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Distributed stabilisation of spatially invariant systems: positive polynomial approach

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Abstract The paper gives a computationally feasible characterisation of spatially distributed controllers stabilising a linear spatially invariant system, that is, a system described by linear partial differential equations with coefficients independent on time and location. With one spatial and one temporal variable such a system can be modelled by a 2-D transfer function. Stabilising distributed feedback controllers are then parametrised as a solution to the Diophantine equation $ax + by = c$ for a given stable bi-variate polynomial c. The paper is built on the relationship between stability of a 2-D polynomial and positiveness of a related polynomial matrix on the unit circle. Such matrices are usually bilinear in the coefficients of the original polynomials. For low-order discrete-time systems it is shown that a linearising factorisation of the polynomial Schur-Cohn matrix exists. For higher order plants and/or controllers such factorisation is not possible as the solution set is non-convex and one has to resort to some relaxation. For continuous-time systems, an analogue factorisation of the polynomial Hermite-Fujiwara matrix is not known. However, for low-order systems and/or controller, positivity conditions on the closed-loop polynomial coefficients can be invoked. Then the computational framework of linear matrix inequalities can be used to describe the stability regions in the parameter space using a convex constraint.

Keywords Multidimensional systems · Algebraic approach · Control design · Positiveness · Convex optimisation

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1 Introduction

Control of *spatially distributed systems* has always been a very active research topic with engineering applications in many areas. Such systems can be mathematically described by *partial differential equations* (PDEs). A vast amount of research has been conducted on control design for PDEs. One group of such design methods relies on the possibility to affect the behaviour of the system (that is, a solution in the domain) by controlling the boundary conditions, so-called *boundary control*, nicely documented in Krstic and Smyshlyaev (2008). In contrast, this paper focuses on systems featuring a dense and regular array of sensors and actuators stretching all over the domain. This grid then enforces a natural spatial discretisation of the system. Provided the parameters of the system do not depend on location, the resulting mathematical model is *shift invariant*. Of course, shift invariance assumes the domain is infinite, that is, the boundaries are at infinite distance, which is not realistic. Nonetheless, the assumption of shift invariance seems a reasonable simplification for design. Moreover, it is assumed that exciting the system at any location, response can only be observed in the nearest neighbourhood, in other words, the dynamics of the system is *localised*.

It turned out as early as in the late 1960s and early 1970s that this type of systems can be studied within a broader class of systems whose coefficients are functions of parameters. The right mathematical concept appeared to be that of linear systems over rings, because the coefficients in the state-space matrices and the coefficients in the transfer functions are elements of a ring. This broader class of systems also includes systems with delays or systems over integers. Among the pioneers in the area of linear systems over (commutative) rings were Kalman and his doctoral student Rouchaleau (1972) and Kamen (1975). Readable surveys were given by Sontag (1976) and Kamen (1978). Specialisation of these general results to spatially distributed systems was given in another survey paper by Bose (1985). Multidimensional systems coming from discretisation of linear PDEs were studied by Brockett and Willems (1974).

A few papers followed in the early 1980s such as Kamen and Khargonekar (1984) and Khargonekar and Sontag (1982), but the interest of the community into this field faded away towards the end of 1980s and throughout 1990s. Surprisingly, the field was revived around the beginning of the new millennium, through the papers by Bamieh et al. (2002), D'Andrea and Dullerud (2003), Gorinevsky (2002), Gorinevsky and Stein (2003), Stewart et al. (2003), Jovanovic and Bamieh (2005) and Stein and Gorinevsky (2005), just to name a few. A related field of repetitive systems Rogers et al. (2007) and using the iterative learning control (ILC) for spatio-temporal dynamics Cichy et al. (2011) also comes along with this revival. Major incentives for this revival came from the availability of both new technologies (MEMS, adaptive optics, networked systems, low cost UAVs or mobile robots, fabrication techniques in silicon industry, …) and new theoretical and computational tools such as powerful convex and non-convex optimisation techniques, namely linear (LMI) and bilinear matrix inequalities and semidefinite programming.

Distinguished feature of this paper is that while majority of the mentioned papers rely on state-space formalism, here the preference is given to input-output description, that is, models are given in the form of a fraction of two multivariate polynomials. Major justification for this preference is availability of some promising results from multidimensional system theory, such as Šebek (1994), which extend the intuitive 1-D polynomial framework pioneered (among a few others) by Kučera (1979). Combined with recent advances in the theory and computation with positive polynomials surveyed in Dumitrescu (2007) and Henrion and Garulli (2005), the polynomial approach appears promising to study the presented class of problems.

The basic idea presented in this paper is that stabilisation of a system modelled by a 2-D spatio-temporal transfer function can be studied by "positivisation" of a related symmetric polynomial matrix on the unit circle. Even though computationally feasible convex LMI conditions are available for testing positiveness of a given polynomial matrix, these tools may become unuseful when the same formalism is used for controller synthesis. The reason is that the polynomial matrix to be tested for positiveness depends bilinearly on the parameters of the system and the controller. However, it will be shown in this paper that thanks to a linearising factorisation of the Schur-Cohn matrix, simple LMI condition can be obtained for special instances of the problem, which gives computationally feasible constraints on stability region in the parameter (or coefficient) space of the characteristic polynomial.

