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Realizing stock market crashes: stochastic cusp catastrophe model of returns under time-varying volatility

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This paper develops a two-step estimation methodology that allows us to apply catastrophe theory to stock market returns with time-varying volatility and to model stock market crashes. In the first step, we utilize high-frequency data to estimate daily realized volatility from returns. Then, we use stochastic cusp catastrophe theory on data normalized by the estimated volatility in the second step to study possible discontinuities in the markets. We support our methodology through simulations in which we discuss the importance of stochastic noise and volatility in a deterministic cusp catastrophe model. The methodology is empirically tested on nearly 27 years of US stock market returns covering several important recessions and crisis periods. While we find that the stock markets showed signs of bifurcation in the first half of the period, catastrophe theory was not able to confirm this behaviour in the second half. Translating the results, we find that the US stock market's downturns were more likely to be driven by the endogenous market forces during the first half of the studied period, while during the second half of the period, exogenous forces seem to be driving the market's instability. The results suggest that the proposed methodology provides an important shift in the application of catastrophe theory to stock markets.

Keywords: Stochastic cusp catastrophe model; Realized volatility; Bifurcations; Stock market crash

1. Introduction

Financial inefficiencies such as under- or over-reactions to information as the causes of extreme events in the stock markets attract researchers across all fields of economics. In one recent contribution, Levy (2008) highlighted the endogeneity of large market crashes as a result of the natural conformity of investors with their peers in a heterogeneity investor population. The stronger the conformity and homogeneity across the market, the more likely the existence of multiple equilibria in the market, which is a prerequisite for a market crash to occur. Gennotte and Leland (1990) presented a model that shares the same notions as Levy (2008) in terms of the effect of small changes when the market is close to a crash point as well as the volatility amplification signalling. In another work, Levy *et al.* (1994) considered the signals produced by dividend yields and assessed the effect of computer trading, which is blamed for making the market more homogeneous and thus more conducive to a crash. Kleidon (1995) summarized and compared several older models from the 1980s and 1990s, and Barlevy and Veronesi (2003) proposed a model based on rational but uninformed traders who can unreasonably panic. Again with this approach, abrupt declines in stock prices can occur without any real change in the underlying fundamentals. Lux (1995) linked the phenomena of market crashes to the process of phase transition from thermodynamics and modelled the emergence of bubbles and crashes as a result of herd behaviour among heterogeneous traders in speculative markets. Finally, a strand of literature documenting precursory patterns and log-periodic signatures years before the largest crashes in the modern history suggested that crashes have an endogenous origin in 'crowd' behaviour and through the interactions of many agents (Sornette and Johansen 1998, Johansen *et al.* 2000, Sornette 2002, 2004). In contrast to many commonly shared beliefs, Didier Sornette and his colleagues argued that exogenous shocks can only serve as triggers and not as the direct causes of crashes and that large crashes are 'outliers'.

Catastrophe theory provides a very different theoretical framework to understand how even small shifts in the

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speculative part of the market can trigger a sudden, discontinuous effect on prices. Catastrophe theory was proposed by French mathematician Thom (1975) with the aim of shedding some light on the 'mystery' of biological morphogenesis. Despite its mathematical virtues, the theory was promptly heavily criticized by Zahler and Sussmann (1977) and Sussmann and Zahler (1978a,b) for its excessive utilization of qualitative approaches, the improper usage of certain statistical methods and for violations of necessary mathematical assumptions in many of its applications. Due to these criticisms, the intellectual bubble and the heyday of the cusp catastrophe approach declined rapidly after the 1970s, although the theory was defended by some researchers, e.g. by Boutot (1993) and the extensive, gradually updated work of Arnold (2004). Nonetheless, the 'fatal' criticism was ridiculed by Rosser (2007, p. 3275 and 3257), who stated that *the baby of catastrophe theory was largely thrown out with the bathwater of its inappropriate applications*, and the author suggested that *economists should re-evaluate the former fad and move it to a more proper valuation*.

The application of catastrophe theory in the social sciences has not been as extensive as in the natural sciences, although it was utilized early in its existence. Zeeman's (1974) cooperation with Thom and his own popularization of the theory through the use of nontechnical examples (Zeeman 1975, Zeeman 1976) led to the development of many applications in the fields of economics, psychology, sociology, political studies and others. Zeeman (1974) also proposed the application of the cusp catastrophe model to stock markets. Translating seven qualitative hypotheses about stock exchanges to the mathematical terminology of catastrophe theory produced one of the first heterogeneous agent models for two main types of investors: fundamentalists and chartists. Heterogeneity and the interactions between these two distinct types of agents attracted wider attention in the behavioural finance literature. Fundamentalists base their expectations about future asset prices on their beliefs about fundamental and economic factors such as dividends, earnings and the macroeconomic environment. In contrast, chartists do not consider fundamentals in their trading strategies at all. Their expectations about future asset prices are based on finding historical patterns in prices. While Zeeman's work was only one qualitative description of observed bull and bear markets, it contained a number of important behavioural elements that were later used in the large volume of literature that focused on heterogeneous agent modelling.† Today, the statistical theory is well developed, and parameterized cusp catastrophe models can be evaluated quantitatively based on data.

The biggest difficulty in the application of catastrophe theory arises from the fact that it stems from deterministic systems. Thus, it is difficult to apply it directly to systems that are subject to random influences, which are common in the behavioural sciences. Cobb and Watson (1980), Cobb (1981) and Cobb and Zacks (1985) provided the necessary bridge and took catastrophe theory from determinism to stochastic systems. While this was an important shift, there are further complications in the theory's application to stock market data. The main restriction of Cobb's method of estimation was the requirement of a constant variance, which forces researchers to assume that the volatility of the stock markets (as the standard deviation of the returns) is constant. Quantitative verification of Zeeman's (1974) hypotheses about the application of the theory to stock market crashes was pioneered by Barunik and Vosvrda (2009), where we fit the cusp model to two separate, large stock market crashes. However, the successful application of Barunik and Vosvrda (2009) brought only preliminary results in a restricted environment. Application of the cusp catastrophe theory on stock market data deserves much more attention. In the current paper, we propose an improved method of application that we believe brings us closer to an answer regarding whether cusp catastrophe theory is capable of explaining stock market crashes.

