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Robust synchronization of a class of chaotic networks ☆

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Abstract

This paper studies synchronization of a dynamical complex network consisting of nodes being generalized Lorenz chaotic systems and connections created with transmitted synchronizing signals. The focus is on the robustness of the network synchronization with respect to its topology. The robustness is analyzed theoretically for the case of two nodes with two-sided (bidirectional) connections, and numerically for various cases with large numbers of nodes. It is shown that, unless a certain minimal coherent topology is present in the network, synchronization is always preserved. While for a minimal network where synchronization is global, the resulting synchrony reduces to semi-global if redundant connections are added.

1. Introduction

The research topic of complex networks has revoked considerable interest in the past few years. Examples of complex networks in interest include the Internet, World Wide Web, food webs, electric power grids, metabolic networks, and biological neural networks, among many others [1,2]. Traditionally, complex networks were studied via random graph theory, introduced by Erdös and Renyí [3], which have been extended to a wide extent in the last decade.

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This paper studies the synchronization phenomenon of dynamical complex networks (DCN), where all nodes are identical chaotic systems (but usually with different parameters and/or initial conditions). Compared to existing results, there are two novel features in our new approach. First, nonlinear synchronizing connections between nodes are allowed; secondly, a directed graph as a model for DCN is considered, in contrast to the general studies where only linear coupling and undirected networks are discussed [4–7]. Note that a general approach to the local synchronization of chaotic systems for any linear coupling scheme was described in [8]. The objective here is to study the synchronizability of the network when some nodes establish or lose certain connections. This notion is referred to as *structural robustness of DCN synchronization*. The motivation comes from the consideration that in a network numerous participants try to synchronize to each other for some reason (e.g. for chaotic secure communication [9,10]), while some participants may connect to or disconnect from some of their partners under certain conditions, but these should not damage the overall synchrony of the network. It will be shown that with an increasing number of connections, synchronization is only semi-global, and it may become even worse as the number of connections continue to increase, in the sense that very high gains (coupling strengths) are needed to maintain the synchrony of the whole network.

This paper aims to study DCN consisting of the so-called generalized Lorenz system (GLS) [11,12]. It was already shown in [12] that two GLS's in master–slave configuration can be synchronized using a single scalar connection. Further, some bidirectionally coupled synchronization results on GLS were obtained in [13], using nonscalar, but linear connections. The present paper will continue the initial study presented in [14], considering both scalar nonlinear bidirectional connection between two GLS's, with mathematical proof of the convergence, and a study of more complex network topologies of up to eight GLS nodes. As an example of good synchronization properties even for larger number of nodes and connections, Fig. 9 shows some possible topologies of eight-node networks and Fig. 10 illustrates their error dynamics. While mathematical proofs for more complex cases are unrealistic, numerical studies show quite interesting behavior, e.g. increasing the numbers of nodes and connections usually leads to increasing sensitivity with respect to initial synchronization errors.

The rest of the paper is organized as follows. Some notions related to the DCN and the synchronization problem are introduced in the next section. The DCN of GLS is then discussed in Section 3, together with theoretical analysis on the DCN with two coupled nodes for its synchronization. Numerical simulations on the GLS-based DCN, with 3, 4 and 8 nodes, are presented in Section 4. Finally, conclusions are given in the last section.

2. Synchronization of dynamical complex networks

Consider a DCN of N identical nonlinear nodes, with each node being a chaotic system, described by

$$\dot{\eta}^i = f(\eta^i) + \sum_{j=1}^{N} c_{ji}\phi(\eta^i, h(\eta^j), L), \tag{1}$$

where $\eta^i = (\eta_1, \eta_2, \ldots, \eta_n,)^\top \in R^n$ is the state vector of node i, $i = 1, \ldots, N$, $L = (l_1, l_2, \ldots, l_n)^\top$ is the vector of coupling gains, $h(\cdot)$ is a scalar synchronizing output of each system, ϕ is nonlinear coupling with $\phi(\eta, h(\eta), L) \equiv 0 \ \forall \eta, L$ and $C = (c_{ij})_{i,j=1,\ldots,n}$ is the adjacency matrix that has no loops, i.e. $i \neq j$. Here, c_{ij} is not always equal to c_{ji}, because the graph is directed, i.e. the adjacency matrix may be nonsymmetrical, but if $c_{ij} = c_{ji} = 1$, then the connection between node i and node j is called as *coupled* or *duplex* coupling. Without loss of generality one can set $c_{ii} = 0\ \forall i \in \{1, 2, \ldots, N\}$, due to the above assumption that $\phi(\eta, h(\eta), L) \equiv 0\ \forall \eta, L$. Network (1) is said to be (*asymptotically*) *synchronized* if, $\forall i, j \in \{1, 2, \ldots, N\}$

$$\lim_{t \to \infty} (\eta^i(t) - \eta^j(t)) = 0. \tag{2}$$

A network, with $c_{ij} = 1\ \forall i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, is called *complete N-node DCN*. A network, where C is a cyclic matrix (i.e. each its row and column has precisely one nonzero entry), and is called *cyclic DCN*. Finally, a network is *disconnected*, if there is re-numbering of the nodes making C block diagonal; otherwise, it is *connected*. One can easily see that the above notions have clear interpretation, e.g. a complete network contains all possible connections (see e.g. the network (h) in Fig. 1), while in a cyclic network each node has exactly one inbound and one outbound connection, so it creates a directed cyclic chain of connections (see e.g. (b) in Fig. 1, or network in Fig. 6). Finally, disconnected network could consist of two or more independent subnetworks.

