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On time transformations for differential equations with state-dependent delay

Research Article

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Abstract: Systems of differential equations with state-dependent delay are considered. The delay dynamically depends on the state, i.e. is governed by an additional differential equation. By applying the time transformations we arrive to constant delay systems and compare the asymptotic properties of the original and transformed systems.

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1. Introduction

Delay differential equations are successfully used to model and study a number of applied problems in physics, biology, and chemistry. Basic mathematical theory of constant delay equations can be found in the classical monographs [6, 7] and references therein.

Differential equations with state-dependent delay (SDD) attracted much attention during last decades and there been obtained many deep results for them (see [9–11, 16] and references therein). Such equations with discrete state-dependent delays are always *nonlinear* by their nature. As described in [9], this type of delay brings additional difficulties in proving such basic properties of solutions as uniqueness and continuous dependence on initial data. The main approach to get the well-posed initial-value problem is to restrict the set of initial functions and hence the set of solutions to C^1-functions [9]. For an alternative approach, where an additional condition on the SDD is used, to get a well-posed initial-value problem in the space of continuous functions see [12, 13]. In this note we rely on the classical approach [9] and

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compare the SDD problem with another one with a constant delay. This constant delay problem is constructed by using the so-called *time transformations* [2, 3]. This transformation could be applied to any particular solution along which the deviating argument is monotone. To be assured that the monotonicity holds for all solutions we concentrate on the system when the state-dependent delay is governed by an additional differential equation and provide a sufficient condition for the monotonicity of the deviating argument. This type of equations is used to describe some models of population dynamics, see [1] and references therein. In [1] one could also find motivations to study this type of SDD and comparison with the frequently used case when SDD is presented by explicit or implicit functionals.

Our main goal in this note is to compare the asymptotic properties of the SDD system with the corresponding ones of the system after the time transformation. We try to find conditions which guarantee that such properties as stability, boundedness and compactness of the initial SDD problem survive under the time transformations.

2. Time transformations

We study the following non-autonomous system with state-dependent delay (see the autonomous case in [1])

$$\dot{y}(t) = f\big(t, y(t), y(t-\eta(t))\big), \qquad t > t^0, \tag{1}$$

$$\dot{\eta}(t) = -\mu(\eta(t)-\widetilde{\eta}) + G(y(t)), \quad t > t^0, \tag{2}$$

with the initial data

$$y(t) = g(t), \qquad t^0 - h \le t \le t^0, \qquad \eta\big(t^0\big) = \eta^0. \tag{3}$$

Here $y \in \mathbb{R}^m$, $\eta \in \mathbb{R}$, $\mu, \eta^0 > 0$, $\widetilde{\eta} > 0$, functions f and G are continuous. The function η is a state-dependent delay since it is a solution of equation (2) where there is a dependence on y.

In the sequel we will denote $h \equiv 2\widetilde{\eta} > 0$ and also $X \equiv C^1([-h,0];\mathbb{R}^m) \times \mathbb{R}$ with the natural norm.

Lemma 2.1.
Let f be continuous function, Lipschitz with respect to the second and third coordinates and G be Lipschitz and $|G(y)| \le \mu\widetilde{\eta}$ for all $y \in \mathbb{R}^m$. Then for any $g \in C^1\big(\big[t^0-h, t^0\big];\mathbb{R}^m\big)$, $\eta\big(t^0\big) = \eta^0 \in [0, 2\widetilde{\eta}]$ the system (1)–(3) has a unique global solution $(y;\eta)$ such that $\eta(t) \in [0, 2\widetilde{\eta}]$ for all $t \ge t^0$. The solution continuously depends on initial data $\big(g;\eta^0\big)$.

Proof. The proof of the existence is simple since the righthand sides of equations (1) & (2) are continuous. Solutions are global due to Lipschitz properties of f and G. The uniqueness follows from the well-known results on the state-dependent delay equations (see e.g. [9]) since we consider Lipschitz initial function g.

Using the property $|G(y)| \le \mu\widetilde{\eta}$, one can easily show that (for any y) any solution of (2) & (3) satisfies $\eta(t) \in [0, 2\widetilde{\eta}]$ provided $\eta\big(t^0\big) = \eta^0 \in [0, 2\widetilde{\eta}]$.

Now we show the continuous dependence on initial data. For the simplicity of presentation we put $t^0 = 0$. Let us consider a pair $\big(\overline{g};\overline{\eta}^0\big) \in C^1([-h,0];\mathbb{R}^m) \times [0,h] \subset X$ and an arbitrary sequence $\big(g^n;\eta^{0,n}\big)$ such that $\big\|\big(g^n;\eta^{0,n}\big) - \big(\overline{g};\overline{\eta}^0\big)\big\|_X \to 0$ as $n \to \infty$. We rewrite the system (1)–(3) in the integral form:

$$y^n(t) \;=\; g^n(0) + \int_0^t f\big(s, y^n(s), y^n(s-\eta^n(s))\big)\,ds, \qquad \eta^n(t) - \widetilde{\eta} \;=\; e^{-\mu t}\big(\eta^{0,n} - \widetilde{\eta}\big) + \int_0^t e^{-\mu(t-s)} G(y^n(s))\,ds.$$

