International Journal of Bifurcation and Chaos, Vol. 23, No. 5 (2013) 1350084 (9 pages)

DOI: 10.1142/S0218127413500843

CHAOS ON HYPERSPACE

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Received May 18, 2012; Revised November 5, 2012

In this paper, the chaotic behavior of a set-valued mapping $F : X \to 2^X$, where X is a compact space, is investigated. The existence of the generalized shadowing property in the hyperspace 2^X is proved. Based on the generalized shadowing property of the set-valued mappings F and the assumption of the existence of an unstable chain recurrent point of the mapping F, it is shown that the Bernoulli system of bi-directional shifts is embedded in the sense of semiconjugacy into the image of mapping F, i.e. Smale's chaos in the set-valued system F is thereby proved.

Keywords: Hyperspace; chaos; shadowing; Bernoulli shift.

1. Introduction

Chaos is a concept that has lasted more than a century. By the end of the 19th century, Poincaré became the first person to predict deterministic chaos, laying thereby the foundations of modern chaos theory [Poincaré, 1890]. These developments have made an impact on almost every field of human endeavor. Chaos theory has been like a sea into which the rivers flow and it tributaries of almost every discipline and subject — from mathematics, physics, astronomy, meteorology [Lorenz, 1963], biology, chemistry, medicine to economics [Lorenz, 1989] and engineering like signal processing [Dedieu & Ogorzalek, 2000], safe communication [Banerjee, 2011], control and anti-control of chaos, see [Čelikovský & Vaněček, 1994; Vaněček & Čelikovský, 1996; Chen & Dong, 1998; Čelikovský & Chen, 2002, 2005], from the study of fluids [Wiggins, 1992] and electrical circuits, e.g. [Matsumoto *et al.*, 1985; Chua *et al.*, 1986], to the study of stock markets and civilizations. The amount of papers dedicated to chaos phenomenon is enormous. Nevertheless, mathematics still retain its abstraction as an umbrella, which generalizes and unifies plenty of different definitions of the term "chaos" that emerge from different areas of research work.

One of the most impressive features of the chaos is that even simple systems, from the point of view of their formal description, can embody a very complicated behavior [Lorenz, 1963]. It shows predictability as a rare phenomenon that operates only within the constraints of what science has filtered out from the rich diversity of our complex world. Chaotic systems can be discrete or continuous, the primary description of the system can be deterministic, stochastic, quantum, etc. Formal description of the system can be done through discrete iterations, ordinary differential equations, partial differential equations, systems of hybrid equations, etc., depending on the particular problem under investigation. During the lengthy period of chaos investigation, several different definitions of chaos have been published, see, e.g. [Wiggins, 1988; Devaney, 1989; Wiggins, 2003; Román-Flores & Chalco-Cano, 2005; Zhang *et al.*, 2006; Guirao *et al.*, 2009], indicating some kind of "subjectivism" in system behavior in the framework of each particular research area.

Initially, through decades, specific single-valued systems were intensively studied. As the numerical complexity of the computations matured and the sensitivity of the computations on numerical errors resulted in practically useful results, the new concept of so-called "shadowing" was introduced. Very roughly, within the shadowing concept, one does not take into account every separate trajectory of the dynamical system but rather a "tube" of trajectories in some neighborhood of the exact, but fictive, trajectory. The neighborhood is defined, e.g. by the precision of numerical representation of the number in the computer [Pilyugin, 1999]. If we adopt such a concept, then we can see that the set-valued (multivalued) approach has to be considered to obtain the solution of the computational/simulation tasks. When the set-valued mappings are analyzed, a certain trade-off approach is often used. Such an approach stems from the so-called induced set-valued maps that are defined as follows: let single-valued map $f : X \to X$ be a homeomorphism, then the induced map F is an extension of f to a subset 2^X of all nonempty compact subsets of X. Further, if X is compact metrizable space and if f is expansive open map, then $F : 2^X \to 2^X$ has the so-called shadowing property [Wu & Xue, 2010]. With such a concept, the induced mapping can inherit some of the "nice" properties of the original mapping f.

A more complicated situation arises when the mapping F is defined as set-valued from the very beginning. As shown, problems arise regarding the basic definitions and properties, like homoclinic/heteroclinic trajectories, which are basically confined in the definition of chaos in the case of single-valued mappings, see [Sander, 1996, 1999]. The approach presented in this paper aims to prove possible existence of chaotic behavior of set-valued mappings using a kind of trade-off. Namely, suppose that the set-valued map F is contracting. As the set-valued mapping in principle does not have an inverse, one cannot expect the existence of homeomorphisms but only continuous surjection. The main idea of continuous surjection construction used here comes from [Stoffer & Palmer, 1999], but, probably the first systematically published results concerning theory and applications of the symbolic dynamics are two papers [Morse & Hedlund, 1938, 1940]. Moreover, the shift mapping between the space of symbols and the space of trajectories that are used in [Stoffer & Palmer, 1999] can be found already in [Lind & Marcus, 1995].