We shall restrict to linear spatially-distributed time-invariant systems described by a *parabolic* PDE, which can be written in the form

$$d\frac{\partial u(x,t)}{\partial t} - c\,\Delta u(x,t) - b\,\nabla u(x,t) - a\,u(x,t) = f(x,t), \tag{1}$$

where $a > 0, b > 0, c > 0, d > 0$ be constants, u be a solution and f right-hand side. Parabolic equations contain first derivation with respect to time. A heat conduction, diffusion, chemical reactions and other irreversible processes are described by a parabolic PDE.

2 Stability of a 2-D transfer function

2.1 Discrete-time case

In the paper we shall concentrate on the linear spatially-distributed time-invariant systems described by constant coefficient PDEs. Assuming infinite spatial domain, a sequence of two z-transforms can be performed to obtain a transfer function: one unilateral, corresponding to the temporal variable t and the other bilateral, corresponding to the spatial variable x. Of course, this assumption is never valid for a physical system, but it simplifies the design a lot, allowing to neglect the influence of boundary conditions. Validity of controllers designed with such approximate models must be checked aposteriori, mostly using simulations.

Let $f(k,l)$ be a real-valued function defined on the Cartesian product $\mathbb{Z}\times\mathbb{Z}$ and $f(k,l) = 0$ for all $k < 0$. The z-transform of $f(k,l)$ is given as

$$F(z,w) = \sum_{k=0}^{+\infty}\sum_{l=-\infty}^{+\infty} f(k,l)\,w^{-l}\,z^{k}. \tag{2}$$

We assume that f is constrained, so that $F(z,w)$ is well defined for (z,w) belonging to some subset of $\mathbb{C}\times\mathbb{C}$. See also Bose (1985).

Using one of the common discretisation schemes for both the temporal and the spatial variables, the system can be described by the transfer function

$$P(z,w) = \frac{b(z,w)}{a(z,w)}, \tag{3}$$

where the variable z corresponds to time shift and the variable w corresponds to a shift along the spatial coordinate axes. Since the system is causal in time and non-causal in space, the polynomial a is *one-sided* in z and *two-sided* in w. For physical systems, it is reasonable to assume their spatial symmetry: the polynomial a can be assumed in the form

$$a(z, w) = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l}\, z^k \left(w^l + w^{-l} \right) \tag{4}$$

and similarly for the polynomial b. Following the systems-over-rings concept, the notation

$$a[w](z) = a_n(w)\, z^n + a_{n-1}(w)\, z^{n-1} + \cdots + a_0(w) \tag{5}$$

can be used to emphasise that the polynomial a can be viewed as a polynomial in z with coefficients being functions of w.

Stability of such systems can be studied by analysing roots of their denominator polynomials, with the first stability criterion given in Justice and Shanks (1973). This is similar to the lumped (1-D) case, but having two variables, the values of the denominator polynomial $a(z, w)$ must be studied on the unit bidisc, bicircle or a combination of the unit disc and circle. It was shown in Goodman (1977) that Shank's theorem loses its validity in the case when the system has a nonessential singularity of the second kind, in which case the numerator can save the stability even if the denominator introduces a singularity on the distinct boundary of stability region. This is also discussed in Jury (1978) or Dudgeon and Mersereau (1984). We disregard these uncommon situations by desiring a closed-loop system with no singularity on the boundary of the stability region. Examining denominator polynomials then suffices. Specialising the general Shank's theorem to systems with spatio-temporal transfer functions, the classical results on stability follows.

Theorem 1 (Justice and Shanks 1973) *Spatially distributed system described by the transfer function* (3) *with the polynomials free of a common factor is BIBO stable if*

$$a(z, w) \neq 0 \quad \textit{for all} \quad \{|w| = 1\} \cap \{|z| \leq 1\}\,.$$

An immediate reformulation goes in the spirit of the concept of systems over rings.

Corollary 1 *Spatially distributed system described by the transfer function* (3) *with the polynomials free of a common factor is BIBO stable if* $a[w](z) = a_n(w)\, z^n + a_{n-1}(w)\, z^{n-1} + \cdots + a_0(w)$ *is stable (roots outside the open unit circle) for all* $|w| = 1$.

A vast number of extensions and simplifications have been proposed in the last decades, such as Strintzis (1977), Mastorakis (1997, 1999), Henrion et al. (2001), Serban and Najim (2007), just to name a few. The key trick used in this paper is described in Šiljak (1975). It consists in establishing an equivalence of stability of a 2-D polynomial and positiveness of a certain symmetric polynomial matrix (Schur-Cohn matrix for discrete-time systems) on the unit circle. Apart from algebraic criteria like Šiljak (1973), the advanced toolset of linear matrix inequalities can be used to test positiveness of a polynomial matrix on the unit circle, see Dumitrescu (2007), Henrion and Garulli (2005), Trentelman and Rapisarda (1999), Genin et al. (2002, 2003). The LMI formalism offers easy extension from analysis to constructive synthesis, which is the topic for the next section. The Schur-Cohn matrix H for a polynomial $a(z, w)$ has the form

$$H(w) = S_1^{\mathrm{T}}(w) S_1(w) - S_2^{\mathrm{T}}(w) S_2(w), \tag{6}$$

where

$$S_1 = \begin{pmatrix} a_0(w) & a_1(w) & \cdots & a_{n-1}(w) \\ 0 & a_0(w) & \cdots & a_{n-2}(w) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0(w) \end{pmatrix}, \qquad S_2 = \begin{pmatrix} a_n(w) & a_{n-1}(w) & \cdots & a_1(w) \\ 0 & a_n(w) & \cdots & a_2(w) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n(w) \end{pmatrix}.$$

See Barnett (1983) for a comprehensive overview. The following lemma formally states the key tool.