Similar equations are for initial data $\big(\overline{g};\overline{\eta}^0\big)$. For the differences of solutions, using the Lipschitz properties of f and G (the corresponding Lipschitz constants are denoted by L_f and L_G) we have

$$|y^n(t) - \overline{y}(t)| \;\le\; |g^n(0) - \overline{g}(0)| + L_f \int_0^t \Big\{\big|y^n(s-\eta^n(s)) - \overline{y}(s-\eta^n(s))\big| + \big|\overline{y}(s-\eta^n(s)) - \overline{y}(s-\overline{\eta}(s))\big| + |y^n(s) - \overline{y}(s)|\Big\}\,ds,$$

$$|\eta^n(t) - \overline{\eta}(t)| \;\le\; e^{-\mu t}\big|\eta^{0,n} - \overline{\eta}^0\big| + L_G \int_0^t |y^n(s) - \overline{y}(s)|\,ds.$$

Let us fix any $T > 0$. Since all solutions are C^1 in time (see e.g. [9]) for $t > 0$, we can denote by $L_{\overline{y},T}$ the Lipschitz constant of the solution $\overline{y}(t)$, $t \in [-h, T]$. Hence $|\overline{y}(s-\eta^n(s)) - \overline{y}(s-\overline{\eta}(s))| \le L_{\overline{y},T}|\eta^n(s)-\overline{\eta}(s)|$ for all $s \in [0, t] \subset [0, T]$. Denoting for short $\beta^n(t) \equiv \max_{\tau\in[0,t]}\{|y^n(\tau)-\overline{y}(\tau)| + |\eta^n(\tau)-\overline{\eta}(\tau)|\}$ and $C^T \equiv 2L_f + L_G + L_f L_{\overline{y},T}$, we obtain

$$0 \le \beta^n(t) \le \beta^n(0) + L_f T \max_{\tau\in[-h,0]}|g^n(\tau)-\overline{g}(\tau)| + C^T \int_0^t \beta^n(s)\, ds.$$

We apply the Gronwall's inequality and get for all $t \in [0, T]$,

$$\max_{\tau\in[0,t]}\left\{|y^n(\tau)-\overline{y}(\tau)| + |\eta^n(\tau)-\overline{\eta}(\tau)|\right\} \le \left\{(1+L_f T)\max_{\tau\in[-h,0]}|g^n(\tau)-\overline{g}(\tau)| + \left|\eta^{0,n}-\overline{\eta}^0\right|\right\}\exp\left(t\cdot(2L_f+L_G+L_f L_{\overline{y},T})\right).$$

The last estimate and equations (1) & (2) give a similar estimate for the time derivatives, so we finally get $|y^n-\overline{y}|_{C^1([0,t];\mathbb{R}^m)} + |\eta^n-\overline{\eta}|_{C^1([0,t];\mathbb{R})} \to 0$ as $n \to +\infty$. This gives the continuous dependence on initial data and completes the proof. □

For any solution $(y;\eta)$ of the system (1)–(3) we call the function σ given by

$$\sigma(t) = t - \eta(t), \qquad t \ge t^0, \tag{4}$$

the *deviating argument* of $(y;\eta)$.

Our goal is to investigate properties connected to the time transformation approach introduced in [2, 3]. We are going to use a function $t = \alpha(s)$ called *time transformation* [3] to convert a particular solution $(y;\eta)$ of the system (1)–(3) into a solution $(z;\chi;\alpha)$ of *constant* delay system

$$\begin{cases} \dot z(s) = f(\alpha(s), z(s), z(s-h))\,\dot\alpha(s), & s \ge s^0,\\ z(s) = \psi(s) \equiv g(\omega(s)), & s^0 - h \le s \le s^0,\\ \dot\chi(s) = -\mu(\chi(s)-\widetilde{\eta})\,\dot\alpha(s) + G(z(s))\,\dot\alpha(s), & \\ \chi(s^0) = \eta^0, & \end{cases} \tag{5}$$

where α satisfies the algebraic equation

$$\begin{cases} \alpha(s) - \chi(s) = \alpha(s-h), & s \ge s^0,\\ \alpha(s) = \omega(s), & s^0 - h \le s \le s^0. \end{cases} \tag{6}$$

Here $\omega\colon \left[s^0-h, s^0\right] \to \mathbb{R}$ is an arbitrary C^1-function with positive derivative and such that $\omega\left(s^0-h\right) = \omega\left(s^0\right) - \eta^0 < t^0$, $\omega\left(s^0\right) = t^0$.

Remark 2.2.

We notice that equation (6) is different from the corresponding rules used in [2, 3] since here we have no given *lag function*.