Time-varying volatility has become an important stylized fact for stock market data, and researchers have recognized that it is an important feature of any modelling strategy. One of the most successful early works of Engle (1982) and Bollerslev (1986) proposed including volatility as a time-varying process in a (generalized) autoregressive conditional heteroskedasticity framework. From that beginning, many models have been developed in the literature to improve the original frameworks. As early as the late 1990s, high frequency data became available to researchers, and this led to another important shift in volatility modelling — realized volatility. A very simple, intuitive approach to compute daily volatility using the sum of squared high-frequency returns was formalized by Andersen *et al.* (2003) and Barndorff-Nielsen and Shephard (2004). While the volatility literature is immense,‡ several researchers have also studied volatility and stock market crashes. For example, Shaffer (1991) argued that volatility might be the cause of a stock market crash. In contrast, Levy (2008) argued that volatility increases before a crash, even when no dramatic information is revealed.

In this study, we utilize the availability of high-frequency data and the popular realized volatility approach to propose a two-step method of estimation that overcomes the difficulties in the application of cusp catastrophe theory to stock market data. Using realized volatility, we estimate stock market returns' volatility, and then we apply the stochastic cusp catastrophe model on volatility-adjusted returns with constant variance. This approach is motivated by the confirmed bimodal distributions of such standardized data in some periods, and it allows us to study whether stock markets are driven into catastrophe endogenously or whether it is simply an effect of volatility. We run simulations that provide strong support for the methodology. The simulations also illustrate the importance of stochastic noise and volatility in the deterministic cusp model.

Using a unique data-set covering almost 27 years of the US stock market evolution, we empirically test the stochastic cusp catastrophe model in a time-varying volatility environment. Moreover, we develop a rolling regression approach to study the dynamics of the model's parameters over a long

†For a recent survey of heterogeneous agent models, see Hommes (2006). A special issue on heterogeneous interacting agents in financial markets edited by Lux and Marchesi (2002) also provides interesting contributions.

‡Andersen *et al.* (2004) provide a very useful and complete review of the methodologies.

period, covering several important recessions and crises. This approach allows us to localize the bifurcation periods.

We need to mention several important works that provided similar results to ours. Creedy and Martin (1993) and Creedy *et al.* (1996) developed a framework for the estimation of non-linear exchange rate models, and they showed that swings in exchange rates can be attributed to bimodality even without the explicit use of catastrophe theory. More recently, Koh *et al.* (2007) proposed using Cardan's discriminant to detect bimodality and confirmed the predictive ability of currency pairs for emerging countries. In our work, we bring new insight to the non-linear phenomena by including time-varying volatility in the modelling strategy.

The paper is organized as follows. The second and the third sections examine the theoretical framework of the stochastic catastrophe theory under time-varying volatility and describe the model's estimation. The fourth section presents the simulations that support our two-step method of estimation, and the fifth section presents the empirical application of the theory on the modelling of stock market crashes. Finally, the last section concludes.

2. Theoretical framework

Catastrophe theory was developed as a deterministic theory for systems that may respond to continuous changes in control variables by a discontinuous change from one equilibrium state to another. A key idea is that the system under study is driven toward an equilibrium state. The behaviour of the dynamical systems under study is completely determined by a so-called potential function, which depends on behavioural and control variables. The behavioural, or state, variable describes the state of the system, while control variables determine the behaviour of the system. The dynamics under catastrophe models can become extremely complex and according to the classification theory of Thom (1975), there are seven different families based on the number of control and dependent variables. We focus on the application of catastrophe theory to model sudden stock market crashes, as qualitatively proposed by Zeeman (1974). In his work, Zeeman used the so-called cusp catastrophe model, which is the simplest specification that gives rise to sudden discontinuities.

2.1. *Deterministic dynamics*

Let us suppose that the process y_t evolves over $t = 1, \ldots, T$ as

$$\mathrm{d}y_t = -\frac{\mathrm{d}V(y_t;\alpha,\beta)}{\mathrm{d}y_t}\mathrm{d}t, \tag{1}$$

where $V(y_t;\alpha,\beta)$ is the potential function describing the dynamics of the state variable y_t controlled by parameters α and β determining the system. When the right-hand side of equation (1) equals zero, $-\mathrm{d}V(y_t;\alpha,\beta)/\mathrm{d}y_t = 0$, the system is in equilibrium. If the system is at a state of non-equilibrium, it will move back to equilibrium where the potential function takes the minimum values with respect to y_t. While the concept of potential function is very general, i.e. it can be a quadratic function yielding equilibrium of a simple flat response surface, one of the most applied potential functions in behavioural sciences, a cusp potential function, is defined as

$$-V(y_t;\alpha,\beta) = -1/4y_t^4 + 1/2\beta y_t^2 + \alpha y_t, \tag{2}$$

with equilibria at

$$-\frac{\mathrm{d}V(y_t;\alpha,\beta)}{\mathrm{d}y_t} = -y_t^3 + \beta y_t + \alpha \tag{3}$$

being equal to zero. The two dimensions of the control space, α and β, further depend on realizations from $i = 1 \ldots, n$ of the independent variables $x_{i,t}$. Thus, it is convenient to think about α and β as functions

$$\alpha_x = \alpha_0 + \alpha_1 x_{1,t} + \ldots + \alpha_n x_{n,t} \tag{4}$$

$$\beta_x = \beta_0 + \beta_1 x_{1,t} + \ldots + \beta_n x_{n,t}. \tag{5}$$

The control functions α_x and β_x are called normal and splitting factors, or asymmetry and bifurcation factors, respectively (Stewart and Peregoy 1983), and they determine the predicted values of y_t given $x_{i,t}$. Therefore, for each combination of values of independent variables, there could be up to three predicted values of the state variable given by roots of

$$-\frac{\mathrm{d}V(y_t;\alpha_x,\beta_x)}{\mathrm{d}y_t} = -y_t^3 + \beta_x y_t + \alpha_x = 0. \tag{6}$$

This equation has one solution if

$$\delta_x = 1/4\alpha_x^2 - 1/27\beta_x^3 \tag{7}$$

is greater than zero, $\delta_x > 0$, and three solutions if $\delta_x < 0$. This construction was first described by the sixteenth-century mathematician Geronimo Cardan and can serve as a statistic for bimodality, one of the catastrophe flags. The set of values for which Cardan's discriminant is equal to zero, $\delta_x = 0$, is the bifurcation set that determines the set of singularity points in the system. In the case of three roots, the central root is called an 'anti-prediction' and is the least probable state of the system. Inside the bifurcation, when $\delta_x < 0$, the surface predicts two possible values of the state variable, which means that in this case, the state variable is bimodal. For an illustration of the deterministic response surface of cusp catastrophe, we borrow from figure 2 in the simulations section, where the deterministic response surface is a smooth pleat.