Lemma 1 *A polynomial $a[w](z)$ of the form (4) is stable if and only if its Schur-Cohn matrix $H(w)$ is positive definite on the unit circle, that is, $H(w) \succ 0$ for all $|w| = 1$.*

The Schur-Cohn matrix is a symmetric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{2m}(w^{2m} + w^{-2m})$ of size n. Using the result stated in Trentelman and Rapisarda (1999) and Genin et al. (2002) the matrix is positive definite for all $|w| = 1$ if and only if there exists a symmetric matrix M of size $2nm$ such that

$$
\begin{aligned}
L(M) = &\left(\begin{array}{c|ccc} H_0 & H_1 & \cdots & H_{2m} \\ \hline H_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{2m} & 0 & \cdots & 0 \end{array}\right) \\
&+ \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) - \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0.
\end{aligned}
\tag{7}
$$

Implementation of this test is easy with the available numerical solvers for linear matrix inequalities, such as a combination of Sturm (1999) and Löfberg (2004).

2.2 Continuous-time case

In the continuous-time case, the situation is similar and yet different. Transfer function can be obtain performing unilateral Laplace transform, corresponding to the temporal variable t, and bilateral z-transform, corresponding to the spatial variable x. Let $f(t, l)$ be a real-valued function defined on the Cartesian product $\mathbb{R} \times \mathbb{Z}$ and $f(t, l) = 0$ for $t < 0$. The joint Laplace and z-transform of $f(t, l)$ is given as

$$F(s, w) = \int\limits_{t=0}^{\infty} \left[\sum_{l=-\infty}^{+\infty} f(t, l)\, w^{-l} \right] e^{-s\,t}\, \mathrm{d}t. \tag{8}$$

We assume that f is constrained, so that $F(s, w)$ is well defined for (s, w) belonging to some subset of $\mathbb{C} \times \mathbb{C}$. See also Bose (1985).

Using one of the common discretisation schemes for the spatial variable, the system can be described by the transfer function

$$P(s, w) = \frac{b(s, w)}{a(s, w)}, \tag{9}$$

where the variable s corresponds to time and the variable w corresponds to shift along the spatial coordinate axes. Since the system is causal in time and non-causal in space, the polynomial a is *one-sided* in s and *two-sided* in w. For physical systems, it is reasonable to assume spatial symmetry: the polynomial a can be assumed in the form

$$a(s, w) = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} s^k \left(w^l + w^{-l} \right), \tag{10}$$

similarly for the polynomial b. Following the systems-over-rings concept, the notation

$$a[w](s) = a_n(w)s^n + a_{n-1}(w)s^{n-1} + \cdots + a_0(w) \tag{11}$$

can be used to emphasise that the polynomial a can be viewed as a polynomial in s with the coefficients being functions of w.

Theorem 2 (Bose 1985, Theorem 4.3, pp. 126) *Spatially distributed system described by the transfer function* (9) *with the polynomials free of a common factor is BIBO stable if*

$$a(s, w) \neq 0 \quad \textit{for all} \quad \{|w| = 1\} \cap \{\Re\{s\} \geq 0\}\,.$$

An immediate reformulation goes in the spirit of the concept of systems over rings.

Corollary 2 *Spatially distributed system described by the transfer function* (9) *with the polynomials free of a common factor is BIBO stable if* $a[w](s) = a_n(w)\, s^n + a_{n-1}(w)\, s^{n-1} + \cdots + a_0(w)$ *is stable (has its roots in the left half-plane) for all* $|w| = 1$.

In the continuous-time case, Hermite-Fujiwara matrix plays a role in the stability testing. Let $a^*(s)$ denote $a(-s)$. The Hermite-Fujiwara matrix is defined as $H = (h_{ij})_{i,j=1,\ldots,n}$, where

$$h_{ij} = (-1)^{j-1} \sum_{k=1}^{m_{ij}} a_{j+k-1}\, a^*_{i-k} - a_{i-k}\, a^*_{j+k-1},$$

where $m_{ij} = \min(i, n - j + 1)$. The following lemma is reformulation of Lemma 1 for the continuous-time case.

Lemma 2 *A polynomial* $a[w](s)$ *of the form* (10) *is stable if and only if its Hermite-Fujiwara matrix* $H(w)$ *is positive definite on the unit circle, that is,* $H(w) \succ 0$ *for all* $|w| = 1$.

The Hermite-Fujiwara matrix is a symmetric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{2m}(w^{2m} + w^{-2m})$ of size n. The same result (Trentelman and Rapisarda 1999; Genin et al. 2002) that was used for testing positiveness of Schur-Cohn matrix on the unit circle can be used for Hermite-Fujiwara matrix.

The LMI formalism offers easy extension from analysis to constructive synthesis. But the obvious obstacle that prevents us from using the LMIs directly is the bilinear dependence of the coefficients of the Schur-Cohn and the Hermite-Fujiwara matrices on the coefficients of the original polynomials (Henrion et al. 1999). The next section offers analysis of possible solutions for low-order systems and/or controllers.

3 Stabilisation via positivity of a polynomial matrix

In this section, we consider a closed-loop denominator polynomial, that is, its coefficients also depend on the parameters of the controller. We consider the control scheme of Fig. 1. Since (1) is of order one in the time variable t, we assume the following.