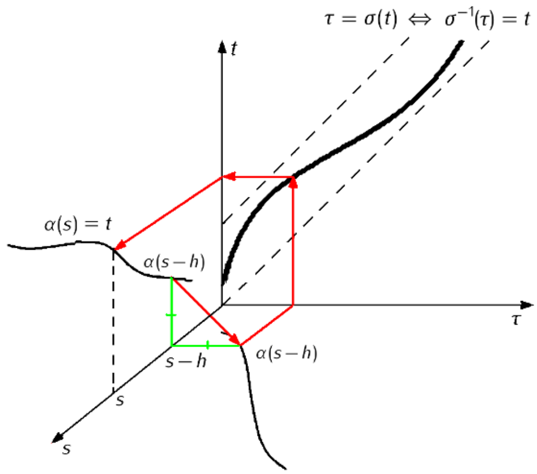
The time transformation α is constructed step by step (see (6) and (4)) by the rule (see more discussion in Remark 2.6)

$$\alpha(s) = \sigma^{-1}(\alpha(s-h)), \qquad s \in \left[s^0 + (k-1)h, s^0 + kh\right], \quad k = 0, 1, 2, \ldots \tag{7}$$

Here we used (see (4)) $\sigma(\alpha(s)) = \alpha(s) - \eta(\alpha(s)) = \alpha(s) - \chi(s) = \alpha(s-h)$, since $\chi(s) = \eta(\alpha(s))$.

Figure 1 shows the connection between s (new time), t (old time) and τ (deviating argument).

Figure 1. The connection between s (new time), t (old time) and τ (deviating argument).



It is clear that one needs the invertibility of σ to define α. In [3] the rule (7) was used *assuming* that $\dot{\sigma}(t) > 0$ (or $\dot{\sigma}(t) < 0$). More precisely, it was used for those solutions along which $\dot{\sigma}(t) > 0$ (or $\dot{\sigma}(t) < 0$). In our study we can give a simple condition which *guarantees* that along *all* solutions we have $\dot{\sigma}(t) > 0$ and hence σ is always invertible. Such a simple condition is $2\mu\widetilde{\eta} < 1$. It is easy to see using (2) that in this case $|\dot{\eta}(t)| \le \mu|\eta - \widetilde{\eta}| + |G(y(t))| \le \mu\widetilde{\eta} + \mu\widetilde{\eta} < 1$. Here we used the assumption $|G(y)| \le \mu\widetilde{\eta}$. Now (4) implies $\dot{\sigma}(t) = 1 - \dot{\eta}(t) > 0$.

Remark 2.3.

Taking into account that the state-dependent delay η takes values in $[0, 2\widetilde{\eta}] = [0, h]$, one could say that the assumption $2\mu\widetilde{\eta} < 1$ means the "slow changing of delay" in the range $[0, 2\widetilde{\eta}]$.

It is important that if $\dot{\sigma}(t) > 0$, then $\dot{\alpha}(s) > 0$ since on the initial time segment $\dot{\alpha}(s) = \dot{\omega}(s) > 0$, $s \in \left[s^0 - h, s^0\right]$ and $\dot{\alpha}(s) = d\sigma^{-1}(\alpha(s-h))/ds = \dot{\alpha}(s-h)/\dot{\sigma}(\alpha(s-h)) = \dot{\alpha}(s-h)/\dot{\sigma}(\sigma(\alpha(s))) > 0$ step by step (see also [3, p. 28]). Here we used $\sigma(\alpha(s)) = \alpha(s-h)$ (see (6) and (7)).

By construction (see [3, Propositions 1 and 2]) the connection between a solution (y, η) of (1)–(3) and the corresponding solution (z, χ, α) of (5) & (6) is given by

$$\begin{aligned} y(t) &= z\left(\alpha^{-1}(t)\right), && t \ge t^0 - h, \\ z(s) &= y(\alpha(s)), && \\ \chi(s) &= \eta(\alpha(s)), && s \ge s^0 - h, \\ t &= \alpha(s), && t^0 = \alpha\left(s^0\right). \end{aligned} \tag{8}$$

Lemma 2.4.

Let f and G be as in Lemma 2.1. Consider a sequence $\left\{\left(g^n; \eta^{0,n}\right)\right\}$ such that $\left\|\left(g^n; \eta^{0,n}\right) - \left(\overline{g}; \overline{\eta}^0\right)\right\|_{C^1([t^0-h,t^0];\mathbb{R}^m)\times\mathbb{R}} \to 0$ and a sequence $\{\omega^n\}$ such that $\|\omega^n - \overline{\omega}\|_{C^1([t^0-h,t^0];\mathbb{R})} \to 0$ as $n \to \infty$. Then for any $S > 0$ the sequence of time transformations α^n uniformly converges to $\overline{\alpha}$, i.e. $\max_{s\in[s^0,s^0+S]} |\alpha^n(s) - \overline{\alpha}(s)| \to 0$ as $n \to \infty$.

Proof. First, using (4), we have from Lemma 2.1 that $\max_{t\in[t^0,t^0+T]} |\sigma^n(t) - \overline{\sigma}(t)| \to 0$ as $n \to \infty$. This convergence and (2) imply that $\|\sigma^n - \overline{\sigma}\|_{C^1([t^0,t^0+T];\mathbb{R})} \to 0$.

