Journal of Convex Analysis
Volume 20 (2013), No. 1, 43–66

# Qualitative Stability of a Class of Non-Monotone Variational Inclusions. Application in Electronics

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Received: October 9, 2011
Revised manuscript received: February 1, 2012

The main concern of this paper is to investigate some stability properties (namely Aubin property and isolated calmness) of a special non-monotone variational inclusion. We provide a characterization of these properties in terms of the problem data and show their importance for the design of electrical circuits involving nonsmooth and non-monotone electronic devices like DIAC (DIode Alternating Current). Circuits with other devices like SCR (Silicon Controlled Rectifiers), Zener diodes, thyristors, varactors and transistors can be analyzed in the same way.

*Keywords:* Hemivariational inequalities, nonsmooth and variational analysis, Aubin property, isolated calmness, non-regular electrical circuits

*1991 Mathematics Subject Classification:* 46N10, 47N10, 47J20, 49J40, 34A60

## 1. Introduction and problem formulation

In this paper, we intend to analyze some qualitative stability properties for a non-monotone variational inclusion of the following form:

$$p \in f(z) + B^T \bar{\partial} J(Bz), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping and $B \in \mathbb{R}^{m \times n}$, $p \in \mathbb{R}^n$ are a given matrix and a given vector with $m \leq n$, respectively. Throughout the paper, we assume

*This author acknowledges the support by the Grant Agency of the Czech Academy of Sciences (grant No. IAA 100750802), by the Grant Agency of the Czech Republic (Grant P201/12/0671) and by the Australian Research Council (Project DP 110102011).

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

that $B$ is *surjective*. The function $J : \mathbb{R}^m \to \mathbb{R}$, $x = (x_1, \ldots, x_m) \mapsto J(x)$ is defined by

$$J(x) = j_1(x_1) + j_2(x_2) + \ldots + j_m(x_m), \quad \forall x = (x_1, \ldots, x_m) \in \mathbb{R}^m, \tag{2}$$

where $j_i : \mathbb{R} \to \mathbb{R}$ are supposed to be locally Lipschitz for every $i = 1, 2, \ldots, m$. For a locally Lipschitz function $\varphi : \mathbb{R}^N \to \mathbb{R}$, $\bar{\partial}\varphi$ denotes its *Clarke generalized gradient* ([8]) defined by

$$\bar{\partial}\varphi(x) := \{p \in \mathbb{R}^N : \varphi^\circ(x; y) \geq \langle p, y \rangle, \ \forall y \in \mathbb{R}^N\},$$

where $\varphi^\circ(x; y)$ stands for the Clarke generalized directional derivative of $\varphi$ at $x$ in the direction $y$ defined by

$$\varphi^\circ(x; y) := \limsup_{\substack{\lambda \to 0^+ \\ z \to x}} \frac{\varphi(z + \lambda y) - \varphi(z)}{\lambda}. \tag{3}$$

Various problems arising in mechanical and electrical engineering can be formulated in the form (1). We will show in this paper that the model (1) is of particular interest in the theory of electrical circuits containing nonsmooth electronic devices like DIACs (DIode Alternating Current), Silicon Controlled Rectifiers, thyristors, varactors and transistors.

Modeling of electrical circuits is usually performed by using Kirchhoff's laws and the Ampere-Volt (or I-V) characteristics for each involved electronic device which are graphs of mappings (possibly set-valued) relating the voltage $V$ to the current $i$. Electrical devices like resistors, inductances and capacitors are usually described by a single-valued (I-V) mapping (linear or non-linear) and are called "smooth electrical devices". On the other hand, semiconductors like diodes, DIACs, Silicon Controlled Rectifiers (SCR) and transistors are defined generally by means of set-valued (I-V) characteristics and are called "nonsmooth electrical devices". The mathematical modeling of electronic circuits involving smooth electrical devices leads generally to algebraic/differential equations that can be studied by using classical mathematical analysis. On the other hand, mathematical modeling of electronic circuits involving nonsmooth electrical devices leads generally to variational/differential inclusions that can be handled by using tools from variational and nonsmooth analysis. An efficient approach for the treatment of monotone set-valued maps in the framework of variational inequalities relies on the notion of convex superpotential introduced by J. J. Moreau [16] and generalized by P. D. Panagiotopoulos [18] to the case of non-monotone set-valued maps by using the generalized gradient of F. H. Clarke [8] for locally Lipschitz functions.

The usage of tools from variational and nonsmooth analysis for the study of electrical circuits is a fairly recent and quite promising topic of research [1, 2, 4, 11]. To the best of our knowledge, this paper is the first work dealing with the qualitative stability of electronic circuits containing nonsmooth electrical devices with non-monotone (I-V)-characteristics. For the monotone case, we can cite e.g. [5], [4], [3], [11].

In electrical engineering, one usually uses a circuit simulator for predicting the behavior of an electronic circuit. Most used software, for the smooth case, are based on various versions of SPICE (Simulation Program with Integrated Circuits Emphasis). The simulation of nonsmooth electronic circuits are well-known to be a difficult task. Due to the lack of smoothness, classical mathematical analysis is not applicable and requires natural extensions.

This paper is devoted to *qualitative stability analysis* of (1) under the structural assumption that

$$f(x) = Mx - q, \quad \forall x \in \mathbb{R}^n, \tag{4}$$

where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ are a given matrix and a given vector. Without loss of generality, we will suppose that $q = 0$.

By stability analysis we mean a study of the local behavior of the *solution map* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$S(p) = \{z \in \mathbb{R}^n \ : \ p \in Mz - q + B^T \partial J(Bz)\}, \tag{5}$$

around a reference point $(\bar{p}, \bar{z}) \in \mathrm{Gr}(S)$.

The structure of the paper is as follows: In Section 2 we recall the definitions of several basic notions from nonsmooth analysis and state some principal results, substantial in our development. Section 3 in devoted to characterizations and criteria (sufficient conditions) for the Aubin property of $S$, whereas in Section 4 we examine in the same way the isolated calmness of $S$. Section 5 deals then with the application of the derived conditions to concrete electronic devices, namely the DIAC.

The used notation is basically standard. For $x, y \in \mathbb{R}^n$, $\langle x, y \rangle$ stands for the Euclidean scalar product on $\mathbb{R}^n$ and $\|x\| = \sqrt{\langle x, x \rangle}$ is the corresponding norm. $\mathbb{B}$ is the unit ball, $\delta_\Omega$ is the indicatory function of a set $\Omega$ and $I_n$ denotes the unit matrix in $\mathbb{R}^n$. For a set-valued map (multifunction) $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$,

$$\mathrm{Gr}(\Phi) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, y \in \Phi(x)\}$$

denotes its graph. The Kuratowski-Painlevé upper (outer) limit of $\Phi$ as $x \to \bar{x}$, denoted by $\mathrm{Limsup}_{x \to \bar{x}} \Phi(x)$, is defined by

$$\operatorname*{Limsup}_{x \to \bar{x}} \Phi(x) := \{x^* \in \mathbb{R}^n \ : \ \exists x_k \to \bar{x}, \, x_k^* \to x^* \text{ with } x_k^* \in \Phi(x_k) \, \forall k \in \mathbb{N}\}. \tag{6}$$

For more details see [23] (Definitions 4.1).

Let $a_1, a_2, \ldots a_p \in \mathbb{R}^n$ some given vectors. The conical hull of $\{a_1, a_2, \ldots a_p\}$ is defined by

$$\operatorname{cone}\{a_1, a_2, \ldots a_p\} = \left\{ \sum_{i=1}^{p} \lambda_i a_i \ : \ \lambda_i \geq 0, \ i = 1, 2, \ldots, p \right\}. \tag{7}$$

## 2. Mathematical tools

In this section, we provide definitions and properties of some essential notions from nonsmooth analysis that will be used throughout this paper. For more details we refer to [10, 13, 23].

