Operator of Composition for Credal Sets

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Abstract

This paper is the first attempt to introduce the operator of composition, already known from probability, possibility and evidence theories, also for credal sets. We prove that the proposed definition preserves all the necessary properties of the operator enabling us to define compositional models as an efficient tool for multidimensional models representation. Theoretical results are accompanied by numerous illustrative examples.

Keywords. Credal sets, graphical models, conditional independence.

1 Introduction

In the second half of 1990's a new approach to efficient representation of multidimensional probability distributions was introduced with the aim to be alternative to Graphical Markov Modeling. This approach is based on the following idea: multidimensional distribution is *composed* from a system of low-dimensional distributions by repetitive application of a special operator of composition, which is also the reason why the models are called *compositional models*. In several papers, in which the properties of the operator and models were studied [4, 5, 6], it was shown (among others) that these models are, in a way, equivalent to Bayesian networks. Roughly speaking, *any multidimensional distribution representable by a Bayesian network can also be represented in the form of a compositional model, and vice versa.*

Later, this compositional models were introduced also in possibility theory [12, 13] (here the models are parameterized by a continuous t-norm) and a few years ago also in evidence theory [8, 9]. In all these frameworks the original idea is kept, but there exist some slight differences among them, as we shall see later.

Although Bayesian networks and compositional models represent the same class of distributions, they do not make it in the same way. Bayesian networks use *conditional distributions* whereas compositional models consist of *unconditional distributions*. Naturally, both types of models contain the same information but while some marginal distributions are explicitly expressed in compositional models, it may happen that their computation from a corresponding Bayesian network is rather computationally expensive. Therefore it appears that some of computational procedures designed for compositional models are (algorithmically) simpler than their Bayesian network counterparts.

Furthermore, the research concerning relationship between compositional models in evidence theory and evidential networks [14] revealed probably a more important thing. Even though any evidential network (with proper conditioning rule and conditional independence concept) can be expressed as a compositional model, if we do it in the opposite way and transform a compositional model into an evidential network, we realize, that the model is more imprecise than the original one. It is caused by the fact that conditioning increases imprecision, and as it is typical not only for evidence theory, but also for other imprecise probability frameworks, compositional models in more general frameworks than evidence theory (e.g. for credal sets) seem to be worth-studying.

The goal of this paper is to show that the operator of composition can also be introduced for credal sets. Moreover, we will show that it keeps the basic properties of its counterparts in other frameworks, and therefore it will enable us to introduce compositional models for multidimensional credal sets.

The contribution is organized as follows. In Section 2 we summarize basic concepts and notation. Definition of the operator of composition is introduced in Section 3, where also its basic properties can be found, while Section 4 is devoted to more advanced properties. Finally, in Section 5 we introduce the concept of so-called *perfect sequences* and demonstrate their importance.

2 Basic Concepts and Notation

In this section we will recall basic concepts and notation necessary for understanding the contribution.

2.1 Variables and Distributions

For an index set $N = \{1, 2, \ldots, n\}$ let $\{X_i\}_{i \in N}$ be a system of variables, each X_i having its values in a finite set $\mathbf{X}_i$ and $\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n$ be the Cartesian product of these sets.

In this paper we will deal with groups of variables on its subspaces. Let us note that X_K will denote a group of variables $\{X_i\}_{i \in K}$ with values in

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i$$

throughout the paper.

Having two probability distributions P_1 and P_2 of X_K we say that P_1 is *absolutely continuous* with respect to P_2 (and denote $P_1 \ll P_2$) if for any $x_K \in \mathbf{X}_K$

$$P_2(x_K) = 0 \Longrightarrow P_1(x_K) = 0.$$

This concept plays an important role in the definition of the operator of composition.

2.2 Credal Sets

A *credal set* $\mathcal{M}(X_K)$ about a group of variables X_K is defined as a closed convex set of probability measures about the values of this variable.

In order to simplify the expression of operations with credal sets, it is often considered [10] that a credal set is the set of probability distributions associated to the probability measures in it. Under such consideration a credal set can be expressed as a *convex hull* of its extreme distributions

$$\mathcal{M}(X_K) = \mathrm{CH}\{\mathrm{ext}(\mathcal{M}(X_K))\}.$$

Consider a credal set about X_K, i.e. $\mathcal{M}(X_K)$. For each $L \subset K$ its *marginal credal set* $\mathcal{M}(X_L)$ is obtained by element-wise marginalization, i.e.

$$\mathcal{M}(X_L) = \mathrm{CH}\{P^{\downarrow L} : P \in \mathrm{ext}(\mathcal{M}(X_K))\}, \quad (1)$$

where $P^{\downarrow L}$ denotes the marginal distribution of P on $\mathbf{X}_L$. If the above introduced notation (1) cannot be used (e.g. to avoid misunderstandings), then we use $\mathcal{M}(X_K)^{\downarrow L}$, or simply $\mathcal{M}^{\downarrow L}$, instead.

Having two credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ about X_K and X_L, respectively (assuming that $K, L \subseteq N$), we say that these credal sets are *projective* if their marginals about common variables coincide, i.e. if

$$\mathcal{M}_1(X_{K \cap L}) = \mathcal{M}_2(X_{K \cap L}).$$

Let us note that if K and L are disjoint, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are projective, as $\mathcal{M}(X_\emptyset) = 1$.