The most important numerical invariant related to orbit growth is topological entropy. It represents the exponential growth rate for the number of orbit segments distinguishable with arbitrarily fine but finite precision. In a sense, the topological entropy describes in a crude but suggestive way the total exponential complexity of the orbit structure with a single number. Even though a lot of results have already been published in the case of the single-valued mappings, the case of the set-valued mappings is still at its infancy. The published results are quite rare, e.g. [Peña & López, 2006; Kwietniak & Oprocha, 2007; Lampart & Raith, 2010].

The main contribution of this paper is the topic discussed in the third section. First, it is shown that under certain assumptions on the set-valued mapping and on the underlaying space, there exists a unique bounded and invariant set. As a second result, the existence of an invariant and bounded set allows us to prove that any contracting mapping into that invariant set necessarily implies the existence of the shadowing property. These two results provide two basic results that enable us to prove the main result, namely the chaotic behavior on the subset of the invariant set, which is formed by *a priori* assumed existence of an unstable chain recurrent trajectory. As set-valued mappings do not have, in general, an inverse, it is shown that there exists a semiconjugacy, i.e. continuous surjection, between that shadowed trajectory and the space of bidirectional shifts. Such a property is well known to be equivalent to the existence of Smale's chaos [Wiggins, 2003].

The rest of the paper is organized as follows. The next section gathers some preliminaries and necessary definitions including some useful results needed later on. Section 3 presents the main results of the paper, including an illustrative example. Some conclusions are drawn in the final section.

2. Preliminaries

Generally, it follows from theoretical research that the existence of a homoclinic trajectory is very often a source of chaotic behavior in nonlinear systems. It is well known that the explicit verification that a system has homoclinic trajectory is a difficult problem even in the case of a single-valued system, and therefore much more in the case of a multi-valued system. There are many papers and books that describe these phenomena. The majority of them *assume the existence* of homoclinic trajectory and only then various kinds of system behavior are derived. The particular cases of systems that explicitly embody homoclinic trajectory including the theoretical proof of its existence are generally very scarce.

Prior to setting up the exact definitions, let us introduce two examples of systems that enjoy homoclinic trajectory. Firstly, let us show the existence of homoclinic trajectory in the single-valued case, see [Hassard & Zhang, 1994].

Example 2.1. Existence of a homoclinic orbit of the Lorenz system.

Consider the Lorenz system

$$\begin{aligned} x' &= s(y-x), \\ y' &= (R-z)x - y, \\ z' &= xy - qz. \end{aligned} \tag{1}$$

Assume that each orbit has the following two properties:

Property P. *Let* $\lim_{t\to-\infty}(x(t), y(t), z(t))^T \to (0,0,0)^T$. *Moreover, there exist numbers* $\tau_1 < s_1 < t_1$ *such that*

$$\begin{aligned} x' &> 0 \quad \text{on } (-\infty, \tau_1), \\ x' &< 0 \quad \text{on } (\tau_1, t_1), \\ x(t_1) &= 0, \\ y' &< 0 \quad \text{on } (\tau_1, s_1), \\ y(s_1) &= 0, \\ y &< 0 \quad \text{on } (s_1, t_1), \\ x &> 0 \quad \text{on } (-\infty, t_1). \end{aligned}$$

Property Q. *Let* $\lim_{t\to-\infty}(x(t), y(t), z(t))^T \to (0,0,0)^T$. *There exists a* τ_1 *such that* $x' > 0$ *on* $(-\infty, \tau_1)$. *Also, the orbit* $(x, y, z)^T$ *does not have Property P.*

We give the conditions for the existence of homoclinic orbit. Let us fix $q > 0$ and $s > 0$ and let $[R_a, R_b]$ be some interval of R values, $R_a > 1$. The Jacobian matrix for system (2) at the origin has two negative and one positive eigenvalues. The unstable manifold at the origin has components $\gamma^+(R)$, $\gamma^-(R)$, where locally $x > 0$ on γ^+ and $x < 0$ on γ^-. We are concerned only with γ^+.

We let $p^+(R) = (\sqrt{q(R-1)}, \sqrt{q(R-1)}, R-1)^T$ denote the stationary point in the positive octant. Let $p(t;R) = (x(t;R), y(t;R), z(t;R))$ denote a family of orbits of the Lorenz system, one for each $R \in [R_a, R_b]$, jointly smooth in t and R, known to exist on the interval $(-\infty, 0]$, and such that $\lim_{t\to-\infty} p(t;R) = (0,0,0)^T$ and $x'(t,R) > 0$ on $(-\infty, 0]$. Then $p(.;R) \in \gamma^+(R)$. It is well known that for fixed parameters q, R, s there is a region V, the interior of an ellipse, which contains the origin and is a (forward) invariant for the flow. Because of the invariant region, each orbit $p(t;R)$ on $(-\infty, 0]$ may be continued to the entire line $(-\infty, +\infty)$. Moreover, the partial derivatives of x, y and z with respect to t of any fixed order, are bound for all t and bounds that are uniform in R may be constructed.