Obviously, a disconnected network can not be synchronized in general. Nevertheless, being connected is only necessary for a network to be synchronizable. This leads to the following definition.

Definition 2.1. The DCN (1) is said to be *synchronizable* if there exists an integer $\mu \in \{1, 2, \ldots, N\}$, such that for every $\sigma \in \{1, 2, \ldots, N\}$ there exists a sequence of integers $\{\kappa_1, \ldots, \kappa_l\}$ satisfying

$$\kappa_1 = \mu, \quad \kappa_l = \sigma, \quad c_{\kappa_1, \kappa_2} = \cdots = c_{\kappa_{l-1}, \kappa_l} = 1.$$

If the above integer μ is unique, then the node with number μ is called the *master* of DCN (1). Synchronizable network is called *minimal*, if removing any connection makes it not synchronizable.

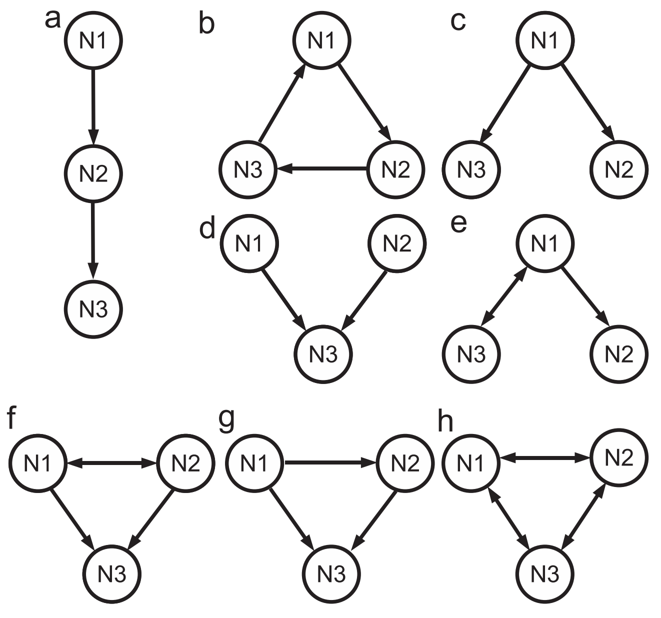

Fig. 1. Some possible topologies of a three-node networks with directed connections among nodes.

Synchronizable network in the terminology of the graph theory is a directed graph that contains a rooted directed spanning tree, where master is a *root*. Each edge (connection) is considered to be directed away from the root [15] and the root is a dynamical system (node, vertex), which influences directly or indirectly all other systems (nodes, vertices) [16]. If a network is a rooted directed spanning tree, then the network is *minimal*. A graph is a directed spanning rooted tree if it is a tree as an undirected graph and there is a directed path from the root (master) to every other node [16]. A minimal network is always unidirectional.

Note that the above synchronizability definition makes sense only for networks being directed graphs. For undirected graphs, it is sufficient to replace it by the simple property of being connected. The following properties obviously hold:

1. *A minimal synchronizable network always has a master.*
2. *A cyclic network is always synchronizable, but never minimal.*

An example of a connected network, which is not synchronizable, is shown in Fig. 2. In Fig. 1, networks (a) and (c) are minimal ones having node 1 as their master. Finally, Fig. 3 gives a full list of all four-nodes minimal synchronizable DCN.

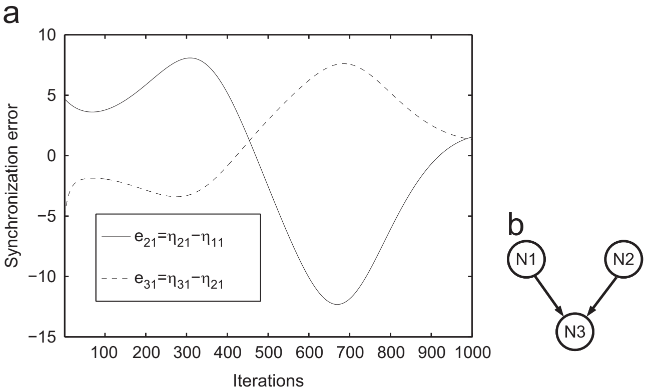

Fig. 2. Lack of synchronizability of a three-node network, with $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$; $l_1, l_2 = -35$ and $\tau = 0.5$, where $\lambda_{1,2,3}$ are eigenvalues of approximate linearization fulfilling the well-known Shilnikov's inequality $-\lambda_2 > \lambda_1 > -\lambda_3$ and τ enables fine chaos tuning. (a) Synchronization errors of a three-node network. (b) Topology of a three-node network.