- *Aubin Property*: Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(S)$. $S$ is said to have the *Aubin property around* $(\bar{x}, \bar{y})$ if there exists a constant $\kappa \geq 0$ and neighborhoods $\mathcal{U} \subset \mathbb{R}^n$ of $\bar{x}$ and $\mathcal{V} \subset \mathbb{R}^m$ of $\bar{y}$ such that
$$S(x') \cap \mathcal{V} \subset S(x) + \kappa \|x' - x\| \mathbb{B}, \text{ for all } x,\, x' \in \mathcal{U}. \tag{8}$$
The Lipschitz modulus of $S$ at $\bar{x}$ for $\bar{y}$, denoted by $\mathrm{lip}(S; (\bar{x}, \bar{y}))$, is defined by
$$\begin{aligned} \mathrm{lip}(S; (\bar{x}, \bar{y})) = \inf\{\kappa \in \mathbb{R}^+ \,:\, \exists\, \mathcal{U} \in \mathcal{N}(\bar{x}),\, \mathcal{V} \in \mathcal{N}(\bar{y}) \\ \text{such that condition (8) is satisfied}\}, \end{aligned} \tag{9}$$
where the notation $\mathcal{N}(x)$ stands the collection of all neighborhoods of $x$.
- *Isolated calmness*: The set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be *isolatedly calm* at $(\bar{x}, \bar{y})$ if there exists a constant $\kappa \geq 0$ and neighborhoods $\mathcal{U} \subset \mathbb{R}^n$ of $\bar{x}$ and $\mathcal{V} \subset \mathbb{R}^m$ of $\bar{y}$ such that
$$S(x) \cap \mathcal{V} \subset \bar{y} + \kappa \|x - \bar{x}\| \mathbb{B}, \text{ for all } x \in \mathcal{U}. \tag{10}$$
We note that the linear mapping associated with every matrix $A \in \mathbb{R}^{m \times n}$ is isolatedly calm at any point. The isolated calmness of its generalized inverse $A^{-1}(\cdot)$ is equivalent to its injectivity, i.e., $A^{-1}(0) = \{0\}$ . More generally, for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ (supposed sufficiently smooth around $\bar{x}$), the inverse $f^{-1}(\cdot)$ is isolatedly calm at $(f(\bar{x}), \bar{x})$ if and only if the derivative mapping $\nabla f(\bar{x})$ is injective.
In [10] it is shown that a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, whose graph is the union of finitely many polyhedral convex sets, is isolatedly calm at $(\bar{x}, \bar{y})$ if and only if $\bar{y}$ is an isolated point of $S(\bar{x})$.
- Let $A \subset \mathbb{R}^n$ be a nonempty subset of $\mathbb{R}^n$ and $\bar{x} \in A$. The *Bouligand tangent (contingent) cone* to $A$ at $\bar{x}$ is defined by
$$T_A(\bar{x}) = \operatorname*{Limsup}_{t \to 0^+} \frac{A - \bar{x}}{t}, \tag{11}$$
or equivalently,
$$T_A(\bar{x}) = \{d \in \mathbb{R}^n \,:\, \exists\, t_k \downarrow 0,\, d_k \to d \,:\, \bar{x} + t_k d_k \in A,\ \forall k \in \mathbb{N}\}. \tag{12}$$
- The *regular (Fréchet) normal cone* to a subset $A \subset \mathbb{R}^n$ at $\bar{x}$, denoted by $\widehat{N}_A(\bar{x})$, is defined by
$$\widehat{N}_A(\bar{x}) = \{x^* \in \mathbb{R}^n \,|\, \langle x^*, x \rangle \leq 0 \text{ for all } x \in T_A(\bar{x})\} \tag{13}$$
$$= \left\{ x^* \in \mathbb{R}^n \,\Big|\, \limsup_{\substack{x \to \bar{x} \\ x \in A}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}. \tag{14}$$
- The *limiting (Mordukhovich) normal cone* to $A$ at $\bar{x}$, denoted by $N_A(\bar{x})$, is the cone
$$N_A(\bar{x}) = \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in A}} \widehat{N}_A(x). \tag{15}$$

Let us remark that if the set $A$ is closed and convex, then the regular and the limiting normal cones coincide with the normal cone in the sense of convex analysis defined by

$$N_A(\bar{x}) := \{x^* \in \mathbb{R}^n \,:\, \langle x^*, x - \bar{x}\rangle \leq 0 \text{ for all } x \in A\}.$$

- A subset $A \subset \mathbb{R}^n$ is called *normally regular* at $\bar{x} \in A$ provided $N_A(\bar{x}) = \widehat{N}_A(\bar{x})$.
- Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be an arbitrary set-valued mapping. For $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$, the *graphical (contingent) derivative* of $\Phi$ at $(\bar{x}, \bar{y})$ is the mapping $D\Phi(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$D\Phi(\bar{x}, \bar{y})(z) = \{w \in \mathbb{R}^m \,:\, (z, w) \in T_{\mathrm{Gr}(\Phi)}(\bar{x}, \bar{y})\}, \quad z \in \mathbb{R}^n. \tag{16}$$

- The *limiting (Mordukhovich) coderivative* of $\Phi$ at $(\bar{x}, \bar{y})$ is the multifunction $D^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined for all $y^* \in \mathbb{R}^m$ by

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n \,:\, (x^*, -y^*) \in N_{\mathrm{Gr}(\Phi)}(\bar{x}, \bar{y})\}.$$

For more details about these notions, we refer the reader to the monographs [13] and [23].

We recall now the following result for the characterization of the Aubin property, known in the literature as the *Mordukhovich criterion*.

**Theorem 2.1 ([12]).** *Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$. Suppose that $\mathrm{Gr}(\Phi)$ is locally closed at $(\bar{x}, \bar{y})$. Then $\Phi$ has the Aubin property around $(\bar{x}, \bar{y})$ if and only if*

$$D^*\Phi(\bar{x}, \bar{y})(0) = \{0\}. \tag{17}$$

*Furthermore, the Lipschitz modulus of $\Phi$ at $\bar{x}$ for $\bar{y}$ is given by*

$$\begin{aligned}\mathrm{lip}(\Phi; (\bar{x}, \bar{y})) &= \|D^*\Phi(\bar{x}, \bar{y})\|^+ \\ &:= \sup\{\|x^*\| \,:\, x^* \in D^*\Phi(\bar{x}, \bar{y})(y^*),\ \|y^*\| \leq 1\}.\end{aligned} \tag{18}$$

To unburden our notation, let us introduce the set-valued mapping $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$Q(z) := B^T \partial J(Bz). \tag{19}$$

On the basis of Theorem 2.1 one can now obtain the following characterization of the Aubin property of $S$.

**Theorem 2.2 ([14]).** *Suppose that $Q$ has a closed graph. Then the set-valued mapping $S$ defined in (5) has the Aubin property around $(\bar{p}, \bar{z})$ if and only if the implication*

$$0 \in M^T b + D^*Q(\bar{z}, \bar{p} - M\bar{z})(b) \Longrightarrow b = 0, \tag{20}$$

*holds true.*

The generalized equation on the left-hand side of (20) is usually called the *adjoint generalized equation*. For the isolated calmness of $S$ we dispose with a similar characterization on the basis of the graphical derivative.

**Theorem 2.3.** *Suppose that $Q$ has a closed graph. Then the set-valued mapping $S$ defined in (5) has the isolated calmness property at $(\bar{p}, \bar{z})$ if and only if the implication*

$$0 \in Mb + DQ(\bar{z}, \bar{p} - M\bar{z})(b) \Longrightarrow b = 0, \tag{21}$$

*holds true.*

**Proof.** The proof of Theorem 2.3 follows easily from [10] (Corollary 4 C.2). In fact, by setting

$$F(z) = Mz + Q(z),$$

one has

$$DS(\bar{p}, \bar{z})(0) = \{0\} \Longleftrightarrow \ker DF(\bar{z}, \bar{p}) = \{0\},$$

and $DF(\bar{z}, \bar{p})(b) = Mb + DQ(\bar{z}, \bar{p} - M\bar{z})(b)$ for all $b \in \mathbb{R}^n$ by virtue of Proposition 4 A.2 in [10]. $\square$

We conclude this preparatory session with a second order chain rule which enables us to compute the coderivative $D^*Q$ of the set-valued mapping $Q$ defined in (19).

