the polyhedral cones formed from the classical inequalities (in each case) and the Ingleton inequality. We want to know whether or not $\overline{\Sigma}_4^{Q} \setminus \Lambda_4^{Q}$ is strictly larger than $\overline{\Sigma}_4^{C} \setminus \Lambda_4^{C}$, *i.e.*, are there quantum states whose entropy vectors do not satisfy the Ingleton inequality and are not equal to any vector in the closure of the classical entropy cone, $\overline{\Sigma}_4^{C}$? If the answer is negative, then 4-party quantum entropy vectors must also satisfy the new non-Shannon inequalities.

It seems that a better understanding of quantum states which do not satisfy (ING) may be the key to determining whether or not quantum states satisfy the classical non-Shannon inequalities.

This question extends naturally to the 5-party case, in which all linear rank inequalities are known from [14]. However, for more parties, one can ask the same question for both the cones associated with inequalities obtained using one common information as in [14], and for the cones obtained using all linear rank inequalities. Although we know from [7, 8] that additional inequalities are required, we do not even have explicit examples to consider.

**Related work** After completion of the present research, we became aware of independent work by Gross and Walter [18], who use discrete phase space methods for stabilizer states to show that the entropies of stabilizer states satisfy all balanced classical entropy inequalities. Indeed, this can also be seen from our formula for the reduced state entropies in Corollary 10.

**Acknowledgements** Portions of this work were done when FM, MBR and AW were participating in workshops at the IMS, National University of Singapore, and at the Center for Sciences "Pedro Pascual" in Benasque, Spain. This collaboration was a direct consequence of their participation in a workshop on matroids held at the Banff International Research Station in 2009.

The work of FM was partially supported by Grant Agency of the Czech Republic under Grant 13-20012S. The work of MBR was partially supported by NSF grant CCF-1018401. NL and AW acknowledge financial support by the European Commission (STREP "QCS" and Integrated Project "QESSENCE"), AW furthermore support by the ERC (Advanced Grant "IRQUAT"), a Royal Society Wolfson Merit Award and a Philip Leverhulme Prize. The Centre for Quantum Technologies is funded by the Singapore Ministry of Education and the National Research Foundation as part of the Research Centres of Excellence programme.

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