The main task in establishing the existence of a homoclinic orbit is to show that for the sequence $p(.;R_j)$ of orbits with Property P constructed in Lemma 2, the sequence $t_{1,j}$ is unbounded. To do that, the following lemma is necessary.

Lemma 1 [Hassard & Zhang, 1994]. *Suppose* $0 < q < 2s$ *and* $1 < R_a < R < R_b$. *Let* $R_j \in (R_a, R_b)$ *and suppose each orbit* $p(.;R_j) = (x(.;R_j), y(.;R_j), z(.;R_j))^T$ *has Property P. Let* $\tau_{1,j} < s_{1,j} < t_{1,j}$ *be from Property P such that* $x'(\tau_{1,j};R_j) = 0$, $y(s_{1,j};R_j) = 0$ *and* $x(t_{1,j};R_j) = 0$. *If the sequences* $\tau_{1,j}$, $s_{1,j}$, $t_{1,j}$ *have finite limits* τ_1, s_1, t_1, *then the orbit* $x(.;R), y(.;R), z(.;R)$ *also has Property P.*

The basic existence result is then the following lemma

Lemma 2 [Hassard & Zhang, 1994]. *Suppose* $0 < q < 2s$ *and* $1 < R_a < R_b$. *Suppose that for* $R = R_b$, *there is an orbit with Property P and for* $R = R_a$ *there is an orbit with Property Q.*

Suppose further that for all $R \in [R_a, R_b]$, $p^+(R)$ *is linearly stable. Then there is a value* R^*, $R_a < R < R_b$ *for which there is an orbit* $p(.;R^*) = (x,y,z)^T$ *with the following properties*:

As $t \to \pm\infty$, $(x,y,z)^T \to (0,0,0)^T$ — *the homoclinic orbit. Also there exist* τ_1, s_1, $-\infty < \tau_1 < s_1 < \infty$ *such that*

$$\begin{aligned} x' &> 0 \quad \text{on } (-\infty, \tau_1), \\ x' &< 0 \quad \text{on } (\tau_1, \infty), \end{aligned}$$

$$y' < 0 \quad on\ [\tau_1, s_1],$$
$$y(s_1) = 0,$$
$$y < 0 \quad on\ (s_1, \infty),$$
$$x > 0 \quad on\ (-\infty, \infty).$$

The authors improved numerically the accuracy of the value R^* for which the homoclinic orbit exists. That allows much more precise construction of the homoclinic orbit numerically.

In the second example, the existence of the homoclinic orbit in the set-valued case of a partial differential inclusion will be presented, see [Kristály *et al.*, 2010; Roubíček, 2005].

Example 2.2. Let $p > 2$ and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

F1: $\lim_{t\to 0} \frac{\max\{|\xi|:\xi\in\partial F(t)\}}{|t|^{p-1}} = 0$,
F2: $\limsup_{|t|\to+\infty} \frac{f(t)}{|t|^p} \le 0$,
F3: There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$, and $F(0) = 0$.

We are going to study the differential inclusion problem

$$-\Delta_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F(u(x)) + \mu\beta(x)\partial G(u(x)) \quad \text{on } \mathbb{R}^N$$
$$u(x) \to 0 \quad \text{as } |x| \to \infty$$

where $p > N \le 2$, the numbers λ, μ are positive, and $G : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz function. Furthermore, we assume that $\beta \in L^1(\mathbb{R}^n)$ is any function, and $\alpha \in L^1(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$, $\alpha \ge 0$, and $\sup_{R>0} \text{essinf}_{|x|\le R}\alpha(x) > 0$.

The functional space where the solutions are going to be sought is the usual Sobolev space $W^{1,p}(\mathbb{R}^n)$, endowed with the norm $\|u\| = (\int_{(\mathbb{R}^n)} \nabla u(x)^p + \int_{(\mathbb{R}^n)} |u(x)|^p)^{1/p}$.

We say that $u \in W^{1,p}(\mathbb{R}^n)$ is a solution of the problem, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for a.e. $x \in \mathbb{R}^N$ such that for all $v \in W^{1,p}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv)\,dx = \lambda\int_{\mathbb{R}^N} \alpha(x)\xi_F v dx + \mu\int_{\mathbb{R}^N} \beta(x)\xi_G v dx.$$

The terms on the right-hand side are well-defined due to Morrey's embedding theorem, i.e. $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ is continuous ($p > N$), we have $u \in L^\infty(\mathbb{R}^n)$. Thus, there exists a compact interval $I_u \subset \mathbb{R}$ such that $u(x) \in I_u$ for a.e. $x \in \mathbb{R}^N$. Since the set-valued mapping ∂F is upper semi-continuous, the set $\partial F(I_u) \subset \mathbb{R}$ is bounded. Let $C_F = \sup|\partial F(I_u)|$. Thus

$$\left|\int_{\mathbb{R}^N} \alpha(x)\xi_F v dx\right| \le C_F\|\alpha\|_{L^1}\|v\|_\infty < \infty.$$

Similar arguments hold for the function G.