Since $d\overline{\sigma}(t)/dt > 0$ on $\left[t^0, t^0+T\right]$, then there is $\overline{\delta} > 0$ such that $d\overline{\sigma}(t)/dt \geq 2\overline{\delta} > 0$ for all $t \in \left[t^0, t^0+T\right]$. This and $\|\sigma^n - \overline{\sigma}\|_{C^1[t^0,t^0+T]} \to 0$ give

$$\frac{d}{dt}\sigma^n(t) \geq \overline{\delta} > 0, \qquad t \in \left[t^0, t^0+T\right], \quad n > n_1. \tag{9}$$

Using the definition of α (see (7)) and the convergence $\omega^n \to \overline{\omega}$ as $n \to \infty$ we only need to show that $\max_{\tau\in[0,T]}\left|(\sigma^n)^{-1}(\tau) - (\overline{\sigma})^{-1}(\tau)\right| \to 0$ as $n \to \infty$. Let us denote $\gamma^n(s) = (\sigma^n)^{-1}(s)$, $\overline{\gamma}(s) = (\overline{\sigma})^{-1}(s)$. Assume opposite, i.e. $\gamma^n(s)$ does not converge to $\overline{\gamma}(s)$ uniformly on some $\left[s^0, s^0+S\right]$. Hence there exists $\varepsilon_0 > 0$ such that for all $N \in \mathbb{N}$ there exist $n_N \geq N$ and $s_{n_N} \in \left[s^0, s^0+S\right]$ such that $|\gamma^n(s_{n_N}) - \overline{\gamma}(s_{n_N})| \geq \varepsilon_0$. Considering $N = 1, 2, \dots$ we get two sequences $\{n_k\}_{k=1}^{\infty}$ and $\{s_{n_k}\}_{k=1}^{\infty} \subset \left[s^0, s^0+S\right]$ such that

$$\left|\gamma^{n_k}(s_{n_k}) - \overline{\gamma}(s_{n_k})\right| \geq \varepsilon_0. \tag{10}$$

Since $\left[s^0, s^0+S\right]$ is compact we have $\widehat{s} \in \left[s^0, s^0+S\right]$ and a subsequence again denoted by $\{n_k\}_{k=1}^{\infty}$ such that $s_{n_k} \to \widehat{s} \in \left[s^0, s^0+S\right]$. We can write

$$\gamma^{n_k}(s_{n_k}) - \overline{\gamma}(s_{n_k}) = \left(\gamma^{n_k}(s_{n_k}) - \gamma^{n_k}(\widehat{s})\right) + \left(\gamma^{n_k}(\widehat{s}) - \overline{\gamma}(\widehat{s})\right) + \left(\overline{\gamma}(\widehat{s}) - \overline{\gamma}(s_{n_k})\right).$$

The last term vanishes due the continuity of $\overline{\gamma}$, the second one due to the point-wise convergence $\gamma^n(s) \to \overline{\gamma}(s)$ for all $s \in \left[s^0, s^0+S\right]$. Hence the only possibility to satisfy (10) is that there is an integer k_1 such that for all $k \geq k_1$ one has

$$\left|\gamma^{n_k}(s_{n_k}) - \gamma^{n_k}(\widehat{s})\right| > \frac{\varepsilon_0}{2}.$$

The last property together with $s_{n_k} \to \widehat{s}$ and differentiability of all γ^{n_k} imply that the derivatives $d\gamma^{n_k}/ds$ are unbounded in a neighborhood of $\widehat{s}$. This contradicts the property $d\sigma^n(t)/dt \geq \overline{\delta} > 0$ (see (9)) since $d\gamma^n(s)/ds = 1/(d\sigma^n(t)/dt)$. $\square$

Now, combining Lemmata 2.1 and 2.4 (and the condition $\mu\widetilde{\eta} < 1/2$) we can formulate the first result on the continuous dependence of the time transformation on initial data.

Theorem 2.5.

Let f be a continuous function, Lipschitz with respect to the second and third coordinates and G be Lipschitz and $|G(y)| \leq \mu\widetilde{\eta} < 1/2$ for all $y \in \mathbb{R}^m$. Consider a sequence $\left\{\left(g^n; \eta^{0,n}\right)\right\}$ such that $\left\|\left(g^n; \eta^{0,n}\right) - \left(\overline{g}; \overline{\eta}^0\right)\right\|_{C^1([t^0-h,t^0];\mathbb{R}^m)\times\mathbb{R}} \to 0$ and a sequence $\{\omega^n\}$ such that $\|\omega^n - \overline{\omega}\|_{C^1[t^0-h,t^0]} \to 0$ as $n \to \infty$. Then the time transformation gives the sequence of the corresponding solutions $\{(z^n, \chi^n, \alpha^n)\}$ of the constant delay system (5) & (6) (see (8)) such that for any $S > 0$ one has

$$\max_{s\in[s^0,s^0+S]}\left\{\|z^n(s) - \overline{z}(s)\| + |\chi^n(s) - \overline{\chi}(s)| + |\alpha^n(s) - \overline{\alpha}(s)|\right\} \to 0 \qquad \text{as} \quad n \to \infty,$$

and $\dot{\alpha}^n(s) > 0$ for $s \in \left[s^0, s^0+S\right]$.

Remark 2.6.