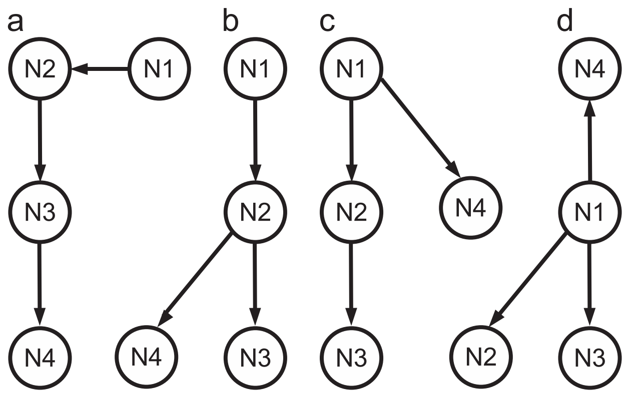

Fig. 3. List of all four-node minimal synchronizable directed networks.

3. Generalized Lorenz systems and their synchronization in DCN

In this section, a DCN with nodes being the so-called generalized Lorenz system (GLS) is studied. GLS is a generalization of the classical Lorenz system containing it as a particular case. Full details about GLS may be found in [11,17], where in particular the so-called *generalized Lorenz canonical form* was introduced there. This form enables to generate rich parameterized chaotic behavior. Here, another canonical form is reviewed in detail with the following theorem established in [12].

Theorem 3.1. *GLS is state equivalent to the following system, further referred to as the observer canonical form*:

$$\frac{d\eta}{dt} = \begin{bmatrix} (\lambda_1+\lambda_2)\eta_1+\eta_2 \\ -\lambda_1\lambda_2\eta_1-(\lambda_1-\lambda_2)\eta_1\eta_3-\frac{(\tau+1)}{2}\eta_1^3 \\ \lambda_3\eta_3+K_1(\tau)\eta_1^2 \end{bmatrix} \tag{3}$$

$$K_1(\tau)=\frac{\lambda_3(\tau+1)-2\tau\lambda_1-2\lambda_2}{2(\lambda_1-\lambda_2)}. \tag{4}$$

An important feature of the above canonical form is that it contains only four parameters, where $\lambda_{1,2,3}$ are eigenvalues of the system linearization fulfilling the well-known Shilnikov's inequality $-\lambda_2>\lambda_1>-\lambda_3$ and τ enables fine chaos tuning. Moreover, the observer canonical form of GLS provides a possibility to synchronize two GLS's coupled in a master–slave configuration using only scalar signal η_1, as in [12].

The main theoretical result of this section is the following theorem that generalizes the mentioned result to the case with symmetric (or duplex) synchronizing connection between two GLS's.

Theorem 3.2. *Consider a DCN consisting of two GLS in the canonical form* (3) *and* (4) *with the states* $\eta,\hat{\eta}$, *outputs* $\eta_1,\hat{\eta}_1$, *and its uniformly bounded trajectory* $\eta(t)$, $t\geq t_0$, *coupled as follows*:

$$\frac{d\hat{\eta}}{dt} = \begin{bmatrix} (\lambda_1+\lambda_2)\hat{\eta}_1+\hat{\eta}_2 \\ -\lambda_1\lambda_2\hat{\eta}_1-(\lambda_1-\lambda_2)\hat{\eta}_1\hat{\eta}_3-\frac{(\tau+1)}{2}\hat{\eta}_1^3 \\ \lambda_3\hat{\eta}_3+K_1(\tau)\hat{\eta}_1^2 \end{bmatrix} + c_{12}\left(\begin{bmatrix} (\lambda_1+\lambda_2)\hat{\eta}_1 \\ -\lambda_1\lambda_2\hat{\eta}_1-(\lambda_1-\lambda_2)\hat{\eta}_1\hat{\eta}_3-\frac{(\tau+1)}{2}\hat{\eta}_1^3 \\ K_1(\tau)\hat{\eta}_1^2 \end{bmatrix} - \begin{bmatrix} (\lambda_1+\lambda_2)\eta_1 \\ -\lambda_1\lambda_2\eta_1-(\lambda_1-\lambda_2)\eta_1\eta_3-\frac{(\tau+1)}{2}\eta_1^3 \\ K_1(\tau)\eta_1^2 \end{bmatrix} + \begin{bmatrix} l_1(\hat{\eta}_1-\eta_1) \\ l_2(\hat{\eta}_1-\eta_1) \\ 0 \end{bmatrix}\right), \tag{5}$$