2.2. *Stochastic dynamics*

Most of the systems in behavioural sciences are subject to noise stemming from measurement errors or the inherent stochastic nature of the system under study. Thus, for real-world applications, it is necessary to add non-deterministic behaviour into the system. Because catastrophe theory was primarily developed to describe deterministic systems, it may not be obvious how to extend the theory to stochastic systems. An important bridge was provided by Cobb and Watson (1980), Cobb (1981) and Cobb and Zacks (1985), who used the Itô stochastic differential equations to establish a link between the potential function of a deterministic catastrophe system and the stationary probability density function of the corresponding stochastic process. This approach in turn led to the definition of a stochastic equilibrium state and bifurcation that was compatible with the deterministic counterpart. Cobb and his colleagues simply added a stochastic Gaussian white noise term to the system

$$\mathrm{d}y_t = -\frac{\mathrm{d}V(y_t;\alpha_x,\beta_x)}{\mathrm{d}y_t}\mathrm{d}t + \sigma_{y_t}\mathrm{d}W_t, \tag{8}$$

where $-\mathrm{d}V(y_t; \alpha_x, \beta_x)/\mathrm{d}y_t$ is the deterministic term, or drift function representing the equilibrium state of the cusp catastrophe, σ_{y_t} is the diffusion function and W_t is a Wiener process. When the diffusion function is constant, $\sigma_{y_t} = \sigma$, and the current measurement scale is not to be nonlinearly transformed, the stochastic potential function is proportional to the deterministic potential function, and the probability distribution function corresponding to the solution from equation (8) converges to a probability distribution function of a limiting stationary stochastic process because the dynamics of y_t are assumed to be much faster than changes in $x_{i,t}$ (Cobb 1981, Cobb and Zacks 1985, Wagenmakers *et al.* 2005). The probability density that describes the distribution of the system's states at any t is then

$$f_s(y|x) = \psi \exp\left(\frac{(-1/4)y^4 + (\beta_x/2)y^2 + \alpha_x y}{\sigma}\right). \quad (9)$$

The constant ψ normalizes the probability distribution function, so its integral over the entire range equals one. As the bifurcation factor β_x changes from negative to positive, the $f_s(y|x)$ changes its shape from unimodal to bimodal. Conversely, α_x causes asymmetry in $f_s(y|x)$.

2.3. *Cusp catastrophe under time-varying volatility*

Stochastic catastrophe theory works only under the assumption that the diffusion function is constant, $\sigma_{y_t} = \sigma$, and the current measurement scale is not to be nonlinearly transformed. While this assumption may be reliable in some applications in the behavioural sciences, it may cause crucial difficulties in others. One of the problematic applications is in modelling stock market crashes because the diffusion function σ, called the volatility of stock market returns, has strong time-varying dynamics, and it clusters over time, which is documented by strong dependence in the squared returns. To illustrate the volatility dynamics, let us borrow the data-set used later in this study. Figure 4 shows the evolution of the S&P 500 stock index returns over almost 27 years and documents how volatility strongly varies over time. One of the possible and very simple solutions in applying cusp catastrophe theory to the stock markets is to consider only a short-time window and to fit the catastrophe model to data where volatility can be assumed to be constant (Barunik and Vosvrda 2009). Although in Barunik and Vosvrda (2009) we were the first to quantitatively apply stochastic catastrophes to explain stock market crashes on localized periods of crashes, this assumption is generally very restrictive.

Here, we propose a more rigorous solution to the problem by utilizing the recently developed concept of realized volatility. This approach allows us to use the previously introduced concepts after estimating the volatility from the returns process consistently, and we are able to estimate the catastrophe model on the process that fulfils the assumptions of the stochastic catastrophe theory. Thus, we assume that stock markets can be described by the cusp catastrophe process subject to time-varying volatility. While this approach represents a great advantage that allows us to apply cusp catastrophe theory to different time periods conveniently, the disadvantage is that the method cannot be generalized to other branches of the behavioural sciences where high-frequency data are not available and therefore realized volatility cannot be computed. Thus, our generalization is mainly restricted to applications on financial data. Still, our main aim is to study stock market crashes, and therefore the advocated approach is very useful in the field of behavioural finance. We now describe the theoretical concept, and in the next sections, we will present the full model and the two-step estimation procedure.

Suppose that the sample path of the corresponding (latent) logarithmic price process p_t is continuous over $t = 1, \ldots, T$ and determined by the stochastic differential equation

$$\mathrm{d}p_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad (10)$$

where μ_t is a drift term, σ_t is the predictable diffusion function or instantaneous volatility, and W_t is a standard Brownian motion. A natural measure of the ex-post return variability over the $[t-h, t]$ time interval, $0 \leq h \leq t \leq T$ is the integrated variance

$$IV_{t,h} = \int_{t-h}^{t} \sigma_\tau^2 \mathrm{d}\tau, \quad (11)$$

which is not directly observed, but as shown by Andersen *et al.* (2003) and Barndorff-Nielsen and Shephard (2004), the corresponding realized volatilities provide its consistent estimates. While it is convenient to work in the continuous time environment, empirical investigations are based on discretely sampled prices, and we are interested in studying h-period continuously compounded discrete-time returns $r_{t,h} = p_t - p_{t-h}$. Andersen *et al.* (2003) and Barndorff-Nielsen and Shephard (2004) showed that daily returns are Gaussian, conditional on an information set generated by the sample paths of μ_t and σ_t, and integrated volatility normalizes the returns as

$$r_{t,h}\left(\int_{t-h}^{t} \sigma_\tau^2 \mathrm{d}\tau\right)^{-1/2} \sim N\left(\int_{t-h}^{t} \mu_\tau \mathrm{d}\tau, 1\right). \quad (12)$$

This result of quadratic variation theory is important to us because we use it to study stochastic cusp catastrophe in an environment where volatility is time varying. In the modern stochastic volatility literature, it is common to assume that stock market returns follow the very general semi-martingale process (as in equation 10), where the drift and volatility functions are predictable and the rest is unpredictable. In the origins of this stream of literature, one of the very first contributions published regarding stochastic volatility by Taylor (1982) assumed that daily returns are the product of a volatility and autoregression process. In our application, we also assume that daily stock market returns are described by a process that is the product of volatility and the cusp catastrophe model.