Besides marginalization we will also need the opposite operation, called vacuous extension. *Vacuous extension* of a credal set $\mathcal{M}(X_L)$ about X_L to a credal set

$$\mathcal{M}(X_K) = \mathcal{M}(X_L)^{\uparrow K}$$

($L \subset K$) is the maximal credal set about X_K such that $\mathcal{M}(X_K)^{\downarrow L} = \mathcal{M}(X_L)$.

Example 1 Let

$$\mathcal{M}(X_1) = \mathrm{CH}(\{[0.2, 0.8], [0.4, 0.6]\})$$

be a credal set about variable X_1. Its vacuous extension $\mathcal{M}(X_1X_2)$ is then

$$\begin{aligned} \mathcal{M}(X_1X_2) \quad = \quad & \mathrm{CH}(\{[0.2, 0, 0.8, 0], [0, 0.2, 0.8, 0], \\ & [0.2, 0, 0, 0.8], [0, 0.2, 0, 0.8], \\ & [0.4, 0, 0.6, 0], [0, 0.4, 0.6, 0], \\ & [0.4, 0, 0, 0.6], [0, 0.4, 0, 0.6]\}), \end{aligned}$$

since evidently

$$\mathcal{M}(X_1X_2)^{\downarrow\{1\}} = \mathrm{CH}(\{[0.2, 0.8], [0.4, 0.6]\}),$$

as desired.

To show, that it is also maximal let us suppose, that there exists a credal set $\mathcal{M}'(X_1X_2)$ containing $\mathcal{M}(X_1X_2)$ and $\mathcal{M}(X_1) = \mathcal{M}'(X_1)$. Then $\mathcal{M}'(X_1X_2)$ must contain at least one $p = (p_1, p_2, p_3, p_4) \notin \mathcal{M}(X_1X_2)$. Nevertheless, it means, that either $p_1 + p_2 < 0.2$ or $p_1 + p_2 > 0.4$ (from which analogous inequalities for $p_3 + p_4$ follow). Therefore, $p^{\downarrow\{1\}} \notin \mathcal{M}(X_1)$ and $\mathcal{M}(X_1X_2)$ is maximal. ♢

The concept of absolute continuity of probability distributions can be generalized for credal sets in the following way. $\mathcal{M}_1(X_K)$ is absolutely continuous with respect to $\mathcal{M}_2(X_K)$, if $P_1 \ll P_2$ for any $P_1 \in \mathcal{M}_1(X_K)$ and $P_2 \in \mathcal{M}_2(X_K)$.

Evidently, it is not the only way how to generalize the concept of absolute continuity to credal sets. It can be done e.g. using lower previsions (but the definitions are not equivalent), nevertheless, the above-presented definition is more suitable for our purpose, as we shall see in the next section.

2.3 Strong Independence

Among numerous definitions of independence for credal sets [2] we have chosen strong independence, as it seems to be most appropriate for multidimensional models.

We say that (groups of) variables X_K and X_L (K and L disjoint) are *strongly independent* with respect to $\mathcal{M}(X_{K\cup L})$ iff (in terms of probability distributions)

$$\mathcal{M}(X_{K\cup L}) = \{P_1 \cdot P_2 : P_1 \in \mathcal{M}(X_K), P_2 \in \mathcal{M}(X_L)\}. \quad (2)$$

Again, there exist several generalizations of this notion to conditional independence, see e.g. [10], but since the following definition is suggested by the authors as the most appropriate for the marginal problem, it seems to be a suitable concept also in our case, since the operator of composition can also be used as a tool for solution of a marginal problem, as shown (in the framework of possibility theory) e.g. in [13].

Given three groups of variables X_K, X_L and X_M (K, L, M be mutually disjoint subsets of N, such that K and L are nonempty), we say that X_K and X_L are *independent on the distribution* [10] given X_M under global set $\mathcal{M}(X_{K\cup L\cup M})$ (in symbols $K \perp\!\!\!\perp L|M[\mathcal{M}]$[1] iff

$$\mathcal{M}(X_{K\cup L\cup M}) = \{(P_1 \cdot P_2)/P_1^{\downarrow M} : P_1 \in \mathcal{M}(X_{K\cup M}), P_2 \in \mathcal{M}(X_{L\cup M}), P_1^{\downarrow M} = P_2^{\downarrow M}\}.$$

This definition is a generalization of stochastic conditional independence: if $\mathcal{M}(X_{K\cup L\cup M})$ is a singleton, then also $\mathcal{M}(X_{K\cup M})$ and $\mathcal{M}(X_{L\cup M})$ are (projective) singletons and the definition collapses into definition of stochastic conditional independence.

3 Operator of Composition and Its Properties

Now, let us start considering how to define composition of two credal sets. Consider two index sets $K, L \subset N$. At this moment we do not put any restrictions on K and L; they may be but need not be disjoint, one may be subset of the other. We even admit that one or both of them are empty.

In order to enable the reader the understanding of this concept, let us first present the definition of composition for precise probabilities [4]. Let P and Q be two probability distributions of (groups of) variables X_K and X_L. Then

$$(P \triangleright Q)(X_{K\cup L}) = \frac{P(X_K) \cdot Q(X_L)}{Q(X_{K\cap L})},$$

[1] If there is no doubt, we will omit $[\mathcal{M}]$.

whenever $P(X_{K\cap L}) \ll Q((X_{K\cap L})$. Otherwise, it remains undefined.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be credal sets about X_K and X_L, respectively. Our goal is to define a new credal set, denoted by $\mathcal{M}_1 \triangleright \mathcal{M}_2$, which will be about $X_{K\cup L}$ and will contain all of the information contained in $\mathcal{M}_1$ and as much as possible of information of $\mathcal{M}_2$ (for the exact meaning see properties *(ii)* and *(iii)* of Lemma 1). The required properties are met by the following definition.