As the conditions of [Pilyugin, 2008; Rieger, 2009] are granted, the set-valued system is contractive and has the shadowing property.

Since $p > N$, then due to [Brézis, 1992], any element $u \in W^{1,p}(\mathbb{R}^n)$ is homoclinic, i.e. $u(x) \to 0$ as $|x| \to \infty$.

Now, some definitions and notions that will be used later on are introduced [Illanes & Nadler, 1999]. Let X be a topological space with topology T, briefly (X, T). A *hyperspace of* X is a specified collection of subsets of X equipped with the so-called Vietoris topology defined later on. The empty set $\emptyset$ is excluded. To avoid some pathology, we restrict our attention to hyperspaces whose elements are closed subsets of X. Finally, let $\mathbb{N}$ stand for the expanded set of natural numbers $0, 1, 2, 3, \ldots$ and let $\mathbb{Z}$ stand for the set of integer numbers.

Definition 2.1. The so-called closed-sets hyperspace of X is the following topological space

$$CL(X) \triangleq \{A \subset X : A \ne \emptyset \text{ and } A \text{ is closed in } X\}$$

equipped by Vietoris topology, which is mentioned in the next definition.

Definition 2.2. Let (X, T) be a topological space. Let $I \subset \mathbb{N}$ be a finite index set and for all I, let $\{U_i \,|\, i \in I\}$ be a set of open subsets of X. Define a set

$$\langle U_i\rangle_{i\in I} \triangleq \left\{U \in CL(X) \,|\, U \subseteqq \bigcup_{i\in I} U_i \text{ and } U \cap U_i \ne \emptyset\ \forall i\right\}$$

for each set $\{U_i \,|\, i \in I\}$. The topology generated by the sets $\langle U_i\rangle_{i\in I}$ is called *Vietoris topology*.

Further, additional variations of the hyperspaces are introduced, each of them having the subspace topology obtained from Vietoris topology on $CL(X)$. Define the following hyperspaces being subspaces of $CL(X)$, with Vietoris topology

$$2^X \triangleq \{A \in CL(X) : A \text{ is compact}\}.$$

$$BC(X) \triangleq \{A \in CL(X) : A \text{ is bounded and closed}\}.$$

$$C(X) \triangleq \{A \in 2^X : A \text{ is connected}\}.$$

Note that $2^X = CL(X)$ when X is compact. Moreover, when X is Hausdorff, then $2^X \triangleq \{A \subset X : A \neq \emptyset, \text{ and } A \text{ being compact}\}$.

Let (X, d) be a metric space. For any $x \in X$ and any $A \in CL(X)$, let

$$d(x, A) \triangleq \inf\{d(x, a) : a \in A\}.$$

For any $r > 0$ and any $A \in CL(X)$, let

$$N_d(r, A) \triangleq \{x \in X : d(x, A) < r\}$$

be *generalized open d-ball in X about A of radius r*. Let (X, d) be a bounded metric space. The *Hausdorff metric for $CL(X)$ induced by d*, which is denoted by H_d, is defined as follows: for any A, $B \in CL(X)$,

$$H_d(A, B) \triangleq \inf\{r > 0 : A \subset N_d(r, B) \text{ and } B \subset N_d(r, A)\}.$$

The source for subsequent definitions can be found in [Pilyugin, 1999]. We slightly modify some of those definitions for the case of set-valued mappings.

Definition 2.3. A finite or infinite sequence $\{x_n\} \subset X$ is called the *chain* for a set-valued mapping $F : X \to 2^X$, if $x_{n+1} \in F(x_n)$ for all n.

Further, we define the *composition* of two set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ as the map $G \circ F : X \rightrightarrows Z$ such that $(G \circ F)(x) = \bigcup_{y \in F(x)} G(y)$. The *inverse* of a set-valued mapping F is a set-valued mapping F^{-1}, defined by $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$. Finally, define inductively F^n as $F^{m+n} \triangleq F^m \circ F^n$ for $m, n \in \mathbb{Z}$.

Definition 2.4. The *orbit* $\mathcal{O}F$ of set-valued mapping F is defined as follows:

$$\mathcal{O}F \triangleq \bigcup_{n \geq 1} F^n.$$

An *n-periodic point* is a point $x \in \mathcal{O}F(x)$ such that there exists $n \geq 1$ so that $x \in F^n(x)$. The number n is called *period*. If $n = 1$, then the point x is called the *fixed point*.

Definition 2.5. A subset $\mathcal{A} \subset X$ is called *positive invariant* with respect to the set-valued mapping F if $F(\mathcal{A}) = \bigcup_{a \in \mathcal{A}} F(a) \subset \mathcal{A}$. If $F(\mathcal{A}) = \mathcal{A}$ then $\mathcal{A}$ is called *F-invariant*.