Assumption 1 The system is of order one in the time variable, hence, the polynomials (4) for a discrete-time system and (10) for a continuous-time system are of degree $n = 1$ in their respective temporal variables z and s.

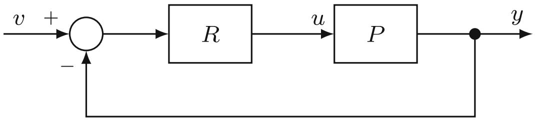

Fig. 1 Standard feedback configuration

3.1 Discrete-time case

A controller is given by the transfer function

$$R(z, w) = \frac{y(z, w)}{x(z, w)}. \tag{12}$$

The characteristic polynomial determining the stability of the closed loop is then $ax + by = c$. Expand this polynomial formally as

$$c = \sum_{k=0}^{\hat{n}} \sum_{l=0}^{\hat{m}} c_{k,l}\, z^k \left(w^l + w^{-l}\right). \tag{13}$$

Extending the well-known results on solvability of a Diophantine equation in the 1-D setting (Kučera 1979), it is clear that the closed-loop polynomial c has to be of degree $2n - 1$ or greater in the variable z to design a realisable controller. Since $n = 1$, $\hat{n} \geq 2n - 1 = 1$. Thus, let $\hat{n} = 1$. Our task simplifies to finding all stable polynomials with the degree equal to 1 in the variable z corresponding to time. The polynomial can have an arbitrary degree in the variable w corresponding to the spatial shift.

Due to the above assumptions, S_1 and S_2 are now scalars. However, the matrix notation is kept in the next lines to highlight the troubles with higher order systems and/or controllers. Without a loss of generality we suppose $a_0(w) > 0$, i. e. $S_1 > 0$. A theorem can be stated

Theorem 3 *A polynomial c of the form* (13) *with $\hat{n} = 1$ is stable if and only if*

$$\begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix} \succ 0, \tag{14}$$

where $H(w, w^{-1}) = S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c.

Proof It follows from Sylvester's criterion that (14) holds if and only if

$$S_1 > 0 \quad \text{and} \quad \det \begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix} > 0.$$

The former condition was assumed (without a loss of generality), the latter is equal to $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}} > 0$. Use of Lemma 1 concludes the proof.

The left-hand side matrix in (14) is a symmetric trigonometric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{\hat{m}}(w^{\hat{m}} + w^{-\hat{m}})$ of size $2\,\hat{n}$. It is positive semidefinite for $|w| = 1$ if and only if there exists a symmetric matrix M of size $2\,\hat{n}\,\hat{m}$ such that

$$
\begin{aligned}
L(M, H_0, H_1, \ldots, H_{\hat{m}}) &= \left(\begin{array}{c|ccc} H_0 & H_1 & \cdots & H_{\hat{m}} \\ \hline H_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{m}} & 0 & \cdots & 0 \end{array}\right) \\
+ \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) &- \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0,
\end{aligned} \tag{15}
$$

where, in contrast with (7), the LMI variable M is not the only variable, the coefficients of H are LMI variables as well. In other words, the coefficients of H are unknown and are subject of design. Theorem 3 allows to complete the following lemma. □

Lemma 3 *Consider a plant described by* (1) *with the transfer function* (3). *A controller with the transfer function* (12) *stabilises the plant if*

$$ax + by = c$$

is a such polynomial that (15) *holds with*

$$H(w) = \begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix},$$

where $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c.

Proof Follows immediately from Theorem 3 and the fact that c is the characteristic polynomial of closed-loop system. □

Remark 1 For $n > 1$ the solution set to the inequality $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}} > 0$ is non-convex, hence approaches for approximation can be used to solve the non-convex polynomial matrix inequalities. See Rami and Henrion (2010) for an early analysis.

3.2 Continuous-time case

In the continuous-time case, a controller is given by the transfer function

$$R(s, w) = \frac{y(s, w)}{x(s, w)}. \tag{16}$$

The characteristic closed-loop polynomial is then $ax + by = c$, or expanding

$$c = \sum_{k=0}^{\hat{n}} \sum_{l=0}^{\hat{m}} c_{k,l}\, s^k \left(w^l + w^{-l}\right). \tag{17}$$

Like in the discrete-time case, the closed-loop polynomial c has to be of degree $2n - 1$ or greater in the variable s to design a realisable controller. Since $n = 1, \hat{n} \geq 2n - 1 = 1$.

However, in the continuous-time case, we can rely on the nice fact that positivity of the coefficients of the polynomial constitutes both the necessary and sufficient conditions for stability and we can avoid building the Hermite-Fujiwara matrix altogether. We assume that the controller (16) is at most first-order in the time variable. Hence, $\hat{n} = 2$ and the closed-loop polynomial (17) is in the form $c = c_2(w)\, s^2 + c_1(w)\, s + c_0(w)$. The degree of c in w can be arbitrary. The sufficient and necessary condition for stability of c can now be expressed by this lemma.

Lemma 4 *A polynomial c of the form* (17) *with* $\hat{n} = 2$ *is stable if and only if*

$$\begin{pmatrix} c_0(w) & 0 & 0 \\ 0 & c_1(w) & 0 \\ 0 & 0 & c_2(w) \end{pmatrix} \succ 0 \tag{18}$$

for all $|w| = 1$.