$$\frac{d\eta}{dt} = \begin{bmatrix} (\lambda_1+\lambda_2)\eta_1+\eta_2 \\ -\lambda_1\lambda_2\eta_1-(\lambda_1-\lambda_2)\eta_1\eta_3-\frac{(\tau+1)}{2}\eta_1^3 \\ \lambda_3\eta_3+K_1(\tau)\eta_1^2 \end{bmatrix} + c_{21}\left(\begin{bmatrix} (\lambda_1+\lambda_2)\eta_1 \\ -\lambda_1\lambda_2\eta_1-(\lambda_1-\lambda_2)\eta_1\eta_3-\frac{(\tau+1)}{2}\eta_1^3 \\ K_1(\tau)\eta_1^2 \end{bmatrix} - \begin{bmatrix} (\lambda_1+\lambda_2)\hat{\eta}_1 \\ -\lambda_1\lambda_2\hat{\eta}_1-(\lambda_1-\lambda_2)\hat{\eta}_1\eta_3-\frac{(\tau+1)}{2}\hat{\eta}_1^3 \\ K_1(\tau)\hat{\eta}_1^2 \end{bmatrix} + \begin{bmatrix} l_1(\eta_1-\hat{\eta}_1) \\ l_2(\eta_1-\hat{\eta}_1) \\ 0 \end{bmatrix}\right), \tag{6}$$

where $l_{1,2}<0$ are gains to be designed. The synchronization connection in Eq. (5) equals to the following equation:

$$\begin{bmatrix} (\lambda_1+\lambda_2)e_1 \\ -\lambda_1\lambda_2 e_1-(\lambda_1-\lambda_2)e_1\hat{\eta}_3-\frac{(\tau+1)}{2}e_1(\hat{\eta}_1^2-\hat{\eta}_1\eta_1+\eta_1^2) \\ K_1(\tau)e_1(\hat{\eta}_1+\eta_1) \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \\ 0 \end{bmatrix} e_1, \tag{7}$$

where $e_1=\hat{\eta}_1-\eta_1$ and $e_1=0$, if $\eta_1=\hat{\eta}_1$. If $e_1=0$, then system (5) is equal to the observer canonical form of GLS (3). Then,

1. *For $c_{12}=0$, $c_{21}=1$ or $c_{21}=0$, $c_{12}=1$, and for all gains $l_{1,2}<0$, one has $\lim_{t\to\infty}(\eta(t)-\hat{\eta}(t))=0$ globally and exponentially.*
2. *For $c_{12}=1$, $c_{21}=1$, and for every bounded region of initial conditions of system (5) and (6), there exist sufficiently large gains $l_{1,2}<0$ such that $\lim_{t\to\infty}(\eta(t)-\hat{\eta}(t))=0$.*

Proof. The first claim is a straightforward consequence of a result in [12], where global synchronization of the master–slave configuration of two GLS was proved. To prove the second claim, denoting $e=(e_1,e_2,e_3)^\top=\eta-\hat{\eta}$, and deducing (6) from Eq. (5), one obtains

$$\dot{e}=\tilde{A}e+\begin{bmatrix} 0 \\ \alpha e_1+\beta_1 e_1^2+\beta_2 e_1^3+\gamma e_3 \\ -K_1(\tau)(2\eta_1(t)e_1+e_1^2) \end{bmatrix}, \tag{8}$$

$$\alpha(t):=\frac{3(\tau+1)\eta_1^2(t)}{2}+(\lambda_1-\lambda_2)\eta_3(t),\quad \beta_2:=\frac{\tau+1}{2},\quad \beta_1(t):=\frac{3(\tau+1)\eta_1(t)}{2},$$

$$\gamma(t):=(\lambda_1-\lambda_2)\eta_1,\quad \tilde{A}=\mathrm{diag}\{\overline{A}(l_1,l_2),\lambda_3\},\quad \hat{A}=\begin{bmatrix} 2l_1-(\lambda_1+\lambda_2) & 1 \\ 2l_2+\lambda_1\lambda_2 & 0 \end{bmatrix}.$$

Notice that $\hat{A}(\theta)$ is Hurwitz $\forall\theta>0$, where

$$\hat{A}(\theta):=\overline{A}(l_1(\theta),l_2(\theta))=\begin{bmatrix} -\theta & 1 \\ -\theta^2 & 0 \end{bmatrix}, \tag{9}$$

where

$$l_1(\theta)=\frac{-\theta+\lambda_1+\lambda_2}{2},\quad l_2(\theta)=\frac{-\theta^2-\lambda_1\lambda_2}{2}.$$

In particular, there exists a matrix S such that

$$S\hat{A}(1)+\hat{A}(1)^\top S=-I_2,\quad S>0,\ S^\top=S,$$

and, moreover, S is a constant matrix independent of θ. Further, consider a Lyapunov function candidate

$$V(e)=[e_1,\theta^{-1}e_2]S[e_1,\theta^{-1}e_2]^\top+\frac{(\theta^{-1}e_3)^2}{2},$$

and compute its full derivative along trajectories of system (8) and (9), to obtain

$$\dot{V}=-\theta(\epsilon_1^2+\epsilon_2^2)+\lambda_3\epsilon_3^2+K_1\epsilon_3(2\eta_1(t)\epsilon_1+\epsilon_1^2)+2[\epsilon_1,\epsilon_2]S[0,\alpha\epsilon_1+\beta_1\epsilon_1^2+\beta_2\epsilon_1^3+\gamma\epsilon_3]^\top,$$

where

$$\epsilon_1 := e_1, \quad \epsilon_2 := \theta^{-1} e_2, \quad \epsilon_3 := \theta^{-1} e_3.$$