Definition 2.6. A finite or infinite sequence $\{x_n\} \subset X$ is called *δ-chain* for a set-valued mapping $F : X \to 2^X$, if for each $\delta > 0$ one has $d(x_{n+1}, F(x_n)) \leq \delta$ for all n. The sequence of points $\{x_n\} \subset X$ is also called *δ-pseudotrajectory* or *δ-pseudo-orbit*.

Definition 2.7. A point $x \in X$ is called *chain recurrent point* if for any $\delta > 0$ there exists a periodic δ-chain through x of the set-valued mapping F. The set $\mathrm{CR}(F)$ consists of all chain recurrent points.

A chain recurrent point $x \in \mathrm{CR}(F)$ is called *unstable* if there is a real number $\epsilon > 0$ such that for any $\delta > 0$ there exists $y \in \mathrm{CR}(F)$ and a natural number n so that when $d(x, y) < \delta$ then $H_d(F^n(x), F^n(y)) > \epsilon$. The just mentioned property of a chain recurrent point being unstable prevents some pathological cases of F like the existence of periodic points of finite period only, e.g. of period-2 only.

Definition 2.8. One says that the set-valued mapping $F : X \to 2^X$ has the *shadowing property on the subset* $A \subset X$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-chain $\{x_n\} \subset A$ there exists a chain $\{y_n\} \subset X$ such that $d(x_n, y_n) \leq \epsilon$ for all n. Alternatively, one says that the chain $\{y_n\}$ *ϵ-shadows* the δ-chain $\{x_n\}$.

A set-valued mapping $F \in BC(X)$ is called a *contracting* one or *contraction*, when for $\forall\, x, y \in X$ there exists a real number $0 < \lambda < 1$ such that $H_d(F(x), F(y)) \leq \lambda d(x, y)$. Further, let, as usual, $\Sigma \triangleq \{s = (s_n)_{n \in \mathbb{Z}} : s_n \in \{0, 1\}\}$ be the space of $0-1$ sequences equipped by the metrics $d(s, t) = \max\{2^{-|n|} : s_n \neq t_n\}$. Notice, that the space (Σ, d) is the compact metric space [Katok & Hasselblatt, 1999].

Definition 2.9. The map $\sigma : \Sigma \to \Sigma$, $(\sigma(s))_n = s_{n+1}\ \forall\, n \in \mathbb{Z}$ is called the *Bernoulli shift map*.

Definition 2.10. Let (X, f) be a topological dynamical system and let $\mathcal{A}$ be an invariant subset of f, i.e. $f(\mathcal{A}) \subseteq \mathcal{A}$. Then $(\mathcal{A}, f|_{\mathcal{A}})$ is said to be a topological dynamical *subsystem* of (X, f).

Definition 2.11. Let (X, f) and (Y, g) be two topological dynamical systems. If there exists a continuous and surjective mapping $\tau : X \to Y$ satisfying $g \circ \tau = \tau \circ f$, then (X, f) and (Y, g) are said to be *topologically semiconjugate* and the mapping τ is said to be a *semiconjugacy*.

Definition 2.12. A dynamical system represented by the set-valued mappings F is said to be *chaotic or chaotic in the sense of Smale* if there is a positive integer n such that the image of F^n admits the Bernoulli shift (Σ, σ), i.e. there is a semiconjugacy between (X, F^n) and (Σ, σ).

3. Main Results

First, let us state and prove two simple preliminary theorems.

Theorem 1. *For any contracting set-valued mappings on a complete metric space there exists a unique bounded and closed invariant set.*

Proof. Let F be a contracting set-valued mapping on X. Let $\hat{F} : BC(X) \to BC(X)$ be the corresponding map on the complete metric space $BC(X)$ such that $\hat{F}(\mathcal{A}) = F(\mathcal{A})$, where $\emptyset \neq \mathcal{A} \subset X$ is bounded and closed with the Hausdorff metric H_d.

As F is contractive, i.e. $H_d(F(x), F(y)) \leq \lambda d(x, y)$, $\forall\, x, y \in X$, where $\lambda > 0$, one can easily see that $H_d(\hat{F}(M_1), \hat{F}(M_2)) \leq \lambda H_d(M_1, M_2)\ \forall\, M_1$, $M_2 \subset X$ that are bounded. So, the mapping $\hat{F}$ is Lipschitz with the same contraction parameter λ as F has. Thus, the mapping $\hat{F}$ has a unique fixed point $\mathcal{A} \in BC(X)$. As a result, the equation $F(\mathcal{A}) = \mathcal{A}$ has a unique bounded solution. ■

Theorem 2. *Any contracting mapping $F : X \to BC(X)$ has the shadowing property.*