We should notice that (5) & (6) is the system of coupled differential and algebraic equations. It is necessary to comment on how to solve it. The way is different from the one of [2, 3] since we have no given lag function (c.f. [3, Section 2.1]). Using (6), we write for $s \in [0, h]$ (and then continue step by step with the step h): $\alpha(s) = \chi(s) + \omega(s-h)$. Then we substitute it into the differential equation for χ in (5) to get $\dot{\chi}(s) = -\mu(\chi(s) - \widetilde{\eta})(\dot{\chi}(s) + \dot{\omega}(s-h)) + G(z(s))(\dot{\chi}(s) + \dot{\omega}(s-h))$. Hence $\dot{\chi}(s)\left[1 + \mu(\chi(s) - \widetilde{\eta}) - G(z(s))\right] = \left\{-\mu(\chi(s) - \widetilde{\eta}) + G(z(s))\right\}\dot{\omega}(s-h)$. We remind that the assumption $2\mu\widetilde{\eta} < 1$ implies $|\mu(\chi(s) - \widetilde{\eta}) - G(z(s))| < 1$. It gives

$$\dot{\chi}(s) = \left\{-\mu(\chi(s) - \widetilde{\eta}) + G(z(s))\right\}\dot{\omega}(s-h)\left[1 + \mu(\chi(s) - \widetilde{\eta}) - G(z(s))\right]^{-1}. \tag{11}$$

Now to rewrite the first equation in (5) we use again $\alpha(s) = \chi(s) + \omega(s-h)$ (and $\dot\alpha(s) = \dot\chi(s) + \dot\omega(s-h)$), substitute it in (5) and use (11). It gives

$$\begin{aligned}\dot z(s) &= f\big(\alpha(s), z(s), z(s-h)\big)\,\dot\alpha(s) = f\big(\chi(s)+\omega(s-h), z(s), z(s-h)\big)(\dot\chi(s)+\dot\omega(s-h))\\ &= f\big(\chi(s)+\omega(s-h), z(s), z(s-h)\big)\left(\frac{-\mu(\chi(s)-\widetilde\eta)+G(z(s))}{1+\mu(\chi(s)-\widetilde\eta)-G(z(s))}+1\right)\cdot\dot\omega(s-h)\\ &= f\big(\chi(s)+\omega(s-h), z(s), z(s-h)\big)\big[1+\mu(\chi(s)-\widetilde\eta)-G(z(s))\big]^{-1}\cdot\dot\omega(s-h).\end{aligned}$$

Hence (5) & (6) can be rewritten for $s\in[0,h]$ (the first step) as

$$\begin{aligned}\dot z(s) &= f\big(\chi(s)+\omega(s-h), z(s), z(s-h)\big)\big[1+\mu(\chi(s)-\widetilde\eta)-G(z(s))\big]^{-1}\cdot\dot\omega(s-h), \qquad z(s)=g(\omega(s)), \qquad s\in[0,h],\\ \dot\chi(s) &= \big\{-\mu(\chi(s)-\widetilde\eta)+G(z(s))\big\}\,\dot\omega(s-h)\big[1+\mu(\chi(s)-\widetilde\eta)-G(z(s))\big]^{-1}, \qquad \chi(0)=\eta^0.\end{aligned}$$

It is easy to see that the last system gives solution $(z(s),\chi(s))$ for $s\in[0,h]$. Then the time transformation α is found by $\alpha(s)=\chi(s)+\omega(s-h)$. In general it reads as

$$\begin{aligned}\dot z(s) &= f\big(\chi(s)+\alpha(s-h), z(s), z(s-h)\big)\big[1+\mu(\chi(s)-\widetilde\eta)-G(z(s))\big]^{-1}\cdot\dot\alpha(s-h), \quad s\ge s^0,\\ &\qquad z(s)=g(\omega(s)), \quad s^0-h\le s\le s^0,\\ \dot\chi(s) &= \big\{-\mu(\chi(s)-\widetilde\eta)+G(z(s))\big\}\big[1+\mu(\chi(s)-\widetilde\eta)-G(z(s))\big]^{-1}\cdot\alpha(s-h), \quad s\ge s^0, \quad \chi\big(s^0\big)=\eta^0,\end{aligned}$$

and it can be solved step by step. Notice, that system (5) & (6) is solved directly, without any references to system (1)–(3) (without using any solution (y,η) of (1)–(3) and without using σ in (4)) since (6) is used instead of (4). Rule (6) contains rule (4), and it is formulated in new time s.

3. Connection between asymptotic properties of systems (1)–(3) and (5) & (6)

In this section we discuss how to determine if some qualitative properties of solutions of the initial state-dependent delay system (1)–(3) survive the time transformation i.e. still be valid for the corresponding solutions of constant delay system (5) & (6). We are also interested to connect the known properties of solutions of (5) & (6) with the ones of (1)–(3).

Let us start with the discussion of the property of the (partial) exponential stability. For the simplicity of presentation we assume that the function $(\overline y(t)\equiv 0; \overline\eta(t))$ is a solution of (1)–(3). Hence, by (8), $(\overline z(s)\equiv 0; \overline\chi(s); \overline\alpha(s))$ will be also a solution of (5) & (6).