Definition 1 For two arbitrary credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ about X_K and X_L, a *composition* $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is defined by one of the following expressions:

[a] if $\mathcal{M}_1(X_{K\cap L}) = \mathcal{M}_2(X_{K\cap L})$, then

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) = \{(P_1 \cdot P_2)/P_2^{\downarrow K\cap L} : P_1 \in \mathcal{M}_1(X_K), P_2 \in \mathcal{M}_2(X_L), P_1^{\downarrow K\cap L} = P_2^{\downarrow K\cap L}\},$$

[b] if $\mathcal{M}_1(X_{K\cap L}) \neq \mathcal{M}_2(X_{K\cap L})$, and $\mathcal{M}_1(X_{K\cap L}) \ll \mathcal{M}_2(X_{K\cap L})$, then

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) = \{(P_1 \cdot P_2)/P_2^{\downarrow K\cap L} : P_1 \in \mathcal{M}_1(X_K), P_2 \in \mathcal{M}(X_L)\}),$$

[c] otherwise

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) = \mathcal{M}_1(X_K)^{\uparrow K\cup L}.$$

From point [b] of the definition one can see the importance of the definition of absolute continuity in the way presented in et the end of Section 2.2. Exactly this definition enables us to define the composition of two credal sets correctly.

The following lemma presents basic properties possessed by this operator of composition.

Lemma 1 *For arbitrary two credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ about X_K and X_L, respectively, the following properties hold true:*

(i) $\mathcal{M}_1 \triangleright \mathcal{M}_2$ *is a credal set about* $X_{K\cup L}$.

(ii) $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_K) = \mathcal{M}_1(X_K)$.

(iii) $\mathcal{M}_1 \triangleright \mathcal{M}_2 = \mathcal{M}_2 \triangleright \mathcal{M}_1 \iff \mathcal{M}_1(X_{K\cap L}) = \mathcal{M}_2(X_{K\cap L})$.

Proof.

(i) To prove that $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is a credal set about $X_{K\cup L}$ we have to distinguish cases [a] and [b] from [c]. In cases [a] and [b] it is enough to show that any $P \in \mathcal{M}_1 \triangleright \mathcal{M}_2$ is a probability distribution on $\mathbf{X}_{K\cup L}$. But it is obvious, as any $P \in (\mathcal{M}_1 \triangleright \mathcal{M}_2)$ is obtained via formula for composition of probability distributions (cf. e.g. [4]). In case [c] it is obvious too, as $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is a vacuous extension of an credal set about X_K to a credal set about $X_{K\cup L}$.

(ii) Again, we have to make the proof separately. If $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L})$ is obtained via [c], then the equality follows directly from the definition of vacuous extension. In cases [a] and [b] marginalization of a credal set is element-wise (as mentioned in the preceding section), therefore, analogous to the proof of *(i)* it is enough to prove that $\left((P_1 \cdot P_2)/P_2^{\downarrow K\cap L}\right)^{\downarrow K} = P_1$ for any $P_1 \in \mathcal{M}_1(X_K)$ and $P_2 \in \mathcal{M}_2(X_L)$. But it immediately follows from the results obtained for precise probabilities (see e.g. [4]).

(iii) First, let us assume that

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) = (\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_{K\cup L}).$$

Then also its marginals must be identical, particularly

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cap L}) = (\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_{K\cap L}).$$

Taking into account *(ii)* of this lemma we have

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cap L}) &= \left(((\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}))^{\downarrow K}\right)^{\downarrow K\cap L} \\ &= ((\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_K))^{\downarrow K\cap L} \\ &= (\mathcal{M}_1(X_K))^{\downarrow K\cap L} = \mathcal{M}_1(X_{K\cap L})\end{aligned}$$

and similarly

$$(\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_{K\cap L}) = \mathcal{M}_2(X_{K\cap L}),$$

from which the desired equality immediately follows.

Let, on the other hand, $\mathcal{M}_1(X_{K\cap L}) = \mathcal{M}_2(X_{K\cap L})$. In this case [a] of Definition 1 is applied and for any distribution P of $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L})$ there exist $P_1 \in \mathcal{M}_1(X_K)$ and $P_2 \in \mathcal{M}_2(X_L)$ such that $P_1^{\downarrow K\cap L} = P_2^{\downarrow K\cap L}$ and $P = (P_1 \cdot P_2)/P_2^{\downarrow K\cap L}$. But simultaneously (due to projectivity) $P = (P_1 \cdot P_2)/P_1^{\downarrow K\cap L}$, which is an element of $(\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_{K\cup L})$. Hence

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) = (\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_{K\cup L}),$$

as desired. □

Let us now illustrate the application of the operator of composition and its properties by three examples. The first shows what happens when $K \cap L = \emptyset$.