Proof. Let $\epsilon > 0$ be an arbitrary real number. Further, let $(x_n)_{n\in\mathbb{N}} \subset X$ be a δ-chain, i.e. $d(x_{n+1}, F(x_n)) \leq \delta$. We define a chain $(y_n)_{n\in\mathbb{N}} \subset X$ such that $y_{n+1} = F(x_n)$ and, due to contractivity of X, $d(x_0, y_0) \leq \epsilon$. We show that the chain $(y_n)_{n\in\mathbb{N}} \subset X$ ϵ-shadows the δ-chain $(x_n)_{n\in\mathbb{N}} \subset X$. As one can see $d(x_1, y_1) = d(x_1, F(y_0))$. Due to δ-chainability of $(x_n)_{n\in\mathbb{N}}$, x_1 lies in the δ-neighborhood of $F(x_0)$, i.e. $x_1 \in U_\delta(x_0)$. So, we can estimate

$$d(x_1, y_1) \leq d(x_1, F(x_0)) + H_d(F(x_0), F(y_0))$$
$$\leq \delta + \lambda d(x_0, y_0) \leq \delta + \lambda\epsilon$$

due to λ-contractivity of the set-valued mapping F. Next, we use the induction. We get

$$d(x_n, y_n) \leq \delta(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1}) + \lambda^n\epsilon.$$

As $0 < \lambda < 1$, the elementary calculations give

$$d(x_n, y_n) \leq \delta\frac{1 - \lambda^{n-1}}{1 - \lambda} + \epsilon \leq \frac{\delta}{1 - \lambda} + \epsilon.$$

So, if one chooses $\delta = \epsilon(1-\lambda)$, we get $d(x_n, y_n) \leq 2\epsilon$ for any $n \in \mathbb{N}$. As ϵ is chosen arbitrarily, we have proven the existence of ϵ-shadowing property. ■

Now, the main contribution of the paper is as follows.

Theorem 3. *Let contracting set-valued mapping F have the shadowing property and suppose there is an unstable chain recurrent point of mapping F. Then F is chaotic.*

Proof. The proof of the chaotic behavior of the system (X, F^n) in the sense of Definition 2.12 will be done in four steps: as a first step, we construct the mapping τ accordingly to Definition 2.11. In analogy with the classical analysis, we have to prove the commutativity of the mappings as the second step, further in the third step, we will prove injectivity of the mapping τ and finally, in the fourth step, we will prove the continuity of the mapping τ.

Step 1. The main idea of the continuous surjection construction comes from [Stoffer & Palmer, 1999]. By assumption, there is an unstable chain recurrent point p, i.e. $\delta > 0$ so that there exists δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{+\infty} \subset \mathcal{A} \subset BC(X)$ (see Theorem 1) such that $\lim_{i\to-\infty} x_i = \lim_{i\to+\infty} x_i = p$.

Let

$$u_1 \triangleq (p, x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, p)$$

be a section of the δ-pseudo-orbit of length $2n + 3$ and let

$$u_0 \triangleq (p, p, p, \ldots, p, p, p, \ldots, p, p)$$

be a section of the same length.

Let

$$s \in (\ldots, s_{-1}; s_0, s_1, \ldots)$$

be an element of the space of bi-infinite sequences of $\{0, 1\}$. We associate with s a δ-pseudo-orbit

$$r_s \triangleq (\ldots, u_{s_{-1}}; u_{s_0}, u_{s_1}, \ldots).$$

Thus, we have

$$r_{\sigma(s)} \triangleq (\ldots, u_{s_{-1}}, u_{s_0}; u_{s_1}, \ldots).$$

Further, let $y = \{y_i\}_{i\in\mathbb{Z}}$ be ϵ-shadowing orbit of r_s. Then $y = \{y_{i+k}\}_{i\in\mathbb{Z}}$ is a ϵ-shadowing orbit of $r_{\sigma(s)}$. Define the mapping τ as follows: $\tau : s \in \Sigma \to \tau(s) = y_{n+1}$ and the ϵ-shadowing orbit that is associated with a δ-pseudo-orbit is unique, we have $\tau(\sigma(s)) = y_{n+1+k}$. Further, formally we have $F^k(\tau(s)) = F^k(y_{n+1}) = y_{n+1+k}$.

Step 2. Let us formally prove the following commutativity property

$$F^k \circ \tau(s) = \tau(s) \circ \sigma(s).$$

Let $s = (\ldots, s_{-1}; s_0, s_1, \ldots)$ and $s' = \sigma(s) = (\ldots, s_{-1}, s_0; s_1, s_2, \ldots)$. Then we have the corresponding δ-pseudo-orbits $r_s = (\ldots, u_{s_{-1}}; u_{s_0}, u_{s_1}, \ldots)$ and $r_{s'} = (\ldots, u_{s_{-1}}, u_{s_0}; u_{s_1}, u_{s_2}, \ldots)$. Let $\{y_j\}_{j\in\mathbb{Z}}$ be an ϵ-shadowing orbit (for the existence, see Theorem 2) of r_s. Similarly, $\{y_{j+k}\}_{j\in\mathbb{Z}}$ is the ϵ-shadowing orbit of $r_{\sigma(s)}$. Now we have $\tau(\sigma(s)) = \tau(y_{n+k}) = y_{n+k+1}$. On the other hand, we have $F^k(\tau(s)) = F^k(y_{n+1}) = y_{n+k+1}$. The commutativity of the mappings is now proved.