Adopting to our case the definition of partial stability (stability with respect to part of the variables) from [15, p. 251] we remind that the solution $(\overline y(t)\equiv 0; \overline\eta(t))$ of (1)–(3) is exponentially y-stable if there exist constants $k_1, k_2>0$ and $k_3>0$ such that $\|y(t)\|\le k_1 e^{-k_2(t-t^0)}\|y_{t^0}\|_{C([-h,0];\mathbb{R}^m)}$ for all $t\ge t^0$ and all solutions satisfying $\|y_{t^0}\|<k_3$. We remind that η-coordinate is bounded ($\eta(\cdot)\in[0,2\widetilde\eta]$). Similarly, we define exponentially z-stable solution $(\overline z(s)\equiv 0; \overline\chi(s); \overline\alpha(s))$ of (5) & (6). Assume one has

$$\|z(s)\|\le D_0 e^{-D_1(s-s^0)}\|z_{s^0}\|_{C([-h,0];\mathbb{R}^m)}, \qquad s\ge s^0, \quad D_0, D_1>0.$$

Hence, by (8), we get

$$\|y(t)\|\le D_0 e^{-D_1(\alpha^{-1}(t)-s^0)}\|z_{s^0}\|_{C([-h,0];\mathbb{R}^m)} = D_0 e^{-D_2(t-t^0)}\|z_{s^0}\|_{C([-h,0];\mathbb{R}^m)} e^{D_2(t-t^0)-D_1(\alpha^{-1}(t)-s^0)}.$$

It is easy to see that if (and only if) $e^{D_2 t - D_1\alpha^{-1}(t)}$ is bounded, then we have

$$\|y(t)\|\le D_3 e^{-D_2(t-t^0)}\|z_{s^0}\|_{C([-h,0];\mathbb{R}^m)}.$$

The above considerations show that the exponential estimate for the solution $z(s)$ implies the exponential estimate for the solution $y(t)$ provided there are positive constants D_1, D_2 such that $D_2 t - D_1 \alpha^{-1}(t) \le D_3$ for all $t \ge t^0$ and some $D_3 \in \mathbb{R}$. Similar estimates give the inverse implication, i.e. the exponential estimate for the solution $y(t)$ implies the exponential estimate for the solution $z(s)$ provided there are positive constants C_1, C_2 such that $C_2 s - C_1 \alpha(s) \le C_3$ for all $s \ge s^0$ and some $C_3 \in \mathbb{R}$. Since by (8), $t = \alpha(s)$, we arrive to the following definition.

Definition 3.1.

We say that "s-time" and "t-time" are *equivalent* if there are constants $A_1, A_2 > 0$, $B_1, B_2 \in \mathbb{R}$ such that $A_1 t + B_1 \le s \le A_2 t + B_2$.

Remark 3.2.

It is evident that in this case we also have $s/A_2 - B_2/A_2 \le t \le s/A_1 - B_1/A_1$.

We saw that in the case of the equivalent "s-time" and "t-time" the exponential estimate survives under the time transformation. Another consequence of the equivalence is that $t \to +\infty$ if and only if $s \to +\infty$, which is clearly important for the study of long-time asymptotic behavior of solutions.

The last result suggests to study the notion of time-equivalence in detail. Let us try to find if in our case we have the equivalence. The rules (7) and (4) show that we need to analyse the function σ^{-1}. Using the property $\sigma(t) \ge t - h$ (bounded delay) and invertibility of σ, we get $\sigma^{-1}(\tau) \le \tau + h$. Hence, by (7), one has $\alpha(s) = \sigma^{-1}(\alpha(s-h)) \le \alpha(s-h) + h$. In particular, $\alpha(h) \le \alpha(0) + h$, $\alpha(2h) \le \alpha(h) + h \le \alpha(0) + 2h$, etc. Hence the property $\alpha(kh) \le \alpha(0) + kh$ and the strict monotonicity of α give the following estimate:

$$\alpha(s) \le \alpha(0) + h + s. \tag{12}$$

Since $\alpha(s) = t$, estimate (12) means the lower bound in Definition 3.1 and the upper bound in Remark 3.2 (with $A_1 = 1$, $B_1 = -(\alpha(0) + h)$).

The complementary bounds in Definition 3.1 and Remark 3.2 are less obvious. For the moment we do not claim that it is true in general case, but present an additional assumption which guarantees the bounds.

Let us assume that the value of delay is bounded from below by a positive constant, say $h_1 > 0$. More precisely, $\eta(t) \ge h_1 \in (0, \widetilde{\eta}]$ for all $t \ge 0$. A sufficient condition for the last property is $|G(y)| \le \mu|\widetilde{\eta} - h_1|$ for all $y \in \mathbb{R}^m$. Under the above condition we have $\sigma^{-1}(\tau) \ge \tau + h_1$. By (7), one has $\alpha(s) = \sigma^{-1}(\alpha(s-h)) \ge \alpha(s-h) + h_1$.

In particular, $\alpha(h) \ge \alpha(0) + h_1$, $\alpha(2h) \ge \alpha(h) + h_1 \ge \alpha(0) + 2h_1$, etc. Hence the property $\alpha(kh) \ge \alpha(0) + kh_1$ and the strict monotonicity of α give the following estimate:

$$\alpha(s) \ge \alpha(0) - h_1 + \frac{h_1}{h} s. \tag{13}$$

Combining (12) and (13) we conclude that "s-time" and "t-time" are equivalent (with $A_1 = 1$, $B_1 = -(\alpha(0) + h)$, $A_2 = h/h_1$, $B_2 = -(\alpha(0) - h_1)h/h_1$ in Definition 3.1 and Remark 3.2). Exactly the same arguments give the following lemma (for the simplicity we put $s^0 = 0$).