**Theorem 2.4.** *Consider the composition $\varphi \circ g$, with $\varphi : \mathbb{R}^m \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$. Let $\tilde{\partial}$ be an arbitrary subdifferential such that $\tilde{\partial}\varphi$ has a closed graph. Finally assume that $g$ is twice continuously differentiable, and on a neighborhood $\mathcal{W}$ of $\bar{x} \in \mathbb{R}^n$ we have*

$$\tilde{\partial}(\varphi \circ g)(x) = (\nabla g(x))^T \tilde{\partial}\varphi(g(x)) \quad \text{for } x \in \mathcal{W}. \tag{22}$$

*Let $\bar{v} \in \tilde{\partial}(\varphi \circ g)(\bar{x})$. If the Jacobian $\nabla g(\bar{x})$ is surjective, then for all $y^* \in \mathbb{R}^n$ one has*

$$\begin{aligned} &D^*\tilde{\partial}(\varphi \circ g)(\bar{x}, \bar{v})(y^*) \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i \nabla^2 g_i(\bar{x})\right) y^* + (\nabla g(\bar{x}))^T D^*\tilde{\partial}\varphi(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x}) y^*), \end{aligned} \tag{23}$$

*where $\bar{\lambda} \in \mathbb{R}^m$ is the unique vector satisfying $(\nabla g(\bar{x}))^T \bar{\lambda} = \bar{v}$, i.e., $\bar{\lambda} = \left[\nabla g(\bar{x})(\nabla g(\bar{x}))^T\right]^{-1} \nabla g(\bar{x}) \bar{v}$.*

**Remark 2.5.** This statement was established in a slightly less general setting in [15] (Theorem 3.4) as an inclusion with the remark that the reverse inclusion is easily seen to hold as well. For the reader's convenience, we provide the proof of the above theorem in the Appendix.

Clearly, the above statement applies, for instance, if $\tilde{\partial}$ is the Clarke subdifferential and $\varphi$ is locally Lipschitz.

## 3. Aubin property of the solution map

In this section we give a characterization of the Aubin property of the map $S$ defined in (5) around the reference pair $(\bar{p}, \bar{z})$. Without any lack of generality we will assume that $q = 0$ and start with the general case without any additional assumptions on the problem data $M$, $B$ and $J$. Thereafter we will try to specialize these conditions under additional assumptions.

In what follows $T \in \mathbb{R}^{m \times n}$ is the matrix defined by

$$T = (BB^T)^{-1} BM^T. \tag{24}$$

Note that the product $BB^T$ is non-singular by virtue of the surjectivity of $B$.

**Theorem 3.1.** *The mapping $S$ in (5) has the Aubin property around $(\bar{p}, \bar{z})$ if and only if one has the implication*

$$((Tb)_i, (Bb)_i) \in -N_{\mathrm{Gr}(\bar{\partial} j_i)}((B\bar{z})_i, \bar{v}_i) \quad \forall i = 1, 2, \ldots, m \Longrightarrow b = 0, \tag{25}$$

*where $\bar{v}$ is the unique solution of the linear system*

$$B^T \bar{v} = \bar{p} - M\bar{z}, \quad \text{i.e., } \bar{v} = \left(BB^T\right)^{-1} \left[B\bar{p} - BM\bar{z}\right]. \tag{26}$$

**Proof.** Our starting point is the implication (20) in Theorem 2.2. So the task is to compute the coderivative of the multifunction $Q$ given by (19). To this aim, we invoke the second-order chain rule (23), whose assumptions are evidently satisfied because $B$ is surjective and $\mathrm{Gr}\,(\bar{\partial} J)$ is closed (due to the assumed Lipschitz continuity of the function $J$). In this case, the adjoint general equation in (20) can be rewritten as

$$0 \in M^T b + B^T D^* \bar{\partial} J(B\bar{z}, \bar{v})(Bb). \tag{27}$$

Using the specific structure of $J$ given in (2), we have for any $x \in \mathbb{R}^m$

$$\bar{\partial} J(x) = \prod_{i=1}^{m} \bar{\partial} j_i(x_i) \quad \text{and}$$

$$\mathrm{Gr}(\bar{\partial} J) = \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \; : \; (x_i, y_i) \in \mathrm{Gr}(\bar{\partial} j_i), \; \forall i = 1, 2, \ldots, m\right\}. \tag{28}$$

The second equality in (28) enables us to employ [23] (Proposition 6.41), which leads for every $b \in \mathbb{R}^m$ to the equivalences

$$\begin{aligned} a \in D^* \bar{\partial} J(x, u)(b) &\iff a_i \in D^* \bar{\partial} j_i(x_i, u_i)(b_i) \quad \forall i = 1, 2, \ldots, m \\ &\iff (a_i, -b_i) \in N_{\mathrm{Gr}(\bar{\partial} j_i)}(x_i, u_i) \quad \forall i = 1, 2, \ldots, m. \end{aligned}$$

The generalized equation (27) attains now the form of the system

$$\begin{aligned} &0 = M^T b + B^T c \\ &(c_i, (-Bb)_i) \in N_{\mathrm{Gr}(\bar{\partial} j_i)}((B\bar{z})_i, \bar{v}_i), \quad \forall i = 1, 2, \ldots, m. \end{aligned} \tag{29}$$

It remains to express the vector $c = (c_1, c_2, \ldots, c_m)^T \in \mathbb{R}^m$ in the form $c = -Tb$ and the proof of Theorem 3.1 is thereby completed. □

**Remark 3.2.** (i) If $n = m$ and $B = I_n$ (the identity matrix), then $T = M^T$. In this case, condition (25) takes the following form

$$\left((M^T b)_i, b_i\right) \in -N_{\mathrm{Gr}(\partial j_i)}\left(\bar{z}_i, \bar{v}_i\right) \ \forall i = 1, 2, \ldots, n \Longrightarrow b = 0. \tag{30}$$

(ii) If $M$ is nonsingular, we may use the equation in (29) in the reverse way to arrive at the condition

$$(c_i, (Rc)_i) \in -N_{\mathrm{Gr}(\partial j_i)}\left((B\bar{z})_i, \bar{v}_i\right) \ \forall i = 1, 2, \ldots, m \Longrightarrow c = 0, \tag{31}$$

where $R := B(M^T)^{-1}B^T$. Indeed, in this case $b = -(M^T)^{-1}B^T c$ and the result follows easily from (29) because

$$b = 0 \Longleftrightarrow c = 0$$

due to the injectivity of $B^T$.
If, in addition, $B = I_n$, then (31) reduces to the form

$$\left(c_i, ((M^T)^{-1}c)_i\right) \in -N_{\mathrm{Gr}(\partial j_i)}\left(\bar{z}_i, \bar{v}_i\right) \ \forall i = 1, 2, \ldots, n \Longrightarrow c = 0. \tag{32}$$

Quite often in applications $M$ or $R$ is a $P$-matrix or a positive semi-definite matrix and then we can sometimes exploit these additional properties in connection with condition (30) and (31). We recall that a matrix $M \in \mathbb{R}^{n \times n}$ is a $P$-matrix if all its principal minors are positive or equivalently

$$(\forall x \in \mathbb{R}^n, x \neq 0)(\exists i \in \{1, ..., n\}) : x_i(Mx)_i > 0.$$