Notice that $\alpha, \beta_{1,2}, \gamma$ are dependent only on system parameters and $\eta(t)$, which is bounded by assumption of the theorem. Therefore, there exist a constant R_2 and a smooth function $R_1(\cdot)$ such that

$$\dot{V} \leq -\theta(\epsilon_1^2 + \epsilon_2^2) + \lambda_3 \epsilon_3^2 + |R_1(\epsilon_1)\epsilon_1(\epsilon_1 + \epsilon_3) + |R_2 \epsilon_2(\epsilon_1 + \epsilon_2 + \epsilon_3)|.$$

Notice that $R_{1,2}$ do not depend on θ. As a consequence, selecting

$$\theta = \theta(e_1) := \max\{|R_1(e_1(t))|, |R_2|\} + R,$$

where $R > 0$ is a big enough constant, guarantees that $\dot{V} \leq -R_3 \|\epsilon\|^2, \ R_3 > 0$. By definition of $V(e)$, there exist real constants $c_2 > c_1 > 0$ such that

$$c_1[e_1^2 + [\theta^{-1} e_2]^2] + [\theta^{-1} e_3]^2/2 \leq \|V(e)\| \leq c_2[e_1^2 + [\theta^{-1} e_2]^2] + [\theta^{-1} e_3]^2/2.$$

As a consequence, it holds that $\forall s > 0$

$$\|e\| \leq s \Rightarrow \|V(e)\| \leq c_2 s, \quad \|V(e)\| \leq s \Rightarrow \left| e_1 \right| \leq \frac{s}{c_1}.$$

Now, semi-global exponential synchronization is achieved in the following way: for any $s > 0$, taking gains (9) with $\theta = \max_{|e_1| \leq s(c_2/c_1)} \theta(e_1)$ guarantees exponential convergence in the region of initial errors $\|e(0)\| \leq s$. Indeed, such a selection of gains guarantees that $\dot{V} \leq -R_3 \|\epsilon\|^2, \ R_3 > 0$, for all $\|e(t)\| \leq s$, since the above inequalities ensure that $\|V(e(t))\| \leq c_2 s$ which, in turn, guarantees that $|e_1| \leq s(c_2/c_1)$. As a consequence, $V(e)$ decreases along trajectories, which guarantees that inequality $\|V(e(t))\| \leq c_2 s$ holds and, consequently, $|e_1(t)| \leq s(c_2/c_1)$. In other words, for any $e(t)$ with $\|e(0)\| \leq s$, it holds that for all $t \geq 0$, $\dot{V} \leq -R_3 \|\epsilon\|^2, \ R_3 > 0$, and therefore $e(t)$ goes to zero exponentially as $t \to \infty$. □

Now, consider a DCN consisting of N nodes, each being a GLS defined by

$$\begin{bmatrix} \dot{\eta}_1^i \\ \dot{\eta}_2^i \\ \dot{\eta}_3^i \end{bmatrix} = \begin{bmatrix} (\lambda_1 + \lambda_2)\eta_1^i + \eta_2^i \\ -\lambda_1 \lambda_2 \eta_1^i - (\lambda_1 - \lambda_2)\eta_1^i \eta_3^i - \frac{(\tau+1)}{2}(\eta_1^i)^3 \\ \lambda_3 \eta_3^i + K_1(\tau)(\eta_1^i)^2 \end{bmatrix} + \sum_{j=1}^{N} c_{ji} \begin{bmatrix} (\lambda_1 + \lambda_2 + l_1)(\eta_1^i - \eta_1^j) \\ \left\{ \begin{array}{c} (-\lambda_1 \lambda_2 + l_2)(\eta_1^i - \eta_1^j) - \\ (\lambda_1 - \lambda_2)(\eta_1^i - \eta_1^j)\eta_3^i - \\ \frac{(\tau+1)}{2}((\eta_1^i)^3 - (\eta_1^j)^3) \end{array} \right\} \\ K_1(\tau)((\eta_1^i)^2 - (\eta_1^j)^2) \end{bmatrix}, \quad (10)$$

with a possibly non-symmetric 0–1 coupling matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1(N-1)} & c_{1N} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2N} \\ c_{31} & c_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & c_{(N-1)(N-1)} & c_{(N-1)N} \\ c_{N1} & c_{N2} & \cdots & c_{N(N-1)} & c_{NN} \end{bmatrix}.$$

As a matter of fact, Theorem 3.2 verifies that synchronization of the 2-node DCN (10), at least semi-globally, does not depend on the topology of its connections, as long as the corresponding DCN remains synchronizable.