It is obvious from the design of the mapping τ that the mapping is a transformation of the space Σ into the $\{y_{j+k}\}_{j\in\mathbb{Z}}$, i.e. into the ϵ-shadowing orbit of $r_{\sigma(s)}$ which is a subsystem of the attractor $\mathcal{A}$ in the sense of Definition 2.10. Accordingly to Definition 2.11, it remains to prove that the mapping $\tau(s) \mapsto y_{n+1}$ is a continuous surjection. To do that, it is necessary to prove that the mapping τ is continuous and injective. This will be done in the following part of the proof.

Step 3. The injectivity of the mapping τ will be proved. Let $s \neq s'$. This implies an existence of $j \in \mathbb{Z}$ such that $s_j \neq s'_j$, thus correspondingly $u_{s_j} \neq u_{s'_j}$. We can assume that $s_j = 0$ and $s'_j = 1$. Then there exist an index $l \in \mathbb{Z}$ so that $(u_s)_l = p$ and $(u_{s'})_l = x_0$. Let y and y' be the ϵ-shadowing orbits of r_s and $r_{s'}$, respectively. As $y, y' \in \mathrm{CR}(F)$ are unstable, then $H_d(F^k(y), F^k(y')) > \epsilon\ \forall\, \epsilon > 0$, and also $y \neq y'$ and thus $\tau(s) = y_{n+1} \neq \tau(s') = y'_{n+1}$. So, the injectivity of τ has been proved.

Step 4. The continuity of the mapping τ is to be proved. Let $\{s^j\}_{j\in\mathbb{N}} \subset \Sigma$ such that $s^j \to s \in \Sigma$. Such a sequence exists due to compactness of Σ. Let r^j, r be δ-pseudo-orbits such that $\tau : s^j \mapsto r^j$, $F^k : r^j \mapsto r$, $\sigma : s^j \mapsto s$, $\tau : s \mapsto r$. Further, let y^j, y be ϵ-shadowing orbits of r^j, r. In order to prove continuity of the mapping τ, we have to prove that $\lim_{j\to\infty} y^j_{n+1} = y_{n+1}$ whenever $s^j \to s \in \Sigma$.

As $H_d(y^j_{n+1}, r^j_{n+1}) \le \epsilon$ due to ϵ-shadowing and $r^j_{n+1} = p$ or $r^j_{n+1} = x_0$ for all j, the sequence $\{y^j_{n+1}\}_{j\in\mathbb{N}}$ is bounded. It is sufficient to prove that y_{n+1} is the unique limit point of $\{y^j_{n+1}\}_{j\in\mathbb{N}}$. Let w_{n+1} be another limit point of $\{y^j_{n+1}\}_{j\in\mathbb{N}}$. That assumption implies an existence of a subsequence $\{y^{j_i}_{n+1}\}_{j\in\mathbb{N}}$ such that $w_{n+1} = \lim_{i\to\infty} y^{j_i}_{n+1}$. Due to action of F^m on w_{n+1} we get $w_{m+n+1} = F^m(w_{n+1})$, $m \in \mathbb{Z}$. Now, we can write:

$$\begin{aligned} &H_d(w_{m+n+1}, y_{m+n+1}) \\ &\quad \le H_d(w_{m+n+1}, y^{j_i}_{m+n+1}) + H_d(y^{j_i}_{m+n+1}, r^{j_i}_{m+n+1}) \\ &\qquad + H_d(r^{j_i}_{m+n+1}, y_{m+n+1}). \end{aligned}$$

We analyze all the three members on the right-hand side. Firstly, as y^j_i ϵ-shadows r^j_i, we get first estimate $H_d(y^{j_i}_{m+n+1}, r^{j_i}_{m+n+1}) \le \epsilon$.

As the next step, one has:

$$\begin{aligned} w_{m+n+1} &= F^m(w_{n+1}) = F^m\left(\lim_{i\to\infty} y^{j_i}_{n+1}\right) \\ &= \lim_{i\to\infty} F^m(y^{j_i}_{n+1}) = \lim_{i\to\infty} y^{j_i}_{m+n+1}. \end{aligned}$$

As a result, we get the second estimate $H_d(w_{m+n+1}, y^{j_i}_{m+n+1}) \le \epsilon\ \forall\, \epsilon > 0$.