**Theorem 3.3.** *Assume that $M$ is nonsingular, $R$ is a $P$-matrix and for all $i = 1, 2, \ldots, m$*

$$N_{\mathrm{Gr}(\partial j_i)}\left((B\bar{z})_i, \bar{v}_i\right) \subset \left\{(a, b) \in \mathbb{R}^2 : ab \leq 0\right\}. \tag{33}$$

*Then $S$ has the Aubin property around $(\bar{p}, \bar{z})$.*

**Proof.** Suppose by contradiction the existence of a nonzero vector $\bar{c} \in \mathbb{R}^m \setminus \{0\}$ such that the implication (31) is not satisfied. By virtue of (33), we have

$$\langle \bar{c}_i, (R\bar{c})_i \rangle \leq 0, \ \forall i = 1, 2, \ldots, m. \tag{34}$$

However, since $R$ is a $P$-matrix, it follows that for each vector $c \in \mathbb{R}^m$, there must be an index $i \in \{1, 2, \ldots, m\}$ such that $\langle c_i, (Rc)_i \rangle > 0$, which contradicts (34) and ensures the validity of (31). $\square$

**Remark 3.4.** In this way, we have established the fact that the Aubin property holds whenever $M$ is nonsingular, $R$ is a $P$-matrix and all functions $j_i$ are convex for $i = 1, 2, \ldots, m$. Observe that $R$ can be a $P$-matrix even if $M^T$ is not. To see this consider e.g. the matrices

$$M^T = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}, \quad (M^T)^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -1 \end{pmatrix}.$$

In some situations the Aubin property of $S$ around $(\bar{p}, \bar{z})$ may be ensured via the (more restrictive) notion of strong metric regularity of the inverse map $S^{-1}$ defined by

$$S^{-1}(z) := Mz + B^T \partial J(Bz)$$

at $(\bar{z}, \bar{p})$, cf. [9, Proposition 3G.1]. This amounts to the statement that the mapping $S$ has a single-valued Lipschitz localization around $(\bar{p}, \bar{z})$. Such a situation arises if, e.g., $B = I_n$, $M$ is symmetric positive definite and $J$ is convex, cf. [23, Proposition 12.54].

We conclude this section with an attempt to combine the positive semi-definitness of $M$ with condition (25).

**Theorem 3.5.** *Assume that $M$ is symmetric and positive semi-definite, $B = I_n$, condition (33) is fulfilled and*

$$\forall b \in \ker M, \ \exists i \in \{1, 2, \ldots, n\} \ \text{ such that } (0, b_i) \notin -N_{\mathrm{Gr}(\partial j_i)}(\bar{z}_i, \bar{v}_i). \tag{35}$$

*Then $S$ has the Aubin property around $(\bar{p}, \bar{z})$.*

**Proof.** Let us analyze a possible violation of condition (30) by a vector $b$. Assume first that $b \in \ker(M) \setminus \{0\}$. Due to assumption (35) such a vector evidently cannot violate this condition. So let now $b \notin \ker(M)$ so that

$$\langle b, Mb \rangle > 0.$$

However, the relations on the left-hand side of implication (30) imply that

$$\langle b, Mb \rangle = \sum_{i=1}^{n} b_i (Mb)_i \leq 0,$$

by virtue of (33). Consequently, this $b$ cannot violate condition (30) as well, and thus the statement has been established. □

We note that all the above conditions become really workable only in the case when we are given the functions $j_i$ and the reference point $(\bar{p}, \bar{z})$. A possible usage of Theorem 3.5 is now illustrated by the following academic example.

**Example 3.6.** Let

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = I_2, \qquad j_1(x_1) = |x_1|,$$

$$j_2(x_2) = |x_2| + \frac{\varepsilon}{2} x_2^2 \ \text{ with some } \varepsilon > 0,$$

and consider the reference point $(\bar{p}, \bar{z})$ with $\bar{p} = (-1, -1)^T$ and $\bar{z} = (0, 0)^T$. Here, one can easily show that, due to the small quadratic term in $j_2$, condition (35) is fulfilled. Since $M$ is positive semi-definite and condition (33) holds true by the convexity of $j_1$ and $j_2$, Theorem 3.5 applies and so the respective $S$ has the Aubin property around $(\bar{p}, \bar{z})$.

In some applications it might be useful to know the value of $\operatorname{lip}(S;(\bar{p},\bar{z}))$. By virtue of (18) and (27) we obtain that

$$\operatorname{lip}(S;(\bar{p},\bar{z})) = \sup\{\|b\| : M^T b + B^T D^* \bar{\partial} J(B\bar{z},\bar{v})(Bb) \cap \mathbb{B} \neq \emptyset\}.$$

To simplify the notation, let us denote $\operatorname{lip}(S;(\bar{p},\bar{z}))$ by $\kappa$. Clearly, $\kappa$ equals the supremal value of the objective in the optimization problem

$$\begin{array}{ll} \text{maximize} & \|b\| \\ \text{subject to} & M^T b + B^T w \in \mathbb{B} \\ & (w, -Bb) \in N_{\operatorname{Gr}(\bar{\partial} J)}(B\bar{z},\bar{v}). \end{array} \tag{36}$$

Due to (2) the last constraint in (36) amounts to

$$(w_i, (-Bb)_i) \in N_{\operatorname{Gr}(\bar{\partial} j_i)}((B\bar{z})_i, \bar{v}_i), \quad i = 1, 2, \ldots, m. \tag{37}$$

Since the functions $j_i$ have typically the form

$$j_i(\cdot) = \nu_i(\cdot) + \delta_{\Gamma_i}(\cdot),$$

where $\Gamma_i$ is a closed interval and $\nu_i$ is piecewise $C^2$ in the sense defined in [15], one can employ the results from [15, Section 4] and conclude that the normal cones in (37) can be expressed as unions of at most three convex polyhedral cones in $\mathbb{R}^2$. So, if $N_{\operatorname{Gr}(\bar{\partial} j_i)}((B\bar{z})_i, \bar{v}_i) = \bigcup_{j=1}^{3} \mathcal{A}_i^j$ for all $i$, we can replace (36) by the disjunctive optimization problem

$$\begin{array}{ll} \text{maximize} & \|b\| \\ \text{subject to} & M^T b + B^T w \in \mathbb{B} \\ & (w_i, (-Bb)_i) \in \bigcup\limits_{j=1}^{3} \mathcal{A}_i^j, \quad i = 1, 2, \ldots, m. \end{array} \tag{38}$$

If problem (38) possesses a solution, say $(\bar{b}, \bar{w})$, then it can be numerically solved by standard techniques and $\kappa = \|\bar{b}\|$. If (38) does not possess a solution and the supremal value of its objective is finite, we get typically only a lower estimate of $\kappa$. Finally, if there is a sequence of feasible points $(b^{(k)}, w^{(k)})$ in (38) with $\|b^{(k)}\| \to +\infty$, then the respective $S$ does not have the Aubin property around $(\bar{p}, \bar{z})$.

In Section 5 we will illustrate the nature of problem (38) by means of a simple academic example.

## 4. Isolated calmness of the solution map

In this section, we give a characterization of the isolated calmness of the map $S$ defined in (5) in terms of the data $(M, B, J)$ involved in problem (1).

We start with an auxiliary lemma which might be helpful also in other situations.