Example 2 Let

$$\mathcal{M}_1(X_1) = \mathrm{CH}\{[0.2, 0.8], [0.7, 0.3]\}$$

and

$$\mathcal{M}_2(X_2) = \mathrm{CH}\{[0.6, 0.4], [1, 0]\}$$

be two credal sets about X_1 and X_2, respectively. Then, as mentioned above, $\mathcal{M}_1(X_1)$ and $\mathcal{M}_2(X_2)$ are projective, and therefore $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is obtained via [a] in Definition 1:

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2)&(X_1X_2) \\ = \{[&0.7 - 0.5\alpha - 0.28\beta + 0.2\alpha\beta, \\ &0.28\beta - 0.2\alpha\beta, \\ &0.3 + 0.5\alpha - 0.12\beta - 0.2\alpha\beta, \\ &0.12\beta + 0.2\alpha\beta], \alpha, \beta \in [0, 1]\},\end{aligned} \tag{3}$$

which is nothing else than strong independence product of $\mathcal{M}_1(X_1)$ and $\mathcal{M}_2(X_2)$. The extreme points of $\mathcal{M}_1 \triangleright \mathcal{M}_2$ are

$$\begin{aligned}&[0.12, 0.08, 0.48, 0.32], [0.2, 0, 0.8, 0], \\ &[0.42, 0.28, 0.18, 0.12], [0.7, 0, 0.3, 0],\end{aligned} \tag{4}$$

nevertheless

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2)&(X_1X_2) \\ \neq \mathrm{CH}\{&[0.12, 0.08, 0.48, 0.32], [0.2, 0, 0.8, 0], \\ &[0.42, 0.28, 0.18, 0.12], [0.7, 0, 0.3, 0]\},\end{aligned}$$

as e.g.

$$\begin{aligned}&[0.41, 0.04, 0.39, 0.16] \\ &\in \mathrm{CH}\{[0.12, 0.08, 0.48, 0.32], [0.2, 0, 0.8, 0], \\ &\qquad [0.42, 0.28, 0.18, 0.12], [0.7, 0, 0.3, 0]\},\end{aligned}$$

but $[0.41, 0.04, 0.39, 0.16] \notin \mathcal{M}_1 \triangleright \mathcal{M}_2$. ♢

It is evident, that one would obtain the same result by application of the formula in [b] (if he/she omits the fact that the condition $\mathcal{M}_1(X_{K\cap L}) \neq \mathcal{M}_2(X_{K\cap L})$ is not fulfilled), as trivially $\mathcal{M}_1(X_{K\cap L})) \ll \mathcal{M}_2(X_{K\cap L})$. Nevertheless, these two cases must be distinguished in general case, as can be seen from the following two examples.

Let us note that in the examples that follow we will prefer to use extreme points of credal sets (4) to their general form (3), as it seems to be more convenient if we want to compare e.g. the resulting credal sets (or their marginals).

Example 3 Let

$$\mathcal{M}_1(X_1X_2) = \text{CH}\{[0.2, 0.8, 0, 0], [0.1, 0.4, 0.1, 0.4], [0.25, 0.25, 0.25, 0.25], [0, 0, 0.5, 0.5]\},$$

and

$$\mathcal{M}_2(X_2X_3) = \text{CH}\{[0.5, 0, 0.5, 0], [0.2, 0.3, 0.2, 0.3], [0.3, 0.3, 0.2, 0.2], [0, 0.6, 0, 0.4]\},$$

be two credal sets about variables X_1X_2 and X_2X_3, respectively. These two credal sets are not projective, as $\mathcal{M}_1(X_2) = \text{CH}\{[0.2, 0.8], [0.5, 0.5]\}$, while $\mathcal{M}_2(X_2) = \text{CH}\{[0.5, 0.5], [0.6, 0.4]\}$. Therefore [b] of Definition 1 should be applied:

$$\begin{aligned}
(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3) \subseteq \text{CH}\{&[0.2, 0, 0.8, 0, 0, 0, 0, 0],\\
&[0.08, 0.12, 0.32, 0.48, 0, 0, 0, 0],\\
&[0.1, 0, 0.4, 0, 0.1, 0, 0.4, 0],\\
&[0.04, 0.06, 0.16, 0.24, 0.04, 0.06, 0.16, 0.24],\\
&[0.1, 0.1, 0.4, 0.4, 0, 0, 0, 0],\\
&[0, 0.2, 0, 0.8, 0, 0, 0, 0],\\
&[0.05, 0.05, 0.2, 0.2, 0.05, 0.05, 0.2, 0.2],\\
&[0, 0.1, 0, 0.4, 0, 0.1, 0, 0.4],\\
&[0.25, 0, 0.25, 0, 0.25, 0, 0.25, 0],\\
&[0.1, 0.15, 0.1, 0.15, 0.1, 0.15, 0.1, 0.15],\\
&[0, 0, 0, 0, 0.5, 0, 0.5, 0],\\
&[0, 0, 0, 0, 0.2, 0.3, 0.2, 0.3]\\
&[0.125, 0.125, 0.125, 0.125,\\
&\quad 0.125, 0.125, 0.125, 0.125],\\
&[0, 0.25, 0, 0.25, 0, 0.25, 0, 0.25],\\
&[0, 0, 0, 0, 0.25, 0.25, 0.25, 0.25],\\
&[0, 0, 0, 0, 0, 0.5, 0, 0.5]\}.
\end{aligned}$$

If we, despite this fact, try to apply [a] of Definition 1, we will realize that only probability distributions P_1 and P_2 from $\mathcal{M}_1(X_1X_2)$ and $\mathcal{M}_2(X_2X_3)$, respectively, with marginal $P_i^{\downarrow\{2\}} = [0.5, 0.5]$ are projective, and therefore we obtain only a subset of $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3)$, namely a subset of

$$\begin{aligned}
\text{CH}\{&[0.25, 0, 0.25, 0, 0.25, 0, 0.25, 0],\\
&[0.1, 0.15, 0.1, 0.15, 0.1, 0.15, 0.1, 0.15],\\
&[0, 0, 0, 0, 0.5, 0, 0.5, 0],\\
&[0, 0, 0, 0, 0.2, 0.3, 0.2, 0.3]\},
\end{aligned}$$

which does not keep the first marginal in contrary to $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3)$, as can easily be checked. ♢