To analyze the last inequality, due to periodicity of the recurrent trajectory, we can express $m + n + 1 = i \cdot k + u$, $u \in (0, 1, \ldots, k-1)$, where $i, u \in \mathbb{Z}$. Consequently, there exists s_i such that $r_{m+n+1} \subset u_{s_i}$ and $r^{j_i}_{m+n+1} \subset u^{j_l}_{s_i}$. As $\lim_{l\to\infty} s^l = s$ there exists a $l > L$ such that $d(s^l, s) \le \frac{1}{2^{|i|+1}}$. Due to the metric on Σ, we get that $s^l_j = s_j\ \forall\, l > L$. Thus for i sufficiently large, we have $s^l_i = s_i$, which implies that $r^{j_l}_{m+n+1} = r_{m+n+1}$. As a consequence, it holds $\forall\, \epsilon > 0$ that $H_d(w_{m+n+1}, y_{m+n+1}) \le \epsilon$. It means that w_{m+n+1} is ϵ-shadowing orbit of y_{m+n+1} $\forall\, m \in \mathbb{Z}$. In other words, $\{w_{n+1}\}_{n\in\mathbb{Z}}$ is an ϵ-shadowing orbit of y. As y has a unique ϵ-shadowing orbit, we get that $w_{n+1} = y_{n+1}$, which we aim to demonstrate. ■

Example 3.1. Let us illustrate the main theorem on an example of a single-valued system.

Let us have, again, the Lorenz system

$$\begin{aligned} x' &= s(y - x), \\ y' &= (R - z)x - y, \qquad (2) \\ z' &= xy - qz, \end{aligned}$$

that defines a dissipative dynamical system. It is known that under the Lorenz equations, the set

$$\Omega = \{(x, y, z) \mid Rx^2 + sy^2 + s(z - 2R)^2 \leq sR^2q^2(q - 1)\}$$

is forward invariant, i.e. any solution that is in Ω at time t_0 remains in Ω for all time $t \geq t_0$. As the divergence of the Lorenz system is equal to $-(s + 1 + q) < 0$, where the numbers s, q are positive, the system is dissipative and contracts the volume in the phase space. This means that the Lorenz system represents a contracting mapping in $\mathbb{R}^3$. According to Theorem 2, the Lorenz system has the shadowing property.

As it has been shown in Example 2.1, there exists a homoclinic orbit with unstable recurrent point at the origin. As the conditions of Theorem 3 are granted, we can state that the Lorenz system embodies chaotic behavior.

Remark 3.1. It has to be said that Theorem 3 guarantees only the existence results. On the other hand, the approach of shadowing orbits methodology implies also very fruitful contribution to the stable numerical methods, see, e.g. [Pilyugin, 1999; Coomes, 1997].

Remark 3.2. As there are no conditions that would generally ensure the existence of a homoclinic orbit, the existence of the associated δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{+\infty} \subset \mathcal{A} \subset BC(X)$ is not generally assured. Thus, we suppose (like in most published papers) the existence of that orbit as a hypothesis.

Remark 3.3. As the set-valued mapping F generally does not have an inverse, the mapping τ represents only a continuous surjection into the attractor $\mathcal{A}$. In other words, the image of the mapping F contains a part, which is homeomorphic with space (Σ, σ).

Remark 3.4. In the frequently used case of the single-valued map $f : X \to X$ that is a homeomorphism, the *induced* set-valued map F is a natural extension of f to 2^X of all nonempty compact subsets of X. Therefore, if X is a compact metrizable space and f is an expansive open map, then $F : 2^X \to 2^X$ has shadowing property. See [Wu & Xue, 2010].

Remark 3.5. We can consider the equivalence between δ-pseudo-orbit of single-valued map f on a metric space X and a relation $g \in X \times X$. Moreover, δ-pseudo-orbit for a map f on X is a finite or infinite sequence $\{x_j\}$ satisfying $d(f(x_j), x_{j+1}) \leq \delta$. One can fatten up the graph of f to the relation g by letting $g = \{(x, y) \mid d(y, f(x)) \leq \delta\}$. So, the orbit for the relation g is exactly the δ-pseudo-orbit for the map f.

Remark 3.6. As stated in the introduction, a kind of middle way, based on the contractibility of the underlaying space and the existence of the shadowing property, has been chosen. The case of the general set-valued mappings is still hard to handle. The meaning of the semiconjugacy of the set-valued mappings and the corresponding shadowing properties, especially chaotic behavior, is to be investigated. The paper [Ishii & Smillie, 2010] seems to be the first step in this direction, which is based on incorporating the additional homotopy properties of the mappings into the entire concept. Anyway, one can still consider chaos notion in the general case of set-valued mappings as an open research area.

4. Conclusions

In this paper, a possible chaotic behavior of the set-valued mappings has been studied and its criteria have been proved. The set-valued mappings are understood here as a "pure" set-valued mapping in the sense that it is not *a priori* induced set-valued map. Further, no existence of a selector is needed here. The shadowing property, generalized to the case of set-valued maps, can be fruitfully used when analyzing or computing the chaotic behavior of the set-valued maps in the future numerically. In addition, due to Remark 3.5, in the case of single-valued mappings with shadowing property, Theorem 3 can be useful to analyze the chaotic behavior of single-valued maps.

Acknowledgment

This work is supported by Czech Science Foundation through the research grant no. 13-20433S.

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