**Lemma 4.1.** *Let $\Xi = A^{-1}(\Gamma)$ where $\Gamma \subset \mathbb{R}^n$ is closed and $A \in \mathbb{R}^{n\times m}$ is a surjective matrix. Let $\Lambda = G(\Xi)$, where $G \in \mathbb{R}^{l\times m}$ is an injective matrix, $\bar{u} \in \Lambda$ and $\bar{v}$ be uniquely given by $G\bar{v} = \bar{u}$. Then one has*

$$T_\Lambda(\bar{u}) = \{Gw \,:\, Aw \in T_\Gamma(A\bar{v})\}. \tag{39}$$

**Proof.** By virtue of the surjectivity of $A$, we have, cf. [23] (Exercise 6.7) that

$$T_\Xi(\bar{v}) = \{w \,:\, Aw \in T_\Gamma(A\bar{v})\}.$$

We have,

$$\{Gw \,:\, w \in T_\Xi(\bar{v})\} \subset T_\Lambda(\bar{u}). \tag{40}$$

Indeed, let $w \in T_\Xi(\bar{v})$. By definition of the tangent cone, there are sequences $v_i \to \bar{v}$ in $\Xi$ and $t_i \downarrow 0^+$ as $i \to +\infty$ such that

$$\frac{v_i - \bar{v}}{t_i} \to w \text{ as } i \to +\infty.$$

Clearly we have $Gv_i \to G\bar{v} = \bar{u}$ as $i \to +\infty$.
Therefore,

$$\frac{Gv_i - \bar{u}}{t_i} \to Gw \text{ as } i \to +\infty.$$

Hence,

$$Gw \in T_\Lambda(\bar{u}).$$

It remains to prove that $T_\Lambda(\bar{u}) \subset \{Gw \,:\, w \in T_\Xi(\bar{v})\}$, which can be conducted by a similar reasoning. Let $k \in T_\Lambda(\bar{u})$, i.e., there are sequences $k_i \to k, \lambda_i \downarrow 0$ such that

$$\bar{u} + \lambda_i k_i = Gw_i$$

for some sequence $(w_i) \subset \Xi$. One has thus

$$k_i = G\frac{w_i - \bar{v}}{\lambda_i}.$$

We show now the boundedness of the sequence $(h_i)$ defined by

$$h_i = \frac{w_i - \bar{v}}{\lambda_i}.$$

Assume by contradiction that $(h_i)$ is unbounded. Hence there exists a subsequence (still denoted by $(h_i)$) such that $\|h_i\| \to +\infty$ as $i \to +\infty$. By passing to a subsequence if necessary, we have

$$\frac{h_i}{\|h_i\|} \to s \text{ with } \|s\| = 1.$$

It follows that

$$Gs = 0,$$

which contradicts the injectivity of $G$. Thus, the sequence $(h_i)$ possesses a convergent subsequence with the limit in $T_\Xi(\bar{v})$. We conclude that $k \in GT_\Xi(\bar{v})$, which completes the proof. □

**Remark 4.2.** The surjectivity of $A$ can be replaced by the weaker qualification condition

$$\left.\begin{array}{l} A^T z = 0 \\ z \in N_\Gamma(A\bar{v}) \end{array}\right\} \Longrightarrow z = 0,$$

provided $\Gamma$ is normally regular at $A\bar{v}$.

On the basis of Theorem 2.3 and equality (39) we are now in a position to state the following characterization of the isolated calmness of $S$ in terms of the problem data. Thereby we set

$$W = (BB^T)^{-1}BM. \tag{41}$$

**Theorem 4.3.** *Assume that for each $i = 1, 2, \ldots, m$ there are neighborhoods $\mathcal{U}_i$ of the points $((B\bar{z})_i, \bar{v}_i)$ such that*

$$\operatorname{Gr}(\overline{\partial} j_i) \cap \mathcal{U}_i = \bigcup_{\nu=1}^{l_i} C_\nu^{(i)}, \tag{42}$$

*where $l_i$ are given integers and the sets $C_\nu^{(i)}$, $\nu = 1, \ldots, l_i$ , are closed and normally regular at $((B\bar{z})_i, \bar{v}_i)$. Then the isolated calmness property of $S$ at $(\bar{p}, \bar{z})$ holds true if and only if one has the implication*

$$((Bb)_i, -(Wb)_i) \in \bigcup_{\nu=1}^{l_i} T_{C_\nu^{(i)}}\left((B\bar{z})_i, \bar{v}_i\right) \;\; \forall i = 1, 2, \ldots, m \Longrightarrow b = 0, \tag{43}$$

*where $\bar{v}$ is given by (26).*

**Proof.** Our starting point is the implication (21) in Theorem 2.3. To compute the graphical derivative of $Q$, we observe that

$$\operatorname{Gr}(Q) = \left\{ G \begin{bmatrix} b \\ c \end{bmatrix} \;:\; A \begin{bmatrix} b \\ c \end{bmatrix} \in \operatorname{Gr}(\overline{\partial} J) \right\},$$

where

$$G = \begin{bmatrix} I_n & 0 \\ 0 & B^T \end{bmatrix}$$

is injective and

$$A = \begin{bmatrix} B & 0 \\ 0 & I_m \end{bmatrix}$$

is surjective. By virtue of (28) one has generally only the inclusion

$$T_{\operatorname{Gr}(\overline{\partial} J)}(B\bar{z}, \bar{v}) \subset \prod_{i=1}^{m} T_{\operatorname{Gr}\overline{\partial} j_i}\left((B\bar{z})_i, \bar{v}_i\right). \tag{44}$$

However, thanks to the structural assumption (42), inclusion (44) becomes equality (cf. [23], Proposition 6.41). This enables us to invoke Lemma 4.1, according to which we have

$$T_{\mathrm{Gr}(Q)}(\bar{z},\bar{p}-M\bar{z}) = \left\{ \begin{bmatrix} b \\ B^T c \end{bmatrix} : ((Bb)_i, c_i) \in \bigcup_{\nu=1}^{l_i} T_{C_{i(\nu)}}((B\bar{z})_i, \bar{v}_i),\ \forall i \right\},$$

i.e.,

$$DQ(\bar{z},\bar{p}-M\bar{z})(b) = \left\{ B^T c : ((Bb)_i, c_i) \in \bigcup_{\nu=1}^{l_i} T_{C_i^{(\nu)}}((B\bar{z})_i, \bar{v}_i),\ \forall i \right\}.$$

The implication (21) can thus be rewritten to the form

$$\left. \begin{array}{l} Mb + B^T c = 0 \\ ((Bb)_i, c_i) \in \bigcup_{\nu=1}^{l_i} T_{C_i^{(\nu)}}((B\bar{z})_i, \bar{v}_i),\ \forall i \end{array} \right\} \Longrightarrow b = 0,$$

from which condition (43) follows because $c = -Wb$. This completes the proof of Theorem 4.3. □

## 5. Illustration in non-regular electrical circuits

The aim of this section is to develop a mathematical model of electronic circuits involving devices like diodes, Zener diodes, DIACs, Silicon controlled rectifiers that are characterized by set-valued monotone or non-monotone ampere-volt characteristics.

Electrical devices like diodes are described in terms of Ampere-Volt characteristic (I, V) which is a multifunction expressing the difference of potential V across the device as a function of current i going through the device (for more details we refer to [2, 4, 6] and references therein).

An electronic circuit is formed by the interconnection of electronic components like generators, resistors, capacitors, inductors, diodes, transistors, etc. The behavior of a circuit is usually described in terms of currents and voltages that can be specified for each involved electrical device. An approach to state a mathematical model that can be used to determine these currents and voltages requires to formulate the ampere-volt characteristic of each electrical device, to write down the Kirchoff's voltage law expressing that the algebraic sum of the voltages between successive nodes in all loops of the circuit are zero and to write down the Kirchhoff's current law stating that the algebraic sum of the currents in all branches which flow to a common node equals zero.

Let $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $E \in \mathbb{R}^{n\times p}$ be given matrices. Let $J : \mathbb{R}^m \to \mathbb{R}$ be a given function. It is assumed that $x \mapsto J(x)$ is locally Lipschitz. Suppose that the state-space equations attain the form

$$(\mathcal{P}) \begin{cases} Ax + Dy_L + Eu = 0 \\ y = Cx \text{ and } y_L \in \overline{\partial} J(y), \end{cases}$$

where $u \in \mathbb{R}^n$ is a given vector (usually $u$ is a control vector that drives the system).