Example 4 Let $\mathcal{M}_1(X_1X_2)$ be as in previous example and

$$\mathcal{M}_2(X_2X_3) = \text{CH}\{[0.2, 0, 0.3, 0.5], [0, 0.2, 0, 0.8], [0.5, 0, 0.5, 0], [0.2, 0.3, 0.2, 0.3]\},$$

be a credal set about variables X_2X_3. Contrary to the previous example these two credal sets are projective, as

$$\mathcal{M}_1(X_2) = \text{CH}\{[0.2, 0.8], [0.5, 0.5]\} = \mathcal{M}_2(\text{X}_2),$$

therefore [a] of Definition 1 should be applied:

$$\begin{aligned}
(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3) \subseteq \text{CH}\{&[0.2, 0, 0.3, 0.5, 0, 0, 0, 0],\\
&[0, 0.2, 0, 0.8, 0, 0, 0, 0],\\
&[0.1, 0, 0.15, 0.25, 0.1, 0, 0.15, 0.25],\\
&[0, 0.1, 0, 0.4, 0, 0.1, 0, 0.4],\\
&[0.25, 0, 0.25, 0, 0.25, 0, 0.25, 0],\\
&[0.1, 0.15, 0.1, 0.15, 0.1, 0.15, 0.1, 0.15],\\
&[0, 0, 0, 0, 0.5, 0, 0.5, 0],\\
&[0, 0, 0, 0, 0.2, 0.3, 0.2, 0.3]\},
\end{aligned}$$

If, instead of it, one used [b] of the same definition, he/she would arrive to the credal set containing in addition the following extreme points

$$\begin{aligned}
&[0.2, 0, 0.8, 0, 0, 0, 0, 0],\\
&[0.08, 0.12, 0.32, 0.48, 0, 0, 0, 0],\\
&[0.1, 0, 0.4, 0, 0.1, 0, 0.4, 0],\\
&[0.04, 0.06, 0.16, 0.24, 0.04, 0.06, 0.16, 0.24],\\
&[0.25, 0, 0.09375, 0.15625, 0.25, 0, 0.09375, 0.15625],\\
&[0, 0.25, 0, 0.25, 0, 0.25, 0, 0.25],\\
&[0, 0, 0, 0, 0.5, 0, 0.1875, 0.3125],\\
&[0, 0, 0, 0, 0, 0.5, 0, 0.5].
\end{aligned}$$

Although both of these composed credal sets keep the first marginal, as can easily be checked, they differ form each other as concerns the second marginal: the correctly composed credal set keeps it, while the other has much bigger marginal, containing in addition the following extreme points:

$$[0.2, 0, 0.8, 0], [0.08, 0.12, 0.32, 0.48], [0.5, 0, 0.1875, 0.3125], [0, 0.5, 0, 0.5].$$

♢

Unfortunately, the definition is not elegant, nevertheless, its basic properties are satisfied and, as we shall see later, it holds also for other properties necessary for the introduction of compositional models.

4 Further Properties

As said in the Introduction, the operator of composition was originally introduced in (precise) probability theory. Nevertheless, any probability distribution may be viewed also as a singleton credal set (i.e. credal set containing a single point). One would expect that the operator of composition we have introduced in this contribution coincides with the probabilistic one if applied to singleton credal sets. And it is the case, as can be seen from the following lemma.

Lemma 2 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two singleton credal sets about X_K and X_L, respectively, where $\mathcal{M}_1(X_{K\cap L})$ is absolutely continuous with respect to $\mathcal{M}_2(X_{K\cap L})$. Then $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L})$ is also a singleton.*

Proof. Let us suppose that $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is not a singleton, i.e. it contains at least two different points. Due to the condition of absolute continuity both these points can be expressed in the form

$$P^i = P_1^i \cdot P_2^i/(P_2^i)^{\downarrow K\cap L}.$$

As $P^1 \neq P^2$, it is evident that either $P_1^1 \neq P_1^2$ or $P_2^1/(P_2^1)^{\downarrow K\cap L} \neq P_2^2/(P_2^2)^{\downarrow K\cap L}$ (and therefore also $P_2^1 \neq P_2^2$), or both. In any case, it is a contradiction as both credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ are singletons. □

The reader should however realize that the definition of the operator of composition for singleton credal sets is not completely equivalent to the definition of composition for probabilistic distributions. They equal each other only in case that the probabilistic version is defined. This is ensured in Lemma 2 by assuming the absolute continuity. In case it does not hold, the probabilistic operator is not defined while its credal version introduced in this paper is always defined (analogous to evidential operator of composition). Nevertheless, in this case, the result is not a singleton credal set. We shall illustrate it by a simple example.

Example 5 Let

$$\mathcal{M}_1(X_1X_2) = \{[0.25, 0.25, 0.25, 0.25]\},$$

and

$$\mathcal{M}_2(X_2X_3) = \{[0.5, 0.5, 0, 0]\},$$

be two singleton credal sets about variables X_1X_2 and X_2X_3, respectively. Let us compute $\mathcal{M}_1 \triangleright \mathcal{M}_2$. As $\mathcal{M}_1(X_2) = \{[0.5, 0.5]\}$, while $\mathcal{M}_2(X_2) = \{[1, 0]\}$, it is evident, that $\mathcal{M}_1$ is not absolutely continuous with respect to $\mathcal{M}_2$. Therefore we have via [c] of Definition 1:

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3) = \mathcal{M}_1(X_1X_2)^{\uparrow\{1,2,3\}}$$

which is evidently not a singleton any more.