Proof Follows from Theorem 2 and well-known sufficient and necessary condition for stability of second degree polynomials. □

The left-hand side of (18) is a symmetric trigonometric polynomial matrix. The formulation (15) can be used to check its positiveness.

Remark 2 Generally, for $\hat{n} > 2$, an analogous condition diag $[c_0(w), c_1(w), \ldots, c_{\hat{n}}(w)] \succ 0$ is no longer sufficient.

We can now complete the following lemma.

Lemma 5 *Consider a plant described by* (1) *with transfer function* (9). *A controller with transfer function* (16) *stabilises the plant if*

$$ax + by = c$$

is a such polynomial that (15) *holds with*

$$H(w) = \begin{pmatrix} c_0(w) & 0 & 0 \\ 0 & c_1(w) & 0 \\ 0 & 0 & c_2(w) \end{pmatrix}$$

where $c = c_2(w)\, s^2 + c_1(w)\, s + c_0(w)$ *is the closed-loop characteristic polynomial.*

Proof Follows immediately from Lemma 4. □

4 Example: controlled heat conduction in aluminium rod

In this section, the above described concept will be demonstrated by means of a numerical example. A distributed controller for a heat conduction in a long thin metal rod will be designed.

4.1 Model of the system

A setup for a controlled heat conduction (or temperature distribution) in a rod equipped with an array of temperature sensors and heaters is sketched in Fig. 2.

4.1.1 Heat equation

The system is described by the well-known *heat equation*, which has the form

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + \hat{q}(t,x), \tag{19}$$

where u denotes temperature (°C), $\hat{q}$ the input heat (°Cs^{-1}), t and x denote time (s) and a spatial coordinate (m), respectively, $\kappa = \frac{\varkappa}{\rho\, c_p}$ is a constant ($\mathrm{m}^2\,\mathrm{s}^{-1}$), where $\varkappa$ is the thermal conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$), ρ is the density ($\mathrm{kg\,m^{-3}}$) and c_p is the heat capacity per unit mass ($\mathrm{J\,K^{-1}\,kg^{-1}}$). Reasonable values are $\varkappa = 230$, $\rho = 2700$ and $c_p = 900$.

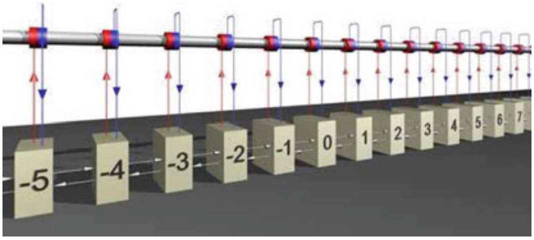

Fig. 2 Distributed control of a distributed parameter system: a rod with an array of heaters and temperature sensors and a distributed controller (an array of controllers)

4.1.2 Discretisation

The corresponding transfer function can be derived as follows, see also Augusta et al. (2007). In the discrete-time case, discretisation of (19) using finite difference methods (Strikwerda 1989) approximates partial derivatives by differences, i. e.

$$\left(\frac{\partial u}{\partial t}\right)_{k,i} = \frac{u_{k+1,i} - u_{k,i}}{T}, \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{k,i} = \frac{u_{k,i-1} - 2\,u_{k,i} + u_{k,i+1}}{h^2},$$

where $T > 0$ is the sampling (time) period and $h > 0$ denotes the distance between the nodes along the rod. Substitution of the above formulae into (19) gives partial recurrence equation

$$u_{k+1,i} = \frac{T\kappa}{h^2} u_{k,i-1} + \left(1 - 2\frac{T\kappa}{h^2}\right) u_{k,i} + \frac{T\kappa}{h^2} u_{k,i+1} + q_{k,i}, \tag{20}$$

where k corresponds to discrete time and i to the coordinate of the node and, for simplicity, $q_{k,i} = T\,\hat{q}_{k,i}$.

In the continuous-time case, we approximate partial derivatives with respect to x only, i. e.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}}{h^2}.$$

Substitution into (19) gives

$$\left(\frac{\partial u(t)}{\partial t}\right)_i = \kappa \frac{u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}}{h^2} + f(t)_i. \tag{21}$$

4.1.3 Selection of sampling rate using von Neumann's stability analysis

Clearly, for any subsequent analysis which should produce acceptable results, the model discretised in both time and space must give a sufficiently accurate approximation to the original model described by (19). A standard tool for this is the von Neumann's analysis of the (numerical) stability of an iterative scheme (20). This will give a relation between T and h to guarantee the convergence.

To proceed, consider the case when zero external heat is applied. Then (20) has the form

$$u_{k+1,i} = \frac{T\kappa}{h^2} u_{k,i-1} + \left(1 - 2\,\frac{T\kappa}{h^2}\right) u_{k,i} + \frac{T\kappa}{h^2} u_{k,i+1}.$$

Also, replace $u_{k,i}$ by $g^k\, e^{j\,i\,\theta}$ to obtain

$$g^{k+1}\, e^{j\,i\,\theta} = \frac{T\kappa}{h^2} g^k \left(e^{j\,(i-1)\,\theta} + e^{j\,(i+1)\,\theta}\right) + \left(1 - 2\,\frac{T\kappa}{h^2}\right) g^k\, e^{j\,i\,\theta},$$

where θ is the spatial frequency and $j = \sqrt{-1}$. The parameter g is termed *the amplification factor* and the recurrence equation is stable if and only if $|g| \leq 1$, see Strikwerda (1989) for details. Using Euler's formula and routine simplification now gives

$$g = \frac{T\kappa}{h^2}\left(e^{-j\theta} + e^{j\theta}\right) + \left(1 - 2\,\frac{T\kappa}{h^2}\right) = \frac{T\kappa}{h^2}\,2\cos\theta + \left(1 - 2\,\frac{T\kappa}{h^2}\right).$$

Hence, $|g| \leq 1$ when $\frac{T}{h^2} < \frac{1}{2\kappa}$. This gives the required relationship between the spatial and time sampling rate.