This framework is particularly useful for the study of non-regular circuits involving nonsmooth electrical devices. In this case, the matrices $A$, $C$, $D$ and $E$ are used to state the Kirchhoff's voltage and current laws in matrix form. In general, $A$ contains some electrical parameters like resistance, capacitance or inductance. The state $x$ denotes a current vector and $y_L$ is a voltage vector corresponding to the electrical devices involved in the circuit.

Suppose that the following key assumption is satisfied:

$$\begin{aligned} &\textit{there exists a symmetric and invertible matrix } R \in \mathbb{R}^{n\times n} \\ &\textit{such that: } R^2 D = C^T \end{aligned} \tag{45}$$

For the connection of assumption (45) with the positive realness (which may be seen as a generalization of the positive definiteness of a matrix to the case of a dynamical system) of the transfer function $H(s) = C(sI_n - A)^{-1}D$, $s \in \mathbb{C}$ associated to problem $(\mathcal{P})$, and the Kalman-Yakubovich-Popov lemma, see Lemma 1 in [4] and references therein. The Kalman-Yakubovich-Popov lemma has been a cornerstone in control and system theory due to its wide range of applications. It relates the frequency domain conditions for positive realness to a set of algebraic equations (Linear Matrix Inequalities describing the state-space representation of the system) and to the dissipativity of the storage function. For more details see [7].

Problem $(\mathcal{P})$ is equivalent to the following variational inclusion:

$$0 \in Ax + D\overline{\partial} J(Cx) + Eu.$$

Setting $z = Rx$, we have

$$\begin{aligned} 0 \in Ax + D\overline{\partial} J(Cx) + Eu \;&\Leftrightarrow\; 0 \in RAR^{-1}Rx + RD\overline{\partial} J(CR^{-1}Rx) + REu \\ &\Leftrightarrow\; 0 \in RAR^{-1}z + R^{-1}R^2 D\overline{\partial} J(CR^{-1}z) + REu \\ &\Leftrightarrow\; 0 \in RAR^{-1}z + R^{-1}C^T\overline{\partial} J(CR^{-1}z) + REu. \end{aligned}$$

This allows us to consider the problem

$$(\mathcal{Q}) \begin{cases} \text{Find } z \in \mathbb{R}^n \text{ such that} \\ 0 \in RAR^{-1}z + REu + R^{-1}C^T\overline{\partial} J(CR^{-1}z). \end{cases}$$

We note that problem $(\mathcal{Q})$ is of the form (1) with $f(z) = RAR^{-1}z + REu$ and $B = CR^{-1}$.

**Proposition 5.1.** *Suppose that assumption (45) is satisfied. If $(x, y_L)$ is a solution of Problem $(\mathcal{P})$ then $z = Rx$ is a solution of Problem $(\mathcal{Q})$. Conversely, if $z$ is a solution of Problem $(\mathcal{Q})$ then there exists a function $y_L$ such that $(R^{-1}z, y_L)$ is a solution of Problem $(\mathcal{P})$.*

**Proof.** We have seen above that if $(x, y_L)$ is a solution of Problem $(\mathcal{P})$ then $z = Rx$ is a solution of Problem $\mathcal{Q}$. Suppose now that $z$ is a solution of Problem $\mathcal{Q}$. Then setting $x = R^{-1}z$, we see as above that:

$$0 \in Ax + D\bar{\partial}J(Cx) + Eu.$$

It follows that there exists a vector $y_L \in \bar{\partial}J(Cx)$ such that:

$$0 = Ax + Dy_L + Eu.$$

By setting $y = Cx$, we have

$$0 = Ax + Dy_L + Eu, \quad y = Cx \text{ and } y_L \in \bar{\partial}J(y). \qquad \square$$

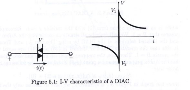


Figure 5.1: I-V characteristic of a DIAC

**Example 5.2. (A diode clipping circuit with a DIAC.)** A DIAC is a two-terminal four-layer semiconductor device that can permit the current to flow in either direction when properly activated. The curve in Figure 5.1 illustrates the I-V characteristic of a DIAC. Here $V_1$ (resp. $V_2 = -V_1$) is the forward (resp. reverse) breakover voltage while $I_1$ (resp. $I_2 = -I_1$) is the forward (resp. reverse) breakover current. For example, for a practical trigger DIAC, $V_1 = 30$ volts and $I_1 = 25\ \mu A$. It is clear that there exists a locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ such that the set-valued map in Figure 5.1 can be written as

$$V \in \bar{\partial}j(i).$$

A DIAC clipper circuit can be used to limit the output voltage signal to a certain level. Let us consider the circuit from Figure 5.2 involving a load resistance $R > 0$, an input-signal source $u$ and the corresponding instantaneous current $i$, a DIAC as a shunt element and a supply voltage $E$. Using Kirchhoff's laws, we have

$$u = V_R + V + E, \tag{46}$$

where $V_R = Ri$ and $V \in \bar{\partial}j(i)$ is the voltage across the resistor and the diode respectively. Hence,

$$0 \in Ri + E - u + \bar{\partial}j(i), \tag{47}$$

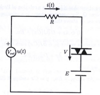


Figure 5.2: A DIAC Clipping circuit

which is an inclusion of the form (1) with the variable $z := i$. For simplicity, we assume that $R = 1$. We will describe all situations concerning the Aubin property and the isolated calmness of the DIAC by using Theorem 3.1 and Theorem 4.3. In this case the operators $T$ and $W$ defined in (24) and (41) are equal to 1 and the matrix $B = 1$. Therefore, the mapping $S$ defined in (5) and associated to (47) has the Aubin property around a point $(\bar{p}, \bar{z})$ if and only if the following condition holds true

$$N_{\mathrm{Gr}(\partial j)}(\bar{z}, \bar{p} - \bar{z}) \cap \{(x, x) \, : \, x \in \mathbb{R}\} = \{0\}. \tag{48}$$

Since the graph in Figure 5.1 is symmetric, we have to discuss only the following three situations:

• Suppose that $\bar{z} = 0$ and $\bar{p} = -1$. Using the (I-V)-characteristic of the DIAC in Figure 5.1, it is clear that the Aubin property will depend on the slope of the curve at $x = 0^-$. Let us suppose that $j''(0^-) = -a$ and we restrict ourselves to the nonmonotone case i.e. $a > 0$. By definitions of the contingent and the limiting normal cones (11)–(15), we have

$$T_{\mathrm{Gr}(\partial j)}(0, -1) = \mathbb{R}_+ \begin{bmatrix} -1 \\ a \end{bmatrix} \cup \mathbb{R}_+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{49}$$

$$N_{\mathrm{Gr}(\partial j)}(0, -1) = \mathbb{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cup \mathbb{R} \begin{bmatrix} -a \\ -1 \end{bmatrix} \cup \operatorname{cone} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -a \\ -1 \end{bmatrix} \right\}. \tag{50}$$

Therefore, condition (48) is satisfied for $(\bar{z}, \bar{p}) = (0, -1)$ if and only if $0 < a < 1$. Consequently, for the DIAC, the map $S$ satisfies the Aubin property around $(-1, 0)$ if and only if $0 < a < 1$.
In this case problem (38) attains the simple form

$$\begin{aligned} &\text{maximize} && |b| \\ &\text{subject to} && -1 \leq b + w \leq 1 \\ & && (w, b) \in \mathbb{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cup \mathbb{R} \begin{bmatrix} -a \\ 1 \end{bmatrix} \cup \operatorname{cone} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -a \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

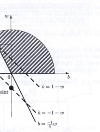


Figure 5.3: Lipschitz modulus of a DIAC

Its solution can be readily found in Figure 5.3, according to which

$$\kappa = \text{lip}(S; (-1,0)) = \frac{1}{1-a}.$$

Using (49) and (43), it is easy to check that for the DIAC, $S$ satisfies the isolated calmness property at $(\bar{p}, \bar{z}) = (-1, 0)$ if and only if $a \neq 1$.
The results about the Aubin property around and the isolated calmness at $(\bar{p}, \bar{z}) = (-1, 0)$ for the DIAC circuit, depicted in Figure 5.2, are summarized in Table 5.1.

| $(\bar{p}, \bar{z}) = (-1, 0)$ | Aubin property | Isolated calmness |
|---|---|---|
| $a = 1$ | No | No |
| $a > 1$ | No | Yes |
| $0 < a < 1$ | Yes | Yes |

Table 5.1: Aubin property and isolated calmness around $(\bar{p}, \bar{z}) = (-1, 0)$.