For DCN with N nodes, the following result is a straightforward consequence of claim 1 of Theorem 3.2.

Theorem 3.3. *Consider DCN (10), which is assumed synchronizable and minimal. Then, it is globally exponentially synchronized.*

Proof. Since DCN is by assumption synchronizable and minimal, each node, except the master node, has exactly one node which sends synchronizing signal to it. It is called the preceding one. Therefore, using Theorem 3.1, case 1 with $c_{12}=1$, $c_{21}=0$, one has that each of these nodes is synchronized with its unique preceding node, therefore all nodes are synchronized with the master node. □

4. Numerical experiments

As indicated by Theorem 3.2 for non-minimal synchronizable DCN's, only semi-global synchronization is possible in general. This theorem, nevertheless, considers a two-node network only. In this section, it is shown experimentally that this property holds even for networks with a larger number of nodes.

More specifically, consider several four-node DCN of GLS in the form of Eq. (10). The first example is presented by Fig. 2, which is not synchronizable in the sense of Definition 2.1. Simulations confirm that the network is indeed not synchronized. In Fig. 4, Theorem 3.2 is illustrated. One can see that two nodes with a duplex connection (i.e. neither is master or slave) are synchronized; but for initial synchronization errors up to 1, quite strong gains are needed. In Fig. 5, a completed connected DCN with four nodes is synchronized, i.e. information is transmitted from any node to all other nodes. Again, strong the synchronizing gains are needed. Fig. 6 shows the special case of a cyclic network, indicating that synchronization persists with the same parameters as in the case of Fig. 5. Fig. 7 presents a network

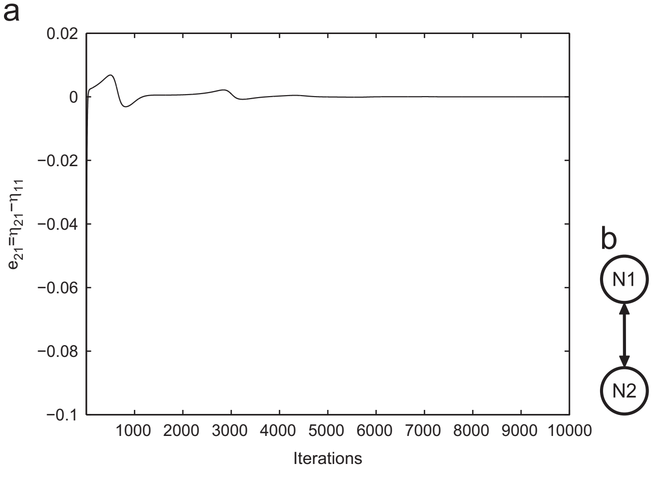

Fig. 4. Synchronization of a two-node network, with $\lambda_1=8$, $\lambda_2=-16$, $\lambda_3=-1$, $c_{ij}=1(i\neq j)$, $l_1, l_2=-40$ and $\tau=0.5$. Initial condition $0\leq[\eta_1^i, \eta_2^i, \eta_3^i]^\top\leq 1$. (a) Synchronization error of a two-node network and (b) topology of a two-node network.

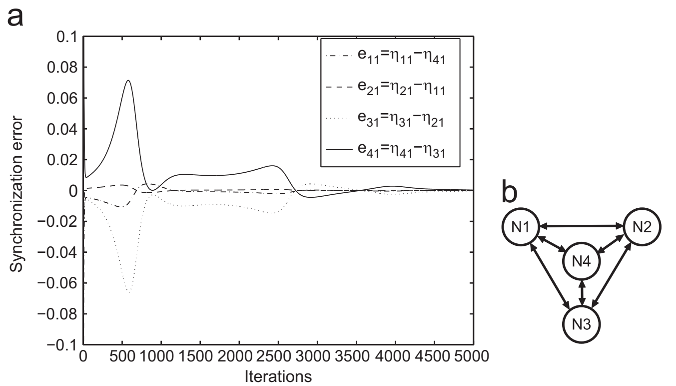

Fig. 5. Synchronization of complete directed network with $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$; $c_{ij} = 1(i \neq j)$; $l_1, l_2 = -40$ and $\tau = 0.5$. (a) Synchronization errors of complete four-node network. (b) Topology of complete four-node network.

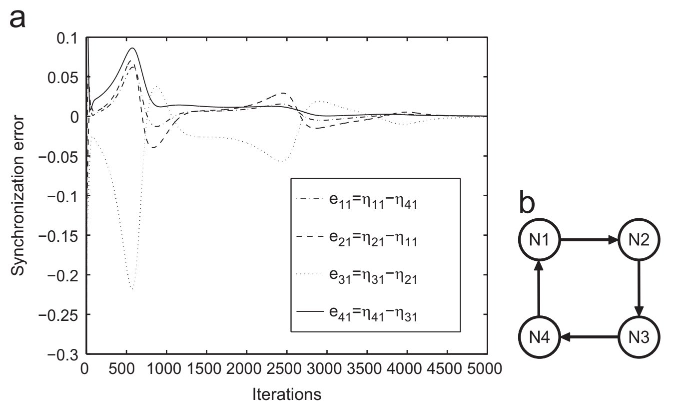

Fig. 6. Synchronization of a network with $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$; $l_1, l_2 = -40$ and $\tau = 0.5$. (a) Synchronization errors of four-node network and (b) topology of cycle four-node network.