Let us remark that $(\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_1X_2X_3)$, in contrast to $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3)$, is a singleton credal set

$$(\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_1X_2X_3) = \{[0.25, 0.25, 0, 0, 0.25, 0.25, 0, 0]\},$$

because $\mathcal{M}_2(X_2)$ is absolutely continuous with respect to $\mathcal{M}_1(X_2)$. ♢

From this example one can see that the operator of composition is not commutative. The following example demonstrates that this operator is neither associative.

Example 6 Let

$$\mathcal{M}_1(X_1) = \mathrm{CH}\{[0.2, 0, 8], [0.7, 0.3]\}$$

and

$$\mathcal{M}_2(X_2) = \{[0.5, 0.5]\}$$

be two credal sets about X_1 and X_2, respectively, and

$$\mathcal{M}_3(X_1X_2) = \mathrm{CH}\{[1, 0, 0, 0], [0, 1, 0, 0] \\ [0, 0, 1, 0], [0, 0, 0, 1]\}.$$

Then $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is obtained via [a] in Definition 1:

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2) = \mathrm{CH}\{[0.1, 0.1, 0.4, 0.4], [0.35, 0.35, 0.15, 0.15]\},$$

due to Definition 1 and $(\mathcal{M}_1 \triangleright \mathcal{M}_2) \triangleright \mathcal{M}_3 = \mathcal{M}_1 \triangleright \mathcal{M}_2$ according to property (2) of Lemma 1. On the other hand

$$(\mathcal{M}_2 \triangleright \mathcal{M}_3)(X_1X_2) = \mathrm{CH}\{[0.5, 0.5, 0, 0], [0.5, 0, 0, 0.5] \\ [0, 0.5, 0.5, 0], [0, 0, 0.5, 0.5]\},$$

via [c] of Definition 1. Now, computing $\mathcal{M}_1 \triangleright (\mathcal{M}_2 \triangleright \mathcal{M}_3)$ we obtain again via [c] of Definition 1

$$(\mathcal{M}_1 \triangleright (\mathcal{M}_2 \triangleright \mathcal{M}_3))(X_1X_2) = \mathrm{CH}\{[0.2, 0, 0.8, 0], [0.2, 0, 0, 0.8], \\ [0, 0.2, 0.8, 0], [0, 0.2, 0, 0.8], \\ [0.7, 0, 0.3, 0], [0.7, 0, 0, 0.3] \\ [0, 0.7, 0.3, 0], [0, 0.7, 0, 0.3]\},$$

which evidently differs from $(\mathcal{M}_1 \triangleright \mathcal{M}_2) \triangleright \mathcal{M}_3$. ♢

The following theorem reveals the relationship between strong independence and the operator of composition. It is, together with Lemma 1, the most important assertion enabling us to introduce multidimensional models.

Theorem 1 *Let $\mathcal{M}$ be a credal set about $X_{K\cup L}$ with marginals $\mathcal{M}(X_K)$ and $\mathcal{M}(X_L)$. Then*

$$\mathcal{M}(X_{K\cup L}) = (\mathcal{M}^{\downarrow K} \triangleright \mathcal{M}^{\downarrow L})(X_{K\cup L}) \qquad (5)$$

iff

$$(K \setminus L) \perp\!\!\!\perp (L \setminus K)|(K \cap L). \qquad (6)$$

Proof. Let us suppose that (5) holds. Since $\mathcal{M}_1(X_K)$ and $\mathcal{M}_2(X_L)$ are projective, [a] of Definition 1 is applied and therefore

$$\begin{aligned}\mathcal{M}(X_{K\cup L}) &\\ &= \{(P_1 \cdot P_2)/P_2^{\downarrow K\cap L} : P_1 \in \mathcal{M}(X_K),\\ &\quad P_2 \in \mathcal{M}(X_L), P_1^{\downarrow K\cap L} = P_2^{\downarrow K\cap L}\}).\end{aligned}$$

To prove (6) means to find for any P from $\mathcal{M}(X_{K\cup L})$ a pair of projective distributions P_1 and P_2 from $\mathcal{M}(X_K)$ and $\mathcal{M}(X_L)$, respectively, such that $P = (P_1 \cdot P_2)/P_1{}^{\downarrow K\cap L}$. But due to condition of projectivity, $\mathcal{M}(X_{K\cup L})$ consists of exactly this type of distributions.

Let on the other hand (6) be satisfied. Then any P from $\mathcal{M}(X_{K\cup L})$ can be expressed as conditional product of its marginals, namely

$$P = (P^{\downarrow K} \cdot P^{\downarrow K})/P^{\downarrow K\cap L},$$

$P^{\downarrow K} \in \mathcal{M}(X_K)$ and $P^{\downarrow L} \in \mathcal{M}(X_L)$. Therefore,

$$\begin{aligned}\mathcal{M}(X_{K\cup L}) &\\ = \quad &\{(P^{\downarrow K} \cdot P^{\downarrow K})/P^{\downarrow K\cap L} : P^{\downarrow K} \in \mathcal{M}_1(X_K),\\ &\quad P^{\downarrow L} \in \mathcal{M}_2(X_L))\},\end{aligned}$$

which concludes the proof. □

5 Compositional models

In this section we will consider repetitive application of the operator of composition with the goal to create a multidimensional model. Since the operator is neither commutative nor associative we have to specify in which order the low-dimensional credal sets are composed together. To make the formulae more transparent we will omit parentheses in case that the operator is to be applied from left to right, i.e., in what follows

$$\begin{aligned}\mathcal{M}_1 &\triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 \triangleright \ldots \triangleright \mathcal{M}_{n-1} \triangleright \mathcal{M}_n\\ &= (\ldots((\mathcal{M}_1 \triangleright \mathcal{M}_2) \triangleright \mathcal{M}_3) \triangleright \ldots \triangleright \mathcal{M}_{n-1}) \triangleright \mathcal{M}_n.\end{aligned}$$

Furthermore, we will always assume $\mathcal{M}_i$ be a credal set about X_{K_i}.