• Suppose now that $\bar{z} < 0$ and set $\bar{p} = \bar{z} + \bar{v}$ with $\bar{v} \in \overline{\partial} j(\bar{z})$. It is easy to show that both conditions (48) as well as (43) are satisfied at $(\bar{p}, \bar{z})$ if and only if $j''(\bar{z}) \neq -1$. Consequently, the map $S$ satisfies the Aubin property around and the isolated calmness at $(\bar{p}, \bar{z})$ with $\bar{z} < 0$ if and only if $j''(\bar{z}) \in ]-1, 0[$.

• Finally, it is easy to check that $S$ has always the Aubin property around and the isolated calmness at $(0, \bar{p})$ with $\bar{p} \in ]-1, 1[$.

| $(\bar{p}, \bar{z})$ with $\bar{z} < 0$ | Aubin property | Isolated calmness |
|---|---|---|
| $j''(\bar{z}) = -1$ | No | No |
| $j''(\bar{z}) \in \mathbb{R}^- \setminus \{-1\}$ | Yes | Yes |

Table 5.2: Aubin property and isolated calmness around $(\bar{p}, \bar{z})$ with $\bar{z} < 0$.

**Remark 5.3.** If the graph in Figure 5.1 was a maximal monotone operator (which is the case of the practical diode model or the Zener diode model or the complete diode model [2]), then the Aubin property around and the isolated calmness at $(-1, 0)$ hold without any restrictions.

**Example 5.4. Double-DIAC clipper circuit.**

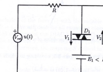


Figure 5.4: A double DIAC Clipping circuit

Let us consider the circuit depicted in Figure 5.4 with a resistor $R > 0$, two DIACs, an input-signal source $u(t)$ and two supply voltages $E_1 < E_2$. The current through the resistor is denoted by $i = i_1 + i_2$. Using Kirchhoff's laws, we have the following system

$$\begin{cases} E_1 + R(i_1 + i_2) - u = +V_1 \\ E_2 + R(i_1 + i_2) - u = -V_2 \end{cases} \tag{51}$$

where $V_1 \in \partial j_1(-i_1)$ and $V_2 \in \partial j_2(i_2)$ are the differences of potential across the DIACs $D_1$ and $D_2$ respectively. We suppose that the graphs of $\partial j_1$ and $\partial j_2$ are of the form depicted in Figure 5.1. In this case, it is easy to see that $\partial j_1(-i_1) = -\partial j_1(i_1)$.

Setting

$$M = \begin{pmatrix} R & R \\ R & R \end{pmatrix}, \quad p = \begin{pmatrix} u - E_1 \\ u - E_2 \end{pmatrix}, \quad z = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$
$$\text{and} \quad J(z) := j_1(i_1) - j_2(i_2), \tag{52}$$

we see that the system in (51) is equivalent to the variational inclusion

$$p \in Mz + \partial J(z), \tag{53}$$

which is of the form (1) with $B = I_2$. We note that here the matrix $M$ is symmetric and positive semidefinite. We will thus apply Theorem 3.1 and Remark 3.2 (i). Let $j_1''(0^-) = -a$ and $j_2''(0^-) = a$ with $a > 0$ and assume that $R = 1$.

- Suppose that the reference point $(\bar{p}, \bar{z})$ is given by $\bar{z} = (0, 0)$ and $\bar{p} = (-1, 1)$, see Figure 5.5.

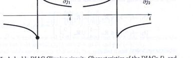


Figure 5.5: A double DIAC Clipping circuit: Characteristics of the DIACs $D_1$ and $D_2$

By Theorem 3.1, the map $S$ has the Aubin property around $(\bar{p}, \bar{z})$ if and only if the following holds

$$\left.\begin{array}{l}(b_1 + b_2, b_1) \in -N_{\mathrm{Gr}(\bar{\partial} j_1)}(0, -1) \\ (b_1 + b_2, b_2) \in -N_{\mathrm{Gr}(\bar{\partial} j_2)}(0, 1)\end{array}\right\} \Longrightarrow (b_1, b_2) = (0, 0),$$

which is equivalent to

$$\left.\begin{array}{l}(b_1, b_2) \in -L_1^{-1} N_{\mathrm{Gr}(\bar{\partial} j_1)}(0, -1) \\ (b_1, b_2) \in -L_2^{-1} N_{\mathrm{Gr}(\bar{\partial} j_2)}(0, 1)\end{array}\right\} \Longrightarrow (b_1, b_2) = (0, 0), \tag{54}$$

with $L_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $L_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $L_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $L_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

A simple computation shows us that the limiting normal cones are given by

$$N_{\mathrm{Gr}(\bar{\partial} j_1)}(0, -1) = \mathbb{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cup \mathbb{R} \begin{bmatrix} -a \\ -1 \end{bmatrix} \cup \mathrm{cone} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -a \\ -1 \end{bmatrix} \right\}, \tag{55}$$

and

$$N_{\mathrm{Gr}(\bar{\partial} j_2)}(0, 1) = \mathbb{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cup \mathbb{R} \begin{bmatrix} -a \\ 1 \end{bmatrix} \cup \mathrm{cone} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -a \\ 1 \end{bmatrix} \right\}. \tag{56}$$

In this case, equation (54) is equivalent to

$$\left.\begin{array}{l}(b_1, b_2) \in \mathbb{R} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cup \mathbb{R} \begin{bmatrix} 1 \\ a-1 \end{bmatrix} \cup \mathrm{cone} \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ a-1 \end{bmatrix} \right\} \\ (b_1, b_2) \in \mathbb{R} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cup \mathbb{R} \begin{bmatrix} 1+a \\ -1 \end{bmatrix} \cup \mathrm{cone} \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1+a \\ -1 \end{bmatrix} \right\}\end{array}\right\} \tag{57}$$

$$\Longrightarrow (b_1, b_2) = (0, 0).$$

A simple computation shows us that (57) does not hold. In fact,

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1+a \\ -1 \end{bmatrix},$$

with $\alpha = a + 1 > 0$ and $\beta = 1$.
Consequently, the Aubin property of $S$ around the point $(\bar{p}, \bar{z})$ with $\bar{z} = (0,0)$ and $\bar{p} = (-1, 1)$ does not hold.
By Theorem 4.3, the isolated calmness property of $S$ at $(\bar{p}, \bar{z})$ is satisfied if and only if the following implication holds true

$$\left.\begin{array}{l} (b_1, -b_1 - b_2) \in \mathbb{R}_+ \begin{bmatrix} -1 \\ a \end{bmatrix} \cup \mathbb{R}_+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ (b_2, -b_1 - b_2) \in \mathbb{R}_+ \begin{bmatrix} -1 \\ -a \end{bmatrix} \cup \mathbb{R}_+ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{array}\right\} \Longrightarrow (b_1, b_2) = (0, 0),$$

which is equivalent to

$$\left.\begin{array}{l} (b_1, b_2) \in \mathbb{R}_+ \begin{bmatrix} -1 \\ 1-a \end{bmatrix} \cup \mathbb{R}_+ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ (b_1, b_2) \in \mathbb{R}_+ \begin{bmatrix} 1+a \\ -1 \end{bmatrix} \cup \mathbb{R}_+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right\} \Longrightarrow (b_1, b_2) = (0, 0). \tag{58}$$

It easy to check that (58) is satisfied whenever $a > 0$. Consequently, the isolated calmness of $S$ at $(\bar{p}, \bar{z})$ is satisfied with $\bar{p} = (-1, 1)$ and $\bar{z} = (0, 0)$.