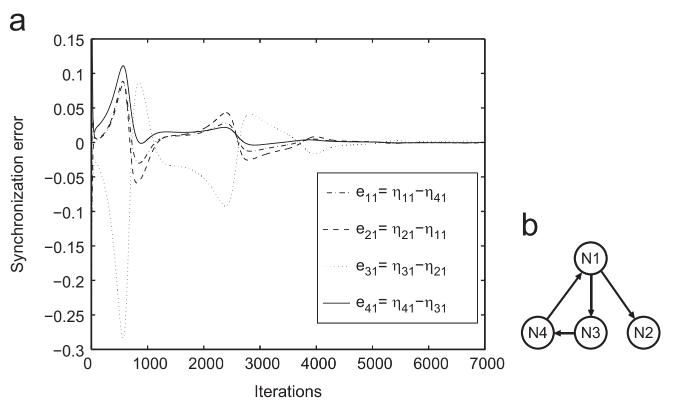

Fig. 7. Synchronization of a network with $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$; $l_1, l_2 = -40$ and $\tau = 0.5$. (a) Synchronization errors of four-node network and (b) topology of four-node network.

with a cycle and master–slave topologies. Fig. 8 shows the special modification of the network illustrated in Fig. 7 with unstable connection from first and third nodes to second node. Here, the network has unstable connection between first and second nodes, and between third and second nodes, respectively. When simulation is starting, connection from first node to second node is stable (switch on) during one thousand of iterations (fixed step size in simulation is equal to 0.001). In the same time,

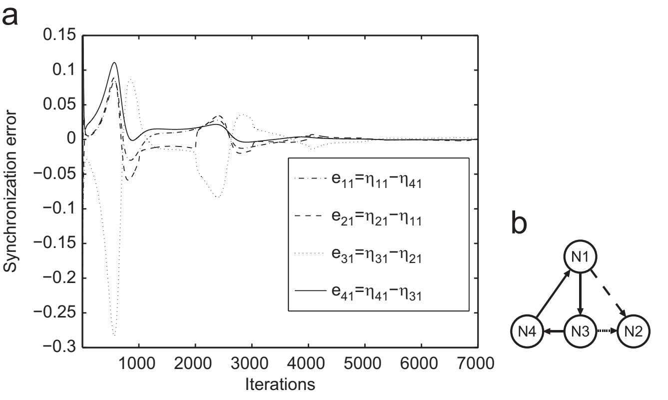

Fig. 8. Synchronization of a network with $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$; $l_1, l_2 = -40$ and $\tau = 0.5$. (a) Synchronization errors of four-node network. (b) Topology of four-node network. Here, the connection from first node to second node (interrupted line) is switched on/off every thousand of iterations. If connection from first node to second node is off, then connection from third node to second node (dotted line) is on. All the time network stays connected.

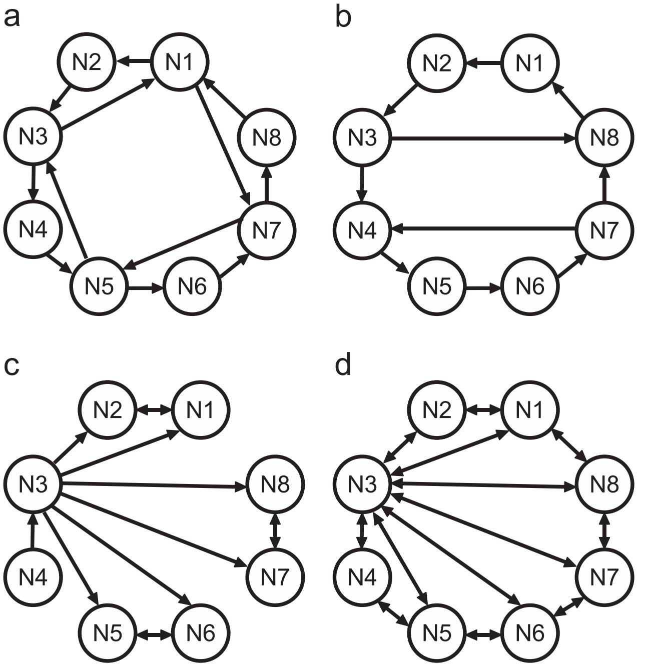

Fig. 9. Some possible topologies of a eight-node networks with directed connections among nodes.