To formulate the approach, we assume that stock returns normalized by their volatility

$$y_t^* = r_t \left(\int_{t-h}^{t} \sigma_\tau^2 \mathrm{d}\tau\right)^{-1/2} \quad (13)$$

follow a stochastic cusp catastrophe process

$$\mathrm{d}y_t^* = -\frac{\mathrm{d}V(y_t^*; \alpha_x, \beta_x)}{\mathrm{d}y_t^*}\mathrm{d}t + \mathrm{d}W_t. \quad (14)$$

It is important to note the difference between equation (14) and equation (8) because there is no longer any diffusion term in the process. Because the diffusion term of y_t^* is constant and now equal to one, Cobb's results can conveniently be used, and we can use the stationary probability distribution function of

y_t^* for the parameter estimation using the maximum likelihood method.

As noted previously, the integrated volatility is not directly observable. However, the now-popular concept of realized volatility and the availability of high-frequency data provide a simple method to accurately measure integrated volatility, which helps us propose a simple and intuitive method to estimate the cusp catastrophe model on stock market returns under highly dynamic volatility.

3. Estimation

A simple, consistent estimator of the integrated variance under the assumption of no microstructure noise contamination in the price process is provided by the well-known realized variance (Andersen *et al.* 2003, Barndorff-Nielsen and Shephard 2004). The realized variance over $[t-h, t]$, for $0 \leq h \leq t \leq T$, is defined by

$$\widehat{RV}_{t,h} = \sum_{i=1}^{N} r^2_{t-h+\left(\frac{i}{N}\right)h}, \tag{15}$$

where N is the number of observations in $[t-h, t]$, and $r_{t-h+\left(\frac{i}{N}\right)h}$ is $i-$th intraday return in the $[t-h, t]$ interval. $\widehat{RV}_t$ converges in probability to the true integrated variance of the process as $N \to \infty$.

$$\widehat{RV}_{t,h} \xrightarrow{p} \int_{t-h}^{t} \sigma_\tau^2 \mathrm{d}\tau, \tag{16}$$

As observed, the log-prices are contaminated with microstructure noise in the real world, and the literature has developed several estimators. While it is important to consider both jumps and microstructure noise in the data, our main interest is in estimating the catastrophe theory and addressing the question whether it can be used to explain the deterministic portion of stock market returns. Thus, we restrict ourselves to the simplest estimator, which uses sparse sampling to deal with the microstructure noise. The extant literature showed support for this simple estimator; most recently, Liu *et al.* (2012) ran a horse race for the most popular estimators and concluded that when simple realized volatility is computed using five-minute sampling, it is very difficult to outperform.

In the first step, we estimate realized volatility from the stock market returns using five-minute data as proposed by the theory, and we normalize the daily returns to obtain returns with constant volatility. While using the daily returns, $h = 1$, and we henceforth drop h for ease of notation.

$$\tilde{r}_t = r_t \widehat{RV}_t^{-1/2} \tag{17}$$

In the second step, we apply the stochastic cusp catastrophe to model the normalized stock market returns. While the state variable of the cusp is a canonical variable, it is an unknown smooth transformation of the actual state variable of the system. As proposed by Grasman *et al.* (2009), we allow for the first-order approximation to the true, smooth transition allowing the measured $\tilde{r}$ to be a

$$y_t = \omega_0 + \omega_1 \tilde{r}_t, \tag{18}$$

with ω_1 as the first-order coefficient of a polynomial approximation. The independent variables are

$$\alpha_x = \alpha_0 + \alpha_1 x_{1,t} + \ldots + \alpha_n x_{n,t} \tag{19}$$

$$\beta_x = \beta_0 + \beta_1 x_{1,t} + \ldots + \beta_n x_{n,t}, \tag{20}$$

Hence, the statistical estimation problem is to estimate $2n+2$ parameters $\{\omega_0, \omega_1, \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \ldots, \beta_n\}$. We estimate the parameters using the maximum likelihood approach of Cobb and Watson (1980) as augmented by Grasman *et al.* (2009). The negative log-likelihood for a sample of observed values $(x_{1,t}, \ldots, x_{n,t}, y_t)$ for $t = 1, \ldots, T$ is simply the logarithm of the probability distribution function in equation (9).

3.1. Statistical evaluation of the fit

To assess the fit of the cusp catastrophe model to the data, a number of diagnostic tools have been suggested. Stewart and Peregoy (1983) proposed a pseudo-R^2 as a measure of the explained variance. However, a difficulty arises here because for a given set of values of the independent variables, the model may predict multiple values for the dependent variable. Because of bimodal density, the expected value is unlikely to be observed because it is an unstable solution at equilibrium. For this reason, two alternatives for the expected value as the predictive value can be used. The first method chooses the mode of the density closest to the state values, which is known as the delay convention; the second method uses the mode at which the density is highest, which is known as the Maxwell convention. Cobb and Watson (1980) and Stewart and Peregoy (1983) suggested using the delay convention where the variance of the error is defined as the variance of the difference between the estimated states and then using the mode of the distribution that is closest to this value. The pseudo-R^2 is defined as $1 - Var(\epsilon)/Var(y)$, where ϵ is error.