The reader familiar with some papers on probabilistic, possibilistic or evidential compositional models knows that one of the most important notions of this theory is that of a so-called *perfect sequence*, which will be now introduced also for credal sets.

Definition 2 A generating sequence of credal sets $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ is called *perfect* if

$$\begin{aligned}\mathcal{M}_1 \triangleright \mathcal{M}_2 &= \mathcal{M}_2 \triangleright \mathcal{M}_1,\\ \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 &= \mathcal{M}_3 \triangleright (\mathcal{M}_1 \triangleright \mathcal{M}_2),\\ &\vdots\\ \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_n &= \mathcal{M}_n \triangleright (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{n-1}).\end{aligned}$$

It is evident that the necessary condition for perfectness is pairwise projectivity of low-dimensional credal sets. However, the following example demonstrates the fact that it need not be sufficient.

Example 7 Let $\mathcal{M}_1(X_1)$ and $\mathcal{M}_2(X_2)$ as in Example 2 and let $\mathcal{M}_3(X_1, X_2)$ be defined as follows:

$$\begin{aligned}\mathcal{M}_3&(X_1, X_2)\\ &= \mathrm{CH}\{[0.1, 0.1, 0.5, 0.3], [0.2, 0, 0.8, 0],\\ &\qquad [0.4, 0.3, 0.2, 0.1], [0.7, 0, 0.3, 0]\}.\end{aligned}$$

It is evident, that $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are pairwise projective, as

$$\mathcal{M}_3(X_1) = \mathrm{CH}\{[0.2, 0.8,], [0.7, 0.3]\} = \mathcal{M}_1(\mathrm{X}_1)$$

and

$$\mathcal{M}_3(X_2) = \mathrm{CH}\{[0.6, 0.4,], [1, 0]\} = \mathcal{M}_2(\mathrm{X}_2)$$

and $\mathcal{M}_1$ and $\mathcal{M}_2$ are trivially projective, as already mentioned above. But they do not form a perfect sequence, as

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3)(X_1 X_2) = (\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1 X_2),$$

whose extreme points are in (4), while

$$(\mathcal{M}_3 \triangleright (\mathcal{M}_1 \triangleright \mathcal{M}_2))(X_1 X_2) = \mathcal{M}_3(X_1 X_3),$$

which is different. ♢

Therefore a stronger condition, expressed by the following assertion, must be fulfilled.

Lemma 3 *A generating sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ is perfect* iff *the pairs of credal sets $\mathcal{M}_j$ and $(\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1})$ are projective, i.e. if*

$$\begin{aligned}\mathcal{M}_j&(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})})\\ &= (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1})(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})}),\end{aligned}$$

for all $j = 2, 3, \ldots, n$.

Proof. This assertion is proved just by a multiple application of assertion (3) of Lemma 1:

$$\mathcal{M}_1 \triangleright \mathcal{M}_2 = \mathcal{M}_2 \triangleright \mathcal{M}_1 \iff \mathcal{M}_1(X_{K_2\cap K_1}) = \mathcal{M}_2(X_{K_2\cap K_1}),$$
$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 = \mathcal{M}_3 \triangleright (\mathcal{M}_1 \triangleright \mathcal{M}_2) \iff (\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K_3\cap(K_1\cup K_2)}) = \mathcal{M}_3(X_{K_3\cap(K_1\cup K_2)}),$$
$$\vdots$$
$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_n = \mathcal{M}_n \triangleright (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}n-1) \iff (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{n-1})(X_{K_n\cap(K_1\cup\ldots\cup K_{n-1})}) = \mathcal{M}_n(X_{K_n\cap(K_1\cup\ldots\cup K_{n-1})}).$$ □

From Definition 2 one can hardly see what are the properties of the perfect sequences; the main one is expressed by the following characterization theorem, which, hopefully, also reveals why we call these sequences perfect.

Theorem 2 *A generating sequence of credal sets* $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ *is perfect* iff *all the credal sets from this sequence are marginal to the composed credal set* $\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_n$*:*

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_n)(X_{K_j}) = \mathcal{M}_j(X_{K_j}),$$

for all $j = 1, \ldots, m$.

Proof. The fact that all credal sets $\mathcal{M}_j$ from perfect sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ are marginals of $(\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_n)$ follows from the fact that $(\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_j)$ is marginal to $(\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_n)$ (due to *(ii)* of Lemma 1) and $\mathcal{M}_j$ is marginal to

$$\mathcal{M}_j \triangleright (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1}) = \mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_j.$$

Suppose now that for all $j = 1, \ldots, n$, $\mathcal{M}_j$ are marginal assignments to $\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_n$. It means that all the assignments from the sequence are pairwise projective, and that each $\mathcal{M}_j$ is projective with any marginal assignment of $\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_n$, and consequently also with $\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1}$. So we get that

$$\mathcal{M}_j(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})}) = (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1})(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})})$$

for all $j = 2, \ldots, n$, which is equivalent, due to Lemma 3, to the fact that $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ is perfect. □

The following (almost trivial) assertion, which brings the sufficient condition for perfectness, resembles assertions concerning decomposable models.