Lemma 3.3.

1) *Assume that along a solution of* (1)–(3) *one has $\eta(t) \le h_2 \le h$, i.e. delay is bounded ($\sigma(t) \ge t - h_2$). Then the corresponding time transformation satisfies $\alpha(s) \le \alpha(0) + h_2 + sh_2/h$, for all $s \ge 0$.*
2) *Assume that along a solution of* (1)–(3) *one has $\eta(t) \ge h_1 > 0$. Then the corresponding time transformation satisfies $\alpha(s) \ge \alpha(0) - h_1 + sh_1/h$, for all $s \ge 0$.*

Remark 3.4.

Both assumptions of Lemma 3.3 are satisfied, for example, provided $|G(y)| \le \mu|\widetilde{\eta} - h_1|$ for all $y \in \mathbb{R}^m$ and $h_1 \in (0, \widetilde{\eta}]$. This is a case of the equivalence of "s-time" and "t-time".

Having the equivalence proved we can use it to compare the asymptotic behavior of the corresponding dynamical systems (processes), constructed by solutions of systems before and after the time transformations. The previous considerations lead to the following

Corollary 3.5.

Let f be continuous function, Lipschitz with respect to the second and third coordinates and G be Lipschitz and $|G(y)| \leq \mu|\widetilde{\eta} - h_1|$ for all $y \in \mathbb{R}^m$ and some $h_1 \in (0, \widetilde{\eta}]$. Let us fix any exponentially y-decaying solution (y, η) of (1)–(3), i.e. $\|y(t)\| \leq k_1 e^{-k_2(t-t^0)} \|y_{t^0}\|_{C([-h,0];\mathbb{R}^m)}$ for all $t \geq t^0$ and fix any $\omega \in C^1\left(\left[s^0 - h, s^0\right]; \mathbb{R}\right)$ with positive derivative and such that $\omega\left(s^0 - h\right) = \omega\left(s^0\right) - \eta^0$. Then the corresponding solution (z, χ, α) of (5) & (6) is exponentially z-decaying.

Let us consider an autonomous case of (1), i.e. the system (cf. (1)–(3))

$$\dot{y}(t) = f^a\left(y(t), y(t - \eta(t))\right), \qquad \dot{\eta}(t) = -\mu(\eta(t) - \widetilde{\eta}) + G(y(t)), \qquad t > 0, \tag{14}$$

with the initial data

$$y(t) = g(t), \quad t \in [-h, 0], \qquad \eta(0) = \eta^0. \tag{15}$$

We can restrict our study (using Lemma 2.1) to the set of initial data

$$X_{f^a} = \left\{\left(g, \eta^0\right) : \dot{g}(0) = f^a\left(g(0), g\left(-\eta^0\right)\right)\right\} \subset C^1([-h, 0]; \mathbb{R}^m) \times [0, h].$$

The set X_{f^a} is an analog to the *solution manifold* used in [16] (see also [9]). We notice that the reason for this restriction is to have C^1 smoothness of solution at zero, i.e. $\dot{y}(0-) = \dot{g}(0-) = \dot{y}(0+) = f^a\left(g(0), g\left(-\eta^0\right)\right)$. It is easy to see that X_{f^a} is invariant.

We define the evolution operator $S^a(t)\colon X_{f^a} \to X_{f^a}$, associated to the system (14)–(15), by the formula $S^a(t)\left(g; \eta^0\right) = (y_t; \eta(t))$, where $(y; \eta)$ is the unique solution of (14)–(15). It is easy to see that under our assumptions the pair (S^a, X_{f^a}) constitutes a dynamical semiflow (in other words, the IVP (14)–(15) is well-posed in X_{f^a}). For more definitions and details on dynamical systems (semiflows) see e.g. [5, 8, 14].

To discuss the properties of solutions to the *non-autonomous* system (5) & (6), let us remind the following definition from [4, pp. 112–119]. Let E be a Banach space. Consider the two-parameter family of maps $\{U(t, \tau)\}$, $U(t, \tau)\colon E \to E$, parameters $\tau \in \mathbb{R}$, $t \geq \tau$.

Definition 3.6 ([4, p. 113]).

A family of maps $\{U(t, \tau)\}$ is called a *process* on E if

(i) $U(\tau, \tau) = I =$ identity,

(ii) $U(t, s) \circ U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau \in \mathbb{R}$.

Since (5) & (6) has constant delay h we have no need to restrict our study to a solution manifold. We define

$$E = C([-h, 0]; \mathbb{R}^m) \times [0, h] \times \left\{\omega \in C^1[-h, 0] : \dot{\omega}(\cdot) > 0,\ \omega(-h) = \omega(0) - \eta^0;\ \dot{\omega}(0)\left[1 + \mu\left(\eta^0 - \widetilde{\eta}\right) - G(\varphi(0))\right] = \dot{\omega}(-h)\right\}$$

and define $U\left(s, s^0\right)\left(\varphi; \eta^0; \omega\right) = (z_s; \chi(s); \alpha_s)$, where $(z; \chi; \alpha)$ is the unique solution of (5) & (6) with initial data $\left(\varphi; \eta^0; \omega\right)$, here $z_{s^0} = \varphi$.