While pseudo-R^2 is problematic due to the nature of the cusp catastrophe model, it should be used in a complementary fashion to other alternatives. To rigorously test the statistical fit of the cusp catastrophe model, we use following steps. First, the cusp fit should be substantially better than multiple linear regression. The cusp fit could be tested by means of a likelihood ratio test, which is asymptotically chi-squared distributed with degrees of freedom equal to the difference in degrees of freedom for two compared models. Second, the ω_1 coefficient should deviate significantly from zero. Otherwise, the y_t in equation (18) would be constant, and the cusp model would not describe the data. Third, the cusp model should show a better fit than the following logistic curve:

$$y_t = \frac{1}{1 + e^{-\alpha_t/\beta_t^2}} + \epsilon_t, \tag{21}$$

for $t = 1, \ldots, T$, where ϵ_t are zero-mean random disturbances. The rationale for choosing to compare the cusp model to this logistic curve is that this function does not possess degenerate points, while it possibly models steep changes in the state variable as a function of changes in the independent variables mimicking the sudden transitions of the cusp. Thus, a comparison of the cusp catastrophe model to the logistic function serves as a good indicator of the presence of bifurcations in the data. While these two models are not nested, Wagenmakers *et al.* (2005) suggested comparing them via information

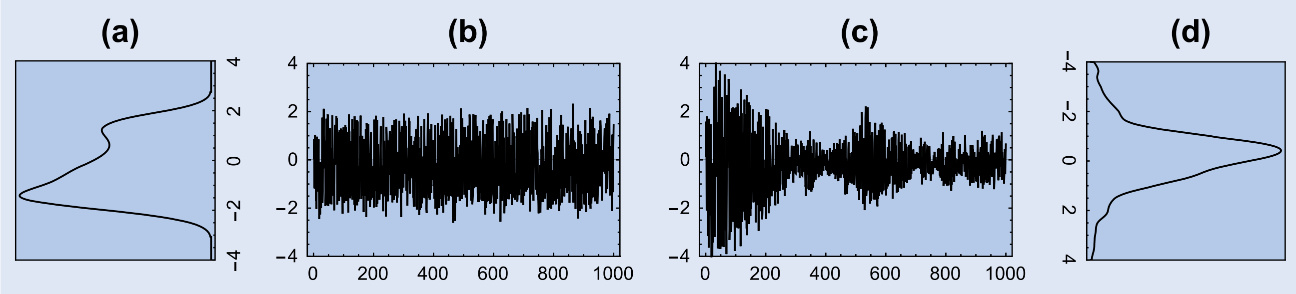

Figure 1. An example of a simulated time series where the cusp surface is subject to noise only, y_t, and noise together with volatility, r_t. (b) simulated returns y_t (a) kernel density estimate of y_t (c) simulated returns r_t contaminated with volatility (d) kernel density estimate of r_t.

criteria, where a stronger Bayesian Information Criterion (BIC) should be required for the decision.

4. Monte Carlo study

To validate our assumptions about the process of generating stock market returns and our two-step estimation procedure, we conduct a Monte Carlo study where we simulate the data from the stochastic cusp catastrophe model, allow for time-varying volatility in the process and estimate the parameters to see whether we can recover the true values.

We simulate the data from the stochastic cusp catastrophe model subject to time-varying volatility as

$$r_t = \sigma_t y_t \tag{22}$$

$$\mathrm{d}\sigma_t^2 = \kappa(\omega - \sigma_t^2)\mathrm{d}t + \gamma \mathrm{d}W_{t,1}, \tag{23}$$

$$\mathrm{d}y_t = (\alpha_t + \beta_t y_t - y_t^3)\mathrm{d}t + \mathrm{d}W_{t,2} \tag{24}$$

where $\mathrm{d}W_{t,1}$ and $\mathrm{d}W_{t,2}$ are standard Brownian motions with zero correlation, $\kappa = 5$, $\omega = 0.04$ and $\gamma = 0.5$. The volatility parameters satisfy Feller's condition $2\kappa\omega \geq \gamma^2$, which keeps the volatility process away from the zero boundary. We set the parameters to values that are reasonable for a stock price, as in Zhang *et al.* (2005).

In the cusp equation, we use two independent variables

$$\alpha_t = \alpha_0 + \alpha_1 x_{t,1} + \alpha_2 x_{t,2} \tag{25}$$

$$\beta_t = \beta_0 + \beta_1 x_{t,1} + \beta_2 x_{t,2} \tag{26}$$

with coefficients $\alpha_2 = \beta_1 = 0$. Hence $x_{t,1}$ drawn from the $U(0, 1)$ distribution drives the asymmetry side, and $x_{t,2}$ drawn from the $U(0, 1)$ distribution drives the bifurcation side of the model. The parameters are set as $\alpha_0 = -2$, $\alpha_1 = 3$, $\beta_0 = -1$ and $\beta_2 = 4$.

In the simulations, we are interested in determining how the cusp catastrophe model performs under time-varying volatility. Thus, we estimate the coefficients on the processes $y_t = r_t/\sigma_t$ and r_t. Figure 1 shows one realization of the simulated returns y_t and r_t. While y_t is the cusp catastrophe subject to noise, r_t is subject to time-varying volatility as well. It is noticeable how time-varying volatility causes the shift from bimodal density to unimodal. More illustrative is figure 2, which shows the cusp catastrophe surface of both processes. While the solution from the deterministic cusp catastrophe is contaminated with noise in the first case, the volatility process in the second case makes it much more difficult to recognize the two states of the system in the bifurcation area. This result causes difficulty in recovering the true parameters.

Table 1 shows the results of the simulation. We simulate the processes 100 times and report the mean and standard deviations from the mean.† The true parameters are easily recovered in the simulations from y_t when the cusp catastrophe is subject to noise only because the mean values are statistically indistinguishable from the true simulated values $\alpha_0 = -2$, $\alpha_1 = 3$, $\beta_0 = -1$ and $\beta_2 = 4$. The fits are reasonable because they explain approximately 60% of the data variation in the noisy environment. Moreover, in the cusp model, we first estimate the full set of parameters $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)$, and then we restrict the parameters $\alpha_2 = \beta_1 = 0$. The estimation easily recovers the true parameters in both cases, while in the unrestricted case, the estimates $\alpha_2 = \beta_1 = 0$ and fits are statistically the same. In comparison, both cusp models perform much better than logistic regression and linear models, which was expected. It is also interesting to note that $\omega_0 = 0$ and $\omega_1 = 1$, which means that the observed data are the true data, and no transformation is needed. These results are important because they confirm that the estimation of the stochastic cusp catastrophe model is valid, and it can be used to quantitatively apply the theory to the data.