4.1.4 Transfer function

Performing the two z-transforms on (20), the transfer function is

$$P = \frac{z}{1 + \left(2\,\frac{T\kappa}{h^2} - 1\right)z - \frac{T\kappa}{h^2}(w + w^{-1})z}, \tag{22}$$

where the system output is the temperature at the particular place and time and the system input is the heat (power brought to the system at the given place and time). Choosing $T = 1$s and $h = \frac{1}{59}$ m in agreement with the above analysis, the transfer function is

$$P = \frac{b(z, w)}{a(z, w)} = \frac{z}{1 - 0.34\,z - 0.32(w + w^{-1})z}. \tag{23}$$

Joint Laplace and z-transform on (21) gives the transfer function

$$P(s, w) = \frac{1}{s - \frac{\kappa}{h^2}\left(w - 2 + w^{-1}\right)}. \tag{24}$$

Choosing $h = \frac{1}{59}$ m, the transfer function is

$$P(s, w) = \frac{1}{s - 0.33\left(w - 2 + w^{-1}\right)}. \tag{25}$$

4.2 Controller design

We will show two approaches to control design using methods presented in this paper. If we want to design a controller with a fixed structure, we can substitute x and y of prescribed degrees into $c = ax + by$ and find coefficients of x and y to c be stable. This will be done in Sect. 4.2.1. Another way is to find a stable closed-loop characteristic polynomial and solving $ax + by = c$ to design a particular controller. This will be proceeded in Sect. 4.2.2.

4.2.1 Discrete-time case

Consider a controller with the transfer function

$$R(z, w) = r_0 + r_1(w + w^{-1}), \tag{26}$$

where r_0 and r_1 are real constants. This controller behaves as a proportional controller in time and a first-order controller in space, that is, it takes the measurement of the temperature

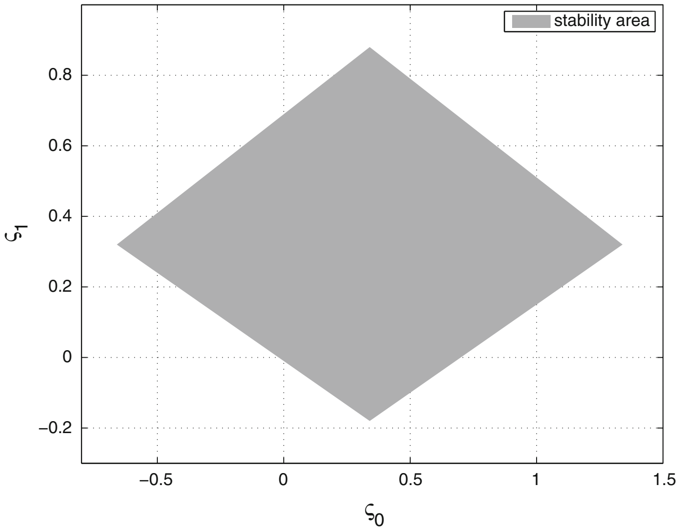

Fig. 3 The values of r_0 and r_1 for which the polynomial (27) is stable

from the neighbouring node into consideration. The characteristic polynomial of closed-loop system has the form

$$c(z, w) = 1 + (r_0 - 0.34)z + (r_1 - 0.32)\,(w + w^{-1})z. \tag{27}$$

Next, we find some constants r_0 and r_1 stabilising the closed-loop system. The method described in Sect. 3.1 gives

$$\begin{aligned} S_1 &= 1, \\ S_2 &= r_0 + \left(r_1 - \frac{8}{25}\right)(w + w^{-1}) - \frac{17}{50}, \\ H_0 &= \begin{pmatrix} 1 & r_0 - \frac{17}{50} \\ r_0 - \frac{17}{50} & 1 \end{pmatrix}, \\ H_1 &= \begin{pmatrix} 0 & r_1 - \frac{8}{25} \\ r_1 - \frac{8}{25} & 0 \end{pmatrix}. \end{aligned}$$

Using SeDuMi and Yalmip we can check that *some* matrix M in (15) exists and r_0, r_1 are, for example, $r_0 = 0.2$, $r_1 = 0.28$. The characteristic polynomial c is then $c = 1 - 0.14\,z - 0.04(w + w^{-1})z$ and the controller (26) is $R = 0.2 + 0.28(w + w^{-1})$.

The Fig. 3 shows the region in the parameter space r_0 and r_1 for which the polynomial (27) is stable. Apparently, the region is convex, which is the nice fact discovered in this paper. The price is, however, that a controller can only be proportional in time.

Although the procedure presented in this paper (find some stable closed-loop polynomial and then solve for the feedback controller that achieves it) is not meant as a truly practical procedure for a controller design but rather as a basic building block for procedures based on convex optimisation, a few numerical simulations of responses follow. Figure 4 gives the response of the uncontrolled system to the initial conditions (temperature profile) in Fig. 5. The closed-loop system response and manipulated variable are shown in Figs. 6 and 7, respectively.