We notice that to come back to the system (14)–(15), using a particular solution of (5) & (6), one restores function g as follows: $g(t) = z_{s^0}(\alpha^{-1}(t))$ for $t \in \left[t^0 - h, t^0\right]$.

Let us continue to discuss which asymptotic properties of (1)–(3) survive the time transformation i.e. still be valid for the corresponding solutions of constant delay system (5) & (6).

One of the important properties of dynamical systems and processes are boundedness of solutions and their compactness (asymptotic compactness) see e.g. [5, 8, 14]. Below we always assume that $t \to +\infty$ if and only if $s \to +\infty$, which is true, for example, in the case of the equivalence of t-time and s-time. One can easily see, by (8), that $\|y(t)\| \le C$, $t \ge t_1$, is equivalent to $\|z(s)\| \le C$, $s \ge s_1$. Hence, the existence of a bounded absorbing set for z_s-coordinate is equivalent to the existence of a bounded absorbing set for y_t-coordinate, both in the space $C([-h,0];\mathbb{R}^m)$. To go further, let us discuss the following additional assumptions on the time transformation α:

- (A1) there exists $C_{1,\alpha} > 0$ such that for all $s \ge s_1$ we have $\dot\alpha(s) \le C_{1,\alpha}$,
- (A1′) there exists $C_{2,\alpha} > 0$ such that for all $t \ge t_1$ we have $\dot\alpha^{-1}(t) \le C_{2,\alpha}$,
- (A2) α is uniformly continuous on $[s_1, +\infty)$,
- (A2′) α^{-1} is uniformly continuous on $[t_1, +\infty)$,
- (A3) $\dot\alpha$ is uniformly continuous on $[s_1, +\infty)$,
- (A3′) $\dot\alpha^{-1}$ is uniformly continuous on $[t_1, +\infty)$.

Using (8), we have $\dot z(s) = \dot y(\alpha(s))\dot\alpha(s)$. Hence, the existence of a bounded absorbing set for the y_t-coordinate in the space $C^1([-h,0];\mathbb{R}^m)$ implies the existence of a bounded absorbing set for the z_s-coordinate, provided (A1) is satisfied. The inverse implication is valid provided (A1′) is satisfied. For the system (1)–(3) the existence of a bounded absorbing set means it is dissipative (for more details on this property see e.g. [5, 8, 14]). Let us now assume that for $t \ge t_1$ the y_t-coordinate belongs to a (pre-) compact set in the space $C([-h,0];\mathbb{R}^m)$. By the Arzelà–Ascoli theorem, the family $\{y_t\}_{t\ge t_1}$ is uniformly bounded and equicontinuous. Using (8), we have $|z(s^1) - z(s^2)| = |y(\alpha(s^1)) - y(\alpha(s^2))|$. This and the above discussion show that for $s \ge s_1$ the z_s-coordinate belongs to a (pre-) compact set in the space $C([-h,0];\mathbb{R}^m)$, provided (A2) is satisfied. The inverse implication is valid provided (A2′) is satisfied. The similar considerations in $C^1([-h,0];\mathbb{R}^m)$ need the estimate $|\dot z(s^1) - \dot z(s^2)| = |\dot y(\alpha(s^1))\dot\alpha(s^1) - \dot y(\alpha(s^2))\dot\alpha(s^2)| \le |\dot y(\alpha(s^1)) - \dot y(\alpha(s^2))||\dot\alpha(s^1)| + |\dot y(\alpha(s^2))||\dot\alpha(s^1) - \dot\alpha(s^2)|$. We see that if for $t \ge t_1$ the y_t-coordinate belongs to a (pre-) compact set in the space $C^1([-h,0];\mathbb{R}^m)$, then for $s \ge s_1$ the z_s-coordinate belongs to a (pre-) compact set in the space $C^1([-h,0];\mathbb{R}^m)$, provided (A1)–(A3) are satisfied. The inverse implication is valid provided (A1′)–(A3′) are satisfied. In particular, we have shown that assumptions (A1)–(A3) and (A1′)–(A3′) connect asymptotic properties of the dynamical system $(S^a(t), X_{f^a})$ and the process $U(s,\tau)\colon E \to E$.

Remark 3.7.

- i) Discussing assumptions (A1)–(A3), one could think that the family $\{\alpha_s\}_{s\ge s_1}$ may belong to a (pre-) compact set in the space $C([-h,0];\mathbb{R})$ or even $C^1([-h,0];\mathbb{R})$, but it is never true since $\alpha(s) = t$ is time, which is naturally unbounded.
- ii) One can see that (A1) implies (A2), but (A2) $\not\Rightarrow$ (A1), similarly (A1′) $\Rightarrow$ (A2′), but (A2′) $\not\Rightarrow$ (A1′).
- iii) We notice that (A1) gives $\alpha(s) \le C_{1,\alpha}s + k_1$ and similarly (A1′) $\Rightarrow \alpha^{-1}(t) \le C_{2,\alpha}t + k_2$. Using Definition 3.1, we see that (A1), (A1′) imply the equivalence of "s-time" and "t-time".

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