The results of the estimation on the r_t process, which is subject to time-varying volatility, reveal that the addition of the volatility process makes it difficult for the maximum likelihood estimation to recover the true parameters. The variances of the estimated parameters are very large, and the means are far away from the true simulated values. Moreover, the fits are statistically weaker, as they explain no more than 38% of the variance in the data. It is also interesting to note that the logistic fit and the linear fit are much closer to the cusp fit.

In conclusion, the simulation results reveal that time-varying volatility in the cusp catastrophe model destroys the ability of the maximum likelihood estimator to recover the cusp potential.

5. Empirical modelling of stock market crashes

Armed with the results from the simulations, we move to the estimation of the cusp catastrophe model on the real-world data

†The distribution of the parameters is Gaussian in both cases, which makes it possible to compare the results within the means and standard deviations.

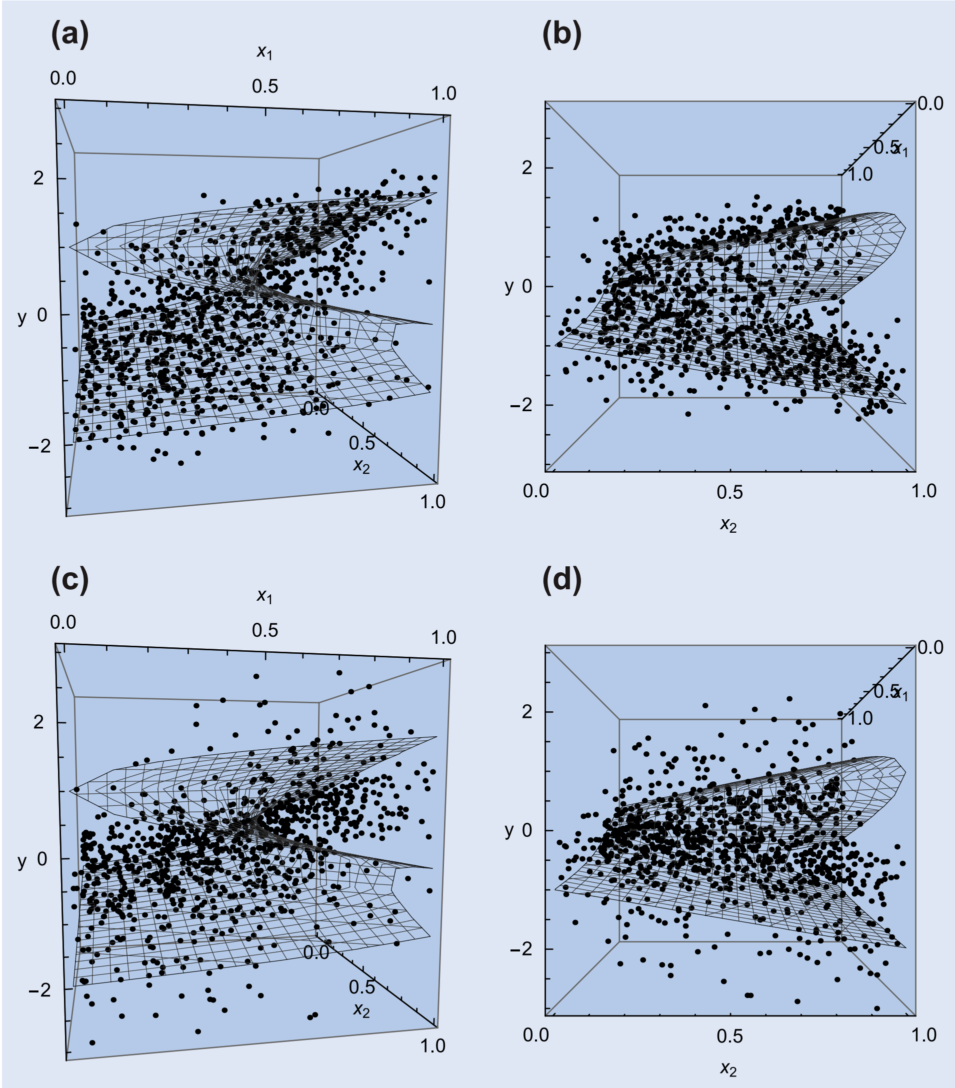

Figure 2. An example of simulated data where the cusp surface is subject to noise only, y_t, and noise with volatility, r_t. Parts (a) and (b) show the cusp deterministic pleat simulated $\{x_1, x_2, y_t\}$ from two different perspectives, and parts (c) and (d) show the cusp deterministic pleat simulated $\{x_1, x_2, r_t\}$ from two different perspectives.

Table 1. Simulation results (a) according to the stochastic cusp catastrophe model and (b) according to the stochastic cusp catastrophe model with process in volatility. The total sample based on 100 random simulations is used, and the sample means and standard deviations (in parentheses) for each value are reported. All figures are rounded to one or three decimal digits.

	(a) Estimates using $y_t = r_t/\sigma_t$				(b) Estimates using r_t			
	Cusp		Linear	Logistic	Cusp		Linear	Logistic
	Unrestricted	Restricted			Unrestricted	Restricted		
α_0	−2.059 (0.143)	−2.066 (0.141)			−1.614 (3.352)	−3.111 (0.531)		
α_1	3.083 (0.195)	3.084 (0.203)			4.355 (1.382)	4.540 (0.690)		
α_2	−0.009 (0.118)				−1.126 (2.121)			
β_0	−1.016 (0.237)	−1.018 (0.187)			−7.088 (3.287)	−5.428 (2.420)		
β_1	0.001 (0.319)				2.394 (1.806)			
β_2	4.003 (0.232)	4.006 (0.223)			5.821 (1.430)	5.544 (1.293)		
ω_0	−0.005 (0.035)	−0.006 (0.025)			0.163 (0.305)	−0.038 (0.053)		
ω_1	1.003 (0.021)	1.003 (0.021) Text			0.539 (0.155)	0.552 (0.154)		
R^2	0.609 (0.022)	0.608 (0.023)	0.397 (0.025)	0.462 (0.027)	0.378 (0.043)	0.379 (0.044)	0.345 (0.034)	0.401 (0.038)
LL	−888.6 (24.3)	−889.4 (24.1)	−1323.3 (21.4)	−1266.4 (23.5)	−1109.4 (52.7)	−1117.6 (57.6)	−1471.7 (212.6)	−1426.5 (211.4)
AIC	1793.1 (48.5)	1790.9 (48.3)	2654.7 (42.7)	2546.9 (47.0)	2234.8 (105.5)	2247.3 (115.3)	2951.3 (425.1)	2867.0 (422.9)
BIC	1832.4 (48.5)	1820.3 (48.3)	2674.3 (42.7)	2581.2 (47.0)	2274.0 (105.5)	2276.7 (115.3)	2970.9 (425.1)	2901.4 (422.9)