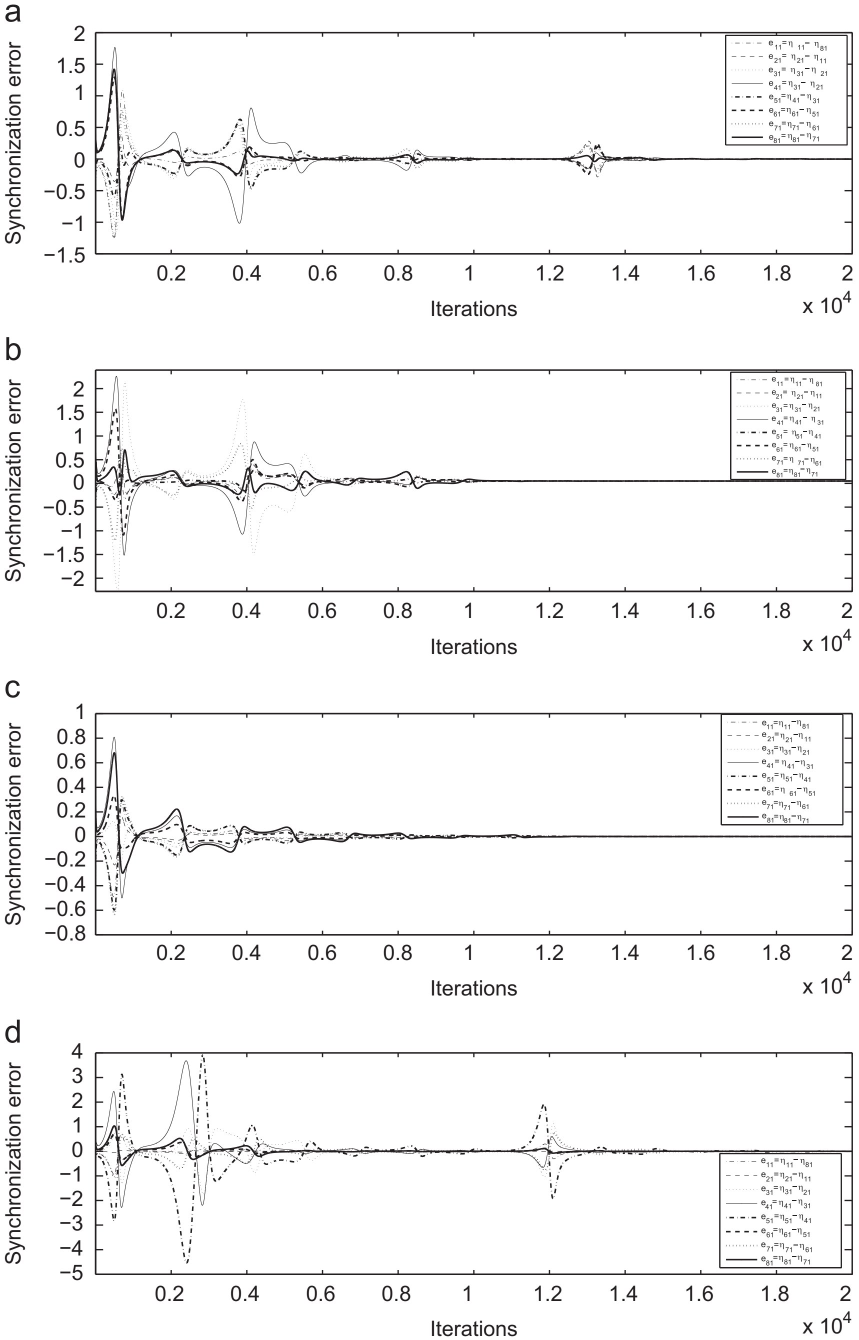

Fig. 10. Synchronization errors of eight-node networks shown in Fig. 9 (a), (b), (c), (d), respectively, with $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$; $l_1, l_2 = -10$ and $\tau = 0.5$.

connection between third and second nodes is switched off, until first and second nodes are connected again. From one thousand to two thousands of iterations the connection between second and third nodes is on, and connection between first and second nodes is off. In every one thousand of iterations connections are switching. All the times, network stays at synchrony. In Fig. 9 some possible topologies of a directed eight-node networks are presented and Fig. 10 illustrates their synchronization errors.

Actually, many more experiments have been carried out, all showing that the above-described a nice robust structural property always holds, subject to the requirement that the network does not lose its synchronizability (see Definition 2.1). Regarding semi-global versus global performance, an interesting observation is that for particular initial conditions and gains, a synchronizable network is always either synchronized or diverging to infinity. This was actually predicted by the proof of Theorem 3.2.

5. Conclusions

The main conclusion drawn from the present investigation is that for general cases one can only guarantee semi-global synchronization while for minimal synchronizable configurations one can always achieve global exponential synchronization. It should be pointed out that both synchronizability and minimality are mere properties of the network topology. In other words, synchronization may be determined from some graph-theoretic properties of the network, once one knows how to synchronize the simple master–slave configuration of two nodes. This therefore is of fundamental importance, at least for chaos synchronization.

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