Theorem 3 *Let a generating sequence of pairwise projective credal sets* $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ *be such that* $K_1, K_2, \ldots, K_n$ *meets the well-known running intersection property:*

$$\forall j = 2, 3, \ldots, n \ \ \exists \ell (1 \le \ell < j) \ \text{such that} \ \ K_j \cap (K_1 \cup \ldots \cup K_{j-1}) \subseteq K_\ell.$$

Then $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ *is perfect.*

Proof. Due to Lemma 3 it is enough to show that for each $j = 2, \ldots, n$ credal set $\mathcal{M}_j$ and the composed credal set $\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1}$ are projective. Let us prove it by induction.

For $j = 2$ the required projectivity is guaranteed by the fact that we assume pairwise projectivity of all $\mathcal{M}_1, \ldots, \mathcal{M}_n$. So we have to prove it for general $j > 2$ under the assumption that the assertion holds for $j-1$, which means (due to Theorem 2) that all $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{j-1}$ are marginal to $\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1}$. Since we assume that $K_1, \ldots, K_n$ meets the running intersection property, there exists $\ell \in \{1, 2, \ldots j-1\}$ such that $K_j \cap (K_1 \cup \ldots \cup K_{j-1}) \subseteq K_\ell$. Therefore $(\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1})(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})})$ and $\mathcal{M}_\ell(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})})$ are the same marginals of $\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1}$ and therefore they have to equal to each other:

$$(\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1})(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})}) = \mathcal{M}_\ell(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})}).$$

However we assume that $\mathcal{M}_j$ and $\mathcal{M}_\ell$ are projective and therefore also

$$(\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{j-1})(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})}) = \mathcal{M}_j(X_{K_j\cap(K_1\cup\ldots\cup K_{j-1})}),$$

as desired. □

It should be stressed at this moment that running intersection property of $K_1, K_2, \ldots, K_n$ is a sufficient condition which guarantees perfectness of a generating sequence of pairwise projective assignments. By no means this condition is necessary as it will be shown in the following example.

Example 8 Simple example is given by two credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ from Example 7 about X_1 and X_2, respectively, and the third credal set $\tilde{\mathcal{M}}_3 = \mathcal{M}_1 \triangleright \mathcal{M}_2$. Considering sequence $\mathcal{M}_1, \mathcal{M}_2, \tilde{\mathcal{M}}_3$, it is evident that $K_1 = \{1\}, K_2 = \{2\}, K_3 = \{1, 2\}$ do not meet the running intersection property. And yet the sequence $\mathcal{M}_1, \mathcal{M}_2, \tilde{\mathcal{M}}_3$ is perfect because all the credal sets are marginal (or equal) to $\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \tilde{\mathcal{M}}_3$. Let us note that if we chose instead of $\tilde{\mathcal{M}}_3$ any other credal set $\mathcal{M}_3$ about X_1X_2 different from $\tilde{\mathcal{M}}_3 = \mathcal{M}_1 \triangleright \mathcal{M}_2$, e.g. that from Example 7 the generating sequence $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ would not be perfect any more. ♢

Therefore we can see that perfectness of a sequence is not only a structural property connected with the properties of $K_1, K_2, \ldots, K_n$ but depends also on specific values of the respective basic assignments.

As said already in the introduction, in precise probability framework any multidimensional distribution representable by a Bayesian network can also be represented in the form of a perfect sequence, and vice versa. For more details the reader is referred to [7], where also an algorithm for transformation of a perfect sequence of probability distributions into a Bayesian network can be found.

Recently we have found out, that in evidence theory transformation from evidential network to a compositional model is exactly the same as in precise probability framework, but the opposite process is a bit different — it may happen that resulting model expressed by evidential network is less precise than that the compositional model [14].

At present we do not know too much about the relationship between compositional models of multidimensional credal sets and credal networks. We conjecture it will be similar to the evidential framework. But it is only a conjecture, the research is just at the beginning. Nevertheless, to clarify this relationship is our first goal.

6 Conclusions

Graphical Markov Models were designed to enable description of real-life problems by probabilistic models. This is because problems of practice lead to multidimensional models, where the number of dimensions is expressed rather in hundreds than in tens. Inspired by the original probabilistic approach the paper is the first attempt to build up compositional models of multidimensional credal sets as an alternative to Graphical Markov Models with imprecision.

We have defined credal set operator of composition manifesting all the main characteristics of its probabilistic pre-image. Even more, there is one point in which the credal set operator of composition is superior to the probabilistic one (similarly to the operator in the evidential framework): thanks to the ability of credal sets to model total ignorance, the operator of composition is for credal sets always defined, which is not the case in the (precise) probabilistic framework.

In the paper we have proved the basic properties of the operator (including the relationship to strong independence) necessary for the introduction of compositional models and their most important special case, *perfect sequence models.*

Naturally, there are still many open problems to be solved. The most important one is a design of efficient computational procedures for this type of models. At this moment we know very little about similarities and differences between the described compositional models and other multidimensional models such as [1, 3, 11], as well as about the relation between the compositional models developed for credal sets and those introduced in possibility [12, 13] and evidence [8, 9] theories.

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