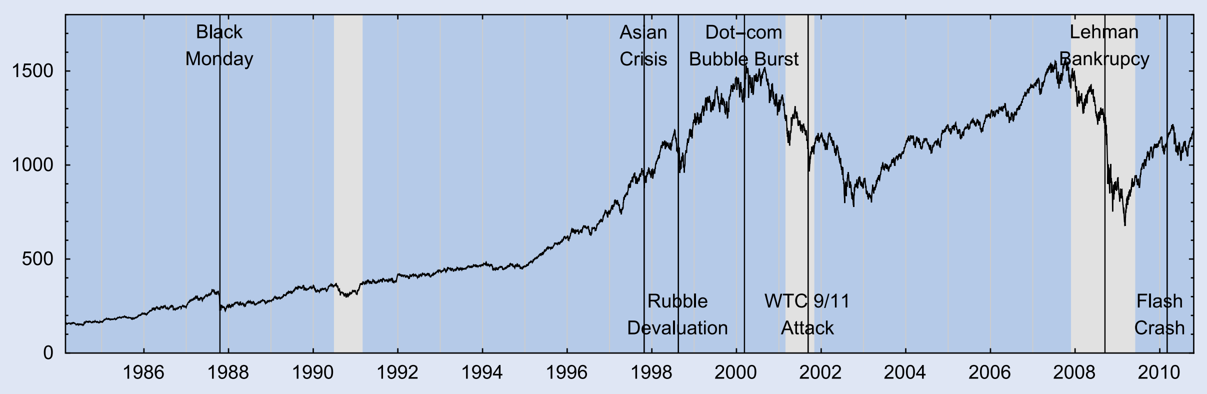

Figure 3. S&P 500 price data. The figure highlights several important recession periods as grey periods and crash events as black lines. The periods are more closely described in the text.

from stock markets. We use long time span for the S&P 500, a broad US stock market index, that covers almost 27 years, from 24 February 1984 to 17 November 2010. Figure 3 plots the prices and depicts the several recessions and crisis periods. According to the National Bureau of Economic Research (NBER), there were three US recessions during the periods of July 1990–March 1991, March 2001–November 2001 and December 2007–June 2009. These recessions are depicted as grey periods. Black lines depict one-day crashes associated with large price drops. Namely, these include Black Monday 1987 (19 October 1987), the Asian Crisis Crash (27 October 1997), the Ruble Devaluation of 1998 (17 August 1998), the Dot-com Bubble Burst (10 March 2000), the World Trade Center Attacks (11 September 2001), the Lehman Brothers Holdings Bankruptcy (15 September 2008) and finally the Flash Crash (6 March 2010). Technically, the largest one-day drops in the studied period occurred on 19 October 1987, 26 October 1987, 29 September 2008, 9 October 2008, 15 October 2008 and 1 December 2008, recording declines of 20.47, 8.28, 8.79, 7.62, 9.03 and 8.93%, respectively.

Let us now look closer at the crashes depicted by figure 3 and discuss their nature. The term Black Monday refers to Monday, 19 October 1987 when stock markets around the world from Hong Kong to Europe and the US crashed in a very short time and recorded the largest one-day drop in history. After this unexpected, severe event, many analysts predicted the most troubled years since the 1930s. However, stock markets gained the losses back and closed the year positively. There has been no consensus opinion on the cause of the crash. Potential causes include programme trading, overvaluation, illiquidity and market psychology. Thus, this crash seems to have had an endogenous cause. Stock markets did not record any large shocks for the next several years until 1996, when the Asian Financial Crisis driven by investors deserting overheated, emerging Asian markets resulted in the 27 October 1997 mini-crash of the US markets. The next year, the Russian government devalued the ruble, defaulted on its domestic debt and declared a moratorium on payments to foreign creditors. These actions caused another international crash on 17 August 1998. These last two shocks are believed to be exogenous to the US stock markets. During the period from 1997–2000, the so-called dot-com bubble emerged, when a group of internet-based companies entered the markets and attracted many investors who were confident in the companies' profits, overlooking their fundamental values. The result was a collapse, or burst bubble, during the period from 2000–2001. Another exogenous shock was brought to stock markets in the 2001 when the World Trade Center (WTC) was attacked and destroyed. While the markets recorded a sudden drop, it should not be attributed to internal forces of the markets. The recent financial crisis of 2007–2008, also known as the Global Financial Crisis, emerged from the bursting of the US housing bubble, which peaked in 2006. In a series of days in September and October 2008, stock markets saw successive large declines. Many analysts believe that this crash was mainly driven by the housing markets, but there is no consensus about the real causes. Finally, our studied period also covers the 6 May 2010 Flash Crash, also known as The Crash of 2:45, in which the Dow Jones Industrial Average plunged approximately 1000 points (9%), only to recover its losses within a few minutes. It was the biggest intraday drop in history, and one of its main possible causes may have been the impact of high-frequency traders or large directional bets.

In terms of Zeeman's (1974) hypotheses, cusp catastrophe theory proposes to model the crashes as endogenous events driven by speculative money. Employing our two-step estimation method, we estimate the cusp model to quantitatively test the theory on the period that covers all of these crashes to determine whether the theory can explain the crashes using our data. An interesting discussion may stem from studying the causality between volatility and crashes. While Levy (2008) provided a modelling approach for increasing volatility before crash events, the crashes are driven endogenously by speculative money in our approach; thus, the sudden discontinuities are not connected to volatility.