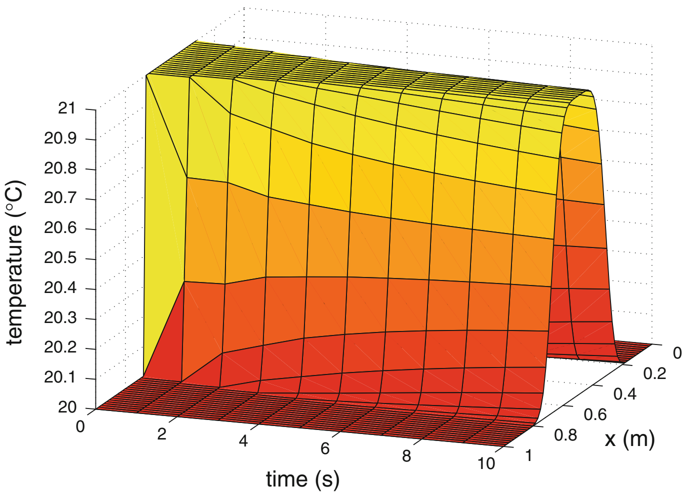

Fig. 4 Response of the uncontrolled system to the initial temperature profile

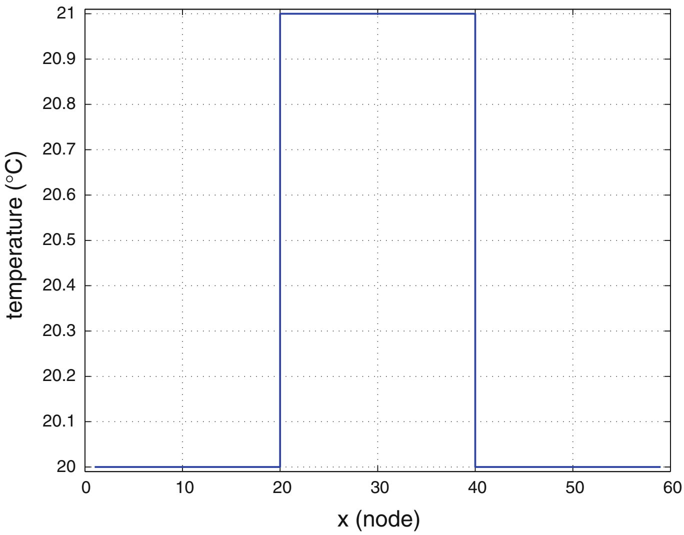

Fig. 5 Initial temperature profile for the numerical example

4.2.2 Continuous-time case

Consider a plant and a controller given by the transfer function (9) and (16), respectively, and the characteristic polynomial of closed-loop system of the form

$$c(s, w) = c_2(w)\, s^2 + c_1(w)\, s + c_0(w), \tag{28}$$

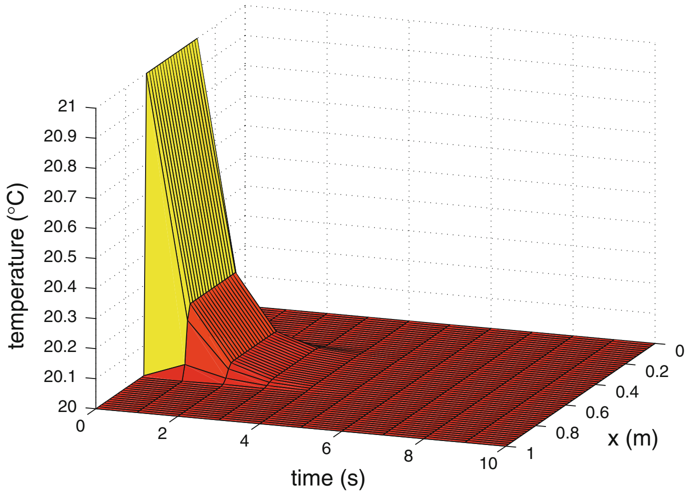

Fig. 6 Response of the controlled system to the initial temperature profile

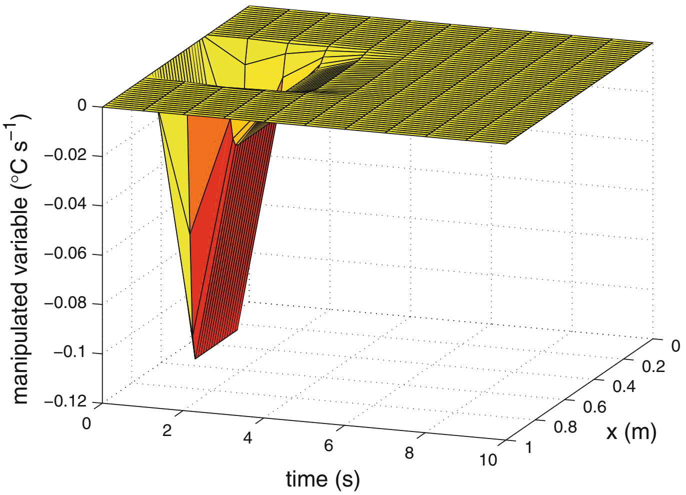

Fig. 7 Controller outputs in response to the initial temperature profile

where the polynomials are

$$c_0(w) = \sum_{l=0}^{\hat{m}} c_{0,l} \left(w^l + w^{-l}\right), \quad c_1(w) = \sum_{l=0}^{\hat{m}} c_{1,l}\, s \left(w^l + w^{-l}\right),$$