• Suppose that the reference point $(\bar{p}, \bar{z})$ is given by $\bar{z} = (0, 0)$ and $\bar{p} = (-1, -1)$, see Figure 5.6.

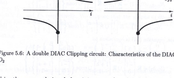


Figure 5.6: A double DIAC Clipping circuit: Characteristics of the DIACs $D_1$ and $D_2$

Using the same analysis as before, it is easy to show that the map $S$ has the Aubin property around $(\bar{p}, \bar{z})$ if and only if the following holds

$$\left.\begin{array}{l}(b_1,b_2) \in \mathbb{R}\begin{bmatrix}0\\-1\end{bmatrix} \cup \mathbb{R}\begin{bmatrix}1\\a-1\end{bmatrix} \cup \operatorname{cone}\left\{\begin{bmatrix}0\\-1\end{bmatrix},\begin{bmatrix}1\\a-1\end{bmatrix}\right\}\\ (b_1,b_2) \in \mathbb{R}\begin{bmatrix}1\\0\end{bmatrix} \cup \mathbb{R}\begin{bmatrix}-1-a\\1\end{bmatrix} \cup \operatorname{cone}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}-1-a\\1\end{bmatrix}\right\}\end{array}\right\} \tag{59}$$

$$\Longrightarrow (b_1,b_2) = (0,0).$$

Since,

$$\begin{bmatrix}a+1\\-1\end{bmatrix} = \alpha\begin{bmatrix}0\\-1\end{bmatrix} + \beta\begin{bmatrix}1\\a-1\end{bmatrix},$$

with $\alpha = a^2 > 0$ and $\beta = a+1 > 0$, we observe that the Aubin property of the map $S$ does not hold around $(\bar{p}, \bar{z})$ with $\bar{z} = (0,0)$ and $\bar{p} = (-1,-1)$.

The isolated calmness property of $S$ at $(\bar{p}, \bar{z})$ is satisfied if and only if the following implication holds true

$$\left.\begin{array}{l}(b_1,b_2) \in \mathbb{R}_+\begin{bmatrix}-1\\1-a\end{bmatrix} \cup \mathbb{R}_+\begin{bmatrix}0\\-1\end{bmatrix}\\ (b_1,b_2) \in \mathbb{R}_+\begin{bmatrix}-1-a\\1\end{bmatrix} \cup \mathbb{R}_+\begin{bmatrix}1\\0\end{bmatrix}\end{array}\right\} \Longrightarrow (b_1,b_2) = (0,0). \tag{60}$$

It is easy to show that if $a > 0$, then (60) is satisfied. Therefore, $S$ satisfies the isolated calmness property at $(\bar{p}, \bar{z})$ with $\bar{z} = (0,0)$ and $\bar{p} = (-1,-1)$.

- Suppose now that the reference point is $(\bar{p}, \bar{z})$ with $\bar{z} = (\bar{z}_1, \bar{z}_2)$, $\bar{z}_1 < 0$ and $\bar{z}_2 < 0$, see Figure 5.7.

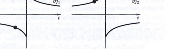


Figure 5.7: A double DIAC Clipping circuit: Characteristics of the DIACs $D_1$ and $D_2$

The map $S$ has the Aubin property around $(\bar{p}, \bar{z})$ if and only if the following holds

$$\left.\begin{aligned} (b_1 + b_2, b_1) &\in -N_{\mathrm{Gr}(\partial j_1)}(\bar{z}_1, \bar{p}_1 - \bar{z}_1 - \bar{z}_2) = \mathbb{R}\begin{bmatrix} a \\ 1 \end{bmatrix} \\ (b_1 + b_2, b_2) &\in -N_{\mathrm{Gr}(\partial j_2)}(\bar{z}_2, \bar{p}_2 - \bar{z}_1 - \bar{z}_2) = \mathbb{R}\begin{bmatrix} -a \\ 1 \end{bmatrix} \end{aligned}\right\} \tag{61}$$
$$\Longrightarrow (b_1, b_2) = (0, 0).$$

In this case we conclude that since $a > 0$, the implication in (61) holds true. Consequently, the map $S$ has the Aubin property around $(\bar{p}, \bar{z})$.

## 6. By the way of conclusion

In this paper, we studied the Aubin property and the isolated calmness of the solution map to a non-monotone variational inclusion with respect to canonical perturbations. If, instead of canonical perturbations, we perturb the matrix $M$ (in a sufficiently smooth way) then, as explained in the theory developed in [10], the conditions in Theorems 3.1, 4.1 will become only sufficient (for the respective stability properties). We showed that our theoretical results are applicable to the mathematical analysis of nonregular circuits involving nonsmooth electrical devices like DIAC (whose I-V characteristic is nonmonotone) which is a topic of major importance in electrical engineering. Some other electronic devices like SCRs, Zener diodes or transistors can be treated in the same way. In order to use the chain rules in subdifferential calculus, we have assumed that the matrix $B$ is surjective; it would be interesting to relax this condition. Another interesting question is to study the (non-isolated) calmness of $S$. This is, however, out of the scope of this work and will probably be the subject of a forthcoming paper.

**Acknowledgements.** The authors would like to express their gratitude to the handling editor and to an anonymous reviewer for their important comments and suggestions.

## 7. Appendix

**Proof of Theorem 2.4.** By the assumption (22), for all $x$ around $\bar{x}$ one has

$$\partial(\varphi \circ g)(x) = (f \circ \widetilde{G})(x),$$

where $\widetilde{G}(x) := (x, G(x))$ and the mappings $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ in the latter representation are defined by

$$f(x, \lambda) := (\nabla g(x))^T \lambda, \; G(x) := \partial \varphi(g(x)).$$

Next we invoke [13] (Lemma 1.126, Statement (1.64)), the assumptions of which are clearly fulfilled. One arrives for all $y^* \in \mathbb{R}^n$ at the equality

$$\begin{aligned} D^*(f \circ \widetilde{G})(\bar{x}, \bar{v})(y^*) &= (\nabla_x f(\bar{x}, \bar{\lambda}))^T y^* + D^* G(\bar{x}, \bar{\lambda})(\nabla g(\bar{x}) y^*) \\ &= \left( \sum_{i=1}^{m} \bar{\lambda}_i \nabla^2 g_i(\bar{x}) \right) y^* + D^* G(\bar{x}, \bar{\lambda})(\nabla g(\bar{x}) y^*), \end{aligned}$$

where $\bar{\lambda} \in \hat{\partial}\varphi(g(\bar{x}))$ is the unique solution of the equation $\bar{v} = (\nabla g(\bar{x}))^T \bar{\lambda}$. It remains thus to compute $D^*G(\bar{x}, \bar{\lambda})$. To this aim we observe that

$$\operatorname{Gr}(G) = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \,\middle|\, \begin{bmatrix} g(x) \\ \lambda \end{bmatrix} \in \operatorname{Gr} \hat{\partial}\varphi \right\}.$$

Hence, by [23] (Exercise 6.7) due to the surjectivity of $\nabla g(\bar{x})$

$$N_{\operatorname{Gr}(G)}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} (\nabla g(\bar{x}))^T & 0 \\ 0 & I_m \end{bmatrix} N_{\operatorname{Gr}\hat{\partial}\varphi}(g(\bar{x}), \bar{\lambda})$$

so that $x^* \in D^*G(\bar{x}, \bar{\lambda})(\nabla g(\bar{x}) y^*)$ amounts to

$$x^* = (\nabla g(\bar{x}))^T a \ \text{ with } a \in D^*\hat{\partial}\varphi(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x}) y^*),$$

and we are done. □

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