$$c_2(w) = \sum_{l=0}^{\hat{m}} c_{2,l}\, s^2 \left(w^l + w^{-l}\right).$$

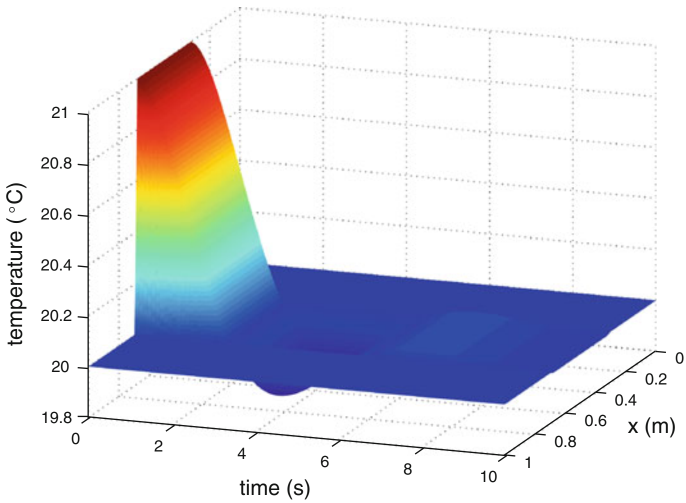

Fig. 8 Response of the controlled system to the initial temperature profile

Build (18) and solve (15). Using SeDuMi/Yalmip confirms that a matrix M exists and returns

$$c_0(w) = 2, \quad c_1(w) = 2, \quad c_2(w) = 2.$$

So, (28) has now the form

$$c(s, w) = 2s^2 + 2s + 2.$$

Now, solve $ax + by = c$ for x and y. A solution is

$$\begin{aligned} x &= \left(0.5w + 3 + 0.5w^{-1}\right) s + \left(0.5w + 3 + 0.5w^{-1}\right) \\ y &= \left(0.165w^2 + 0.66w - 1.65 + 0.66w^{-1} + 0.165w^{-2}\right) s \\ &\quad + \left(0.165w^2 + 0.66w - 1.65 + 0.66w^{-1} + 0.165w^{-2}\right). \end{aligned}$$

Responses to the initial conditions given by Fig. 5 of the closed-loop system are shown in Figs. 8 and 9.

4.3 Remarks

In this section we showed how to find a stabilisable controller of a spatially invariant system. A similar problem is solved, e. g., by Cichy et al. (2011), where a spatially distributed system is described by state-space equations and a method based on ILC is used for a control design. In comparison with Cichy et al. (2011), our method does not offer a reference tracking but gives results directly usable in distributed control techniques. From a computing point of view, both design algorithms consist in solving an LMI. Our method solves an LMI of half dimension compared to Cichy's method but with higher number of variables.

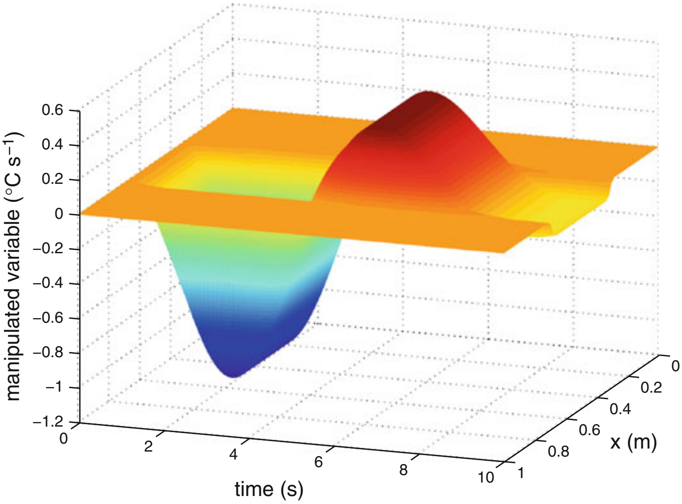

Fig. 9 Controller outputs in response to the initial temperature profile

5 Conclusions

A convex characterisation of linear, time and shift invariant, spatially distributed controllers stabilising a linear spatially invariant system was searched for in this paper. It turns out that a stabilising region in the coefficient space of the closed-loop characteristic polynomials can be characterised using an LMI condition only if both the system and the controller are of low order in the time variable. Namely, the closed-loop characteristic polynomial can be of order 1 with respect to the time variable in the discrete-time case and of order 2 in the continuous-time case. Although looking overly restrictive, a numerical example demonstrates that a physically motivated problem from the domain of spatial temperature profile control can fit realistically.

The underlying technique is based on an LMI condition on positiveness of a polynomial matrix on the unit circle. Complementing this with the 2-D Diophantine equation $ax+by=c$, a stabilising controller is fully characterised. Alternatively, a structure of a distributed feedback controller can be specified first, and then a "stabilising" region in the controller parameter space can be characterised using convex constraints.

The paper showed possibilities and limitations of the transfer function formalism for analysis (and potentially also synthesis) of spatially invariant systems. The main restriction comes from non-convexity of the set of closed-loop polynomial coefficients (hence also the controller parameters) for higher order systems and/or controllers, should these be needed. This hopefully gives some incentives for exploring relaxations to the non-convex Schur-Cohn and Hermite-Fujiwara inequalities.

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