5.1. Data description

For our two-step estimation procedure, we need two sets of data. The first set consists of high-frequency trading data, which are used to estimate the volatility of returns. The second set consists of data on sentiment. Let us describe both data-sets used. For the realized volatility estimation, we use the S&P 500 futures traded on the Chicago Mercantile Exchange (CME).† The sample period extends from 24 February 1984 to 17 November 2010. Although after the introduction of the CME Globex(R) electronic trading platform on Monday, 18

†The data were provided by Tick Data, Inc.

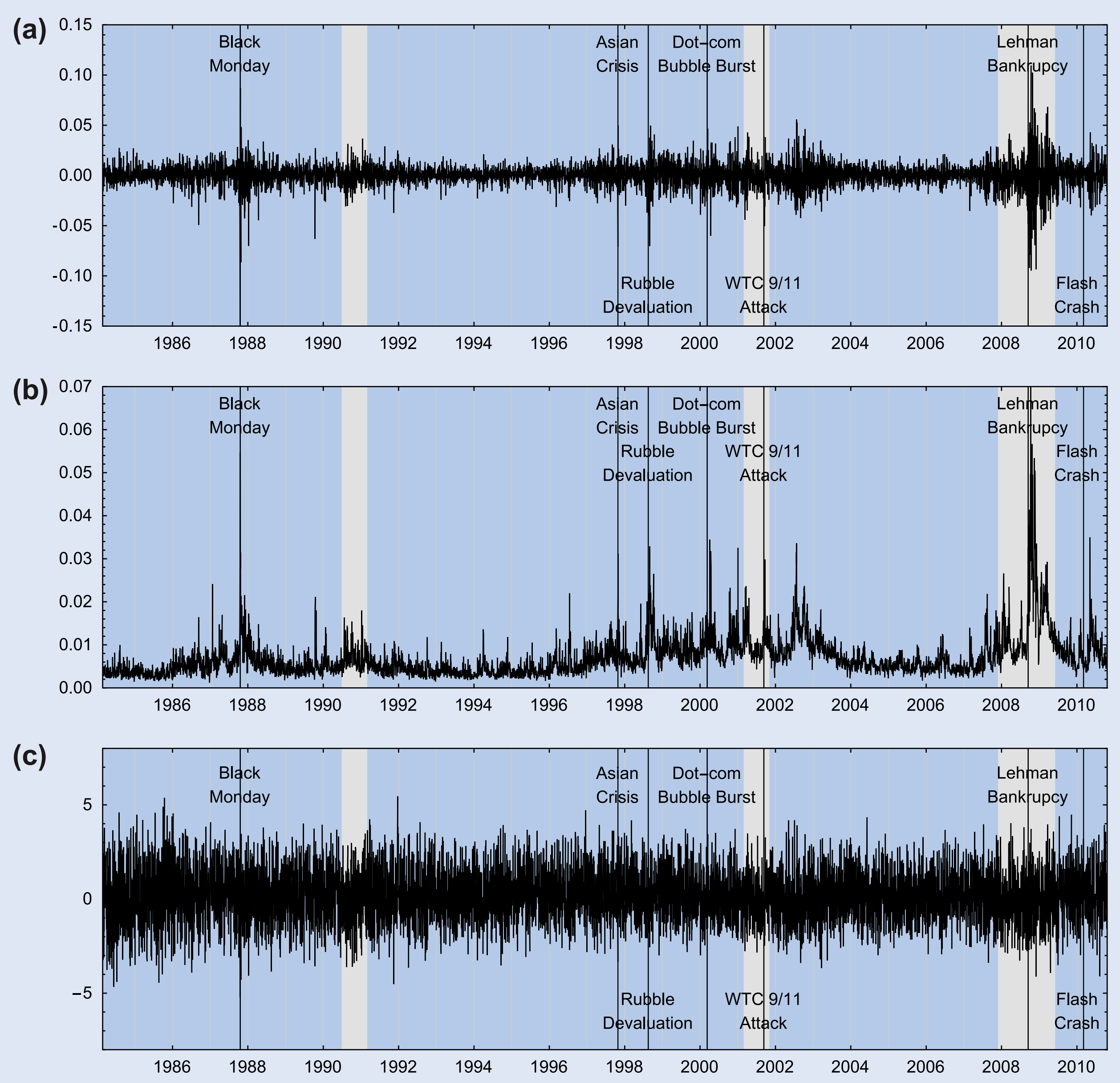

Figure 4. S&P 500 (a) returns r_t, (b) realized volatility RV_t, and (c) standardized returns $r_t RV_t^{-1/2}$. The figure highlights several important recession periods as grey periods and crash events as black lines. The periods are more closely described in the text.

December 2006, CME started to offer nearly continuous trading, we restrict the analysis to the intraday returns with five-minute frequencies within the business hours of the New York Stock Exchange (NYSE) because the most liquidity of the S&P 500 futures came from the period when the US markets were open. We eliminate transactions executed on Saturdays and Sundays, US federal holidays, 24–26 December and 31 December to 2 January because of the low activity on those days, which could lead to estimation bias.

Using the realized volatility estimator, we then measure the volatility of the stock market returns as the sum of the squared five-minute intraday returns. In this way, we obtain 6739 daily volatility estimates. Figure 4 shows the estimated volatility 4(b) together with the daily returns 4(a). It can be immediately observed that the volatility of the S&P 500 is strongly time varying over the very long period.

For the state (behavioural) variable of the cusp model, we use the S&P 500 daily returns standardized by the estimated daily realized volatility according to equation (17). By standardization, we obtain stationary data depicted in figure 4(c).

In choosing the control variables, we follow the successful method from our previous application in Barunik and Vosvrda (2009), where we compared several measures of control variables and showed that fundamentalists, or the asymmetry side of the market, are best described by the ratio of advancing and declining stock volume, and chartists, or the bifurcation side of the model, are best described by the OEX put/call ratio.† The variables related to the trading volume generally correlate with the volatility and therefore are considered good measures of the trading activity of large funds and other institutional investors. Trading volume indicators thus represent the fundamental side

†The data were provided by Pinnacle Data Corp.

Table 2. Estimation results on the S&P 500 stock market data. The full sample extends from 24 February 1984 to 17 November 2010. The left side of the table presents the estimation results on the normalized returns $r_t \widehat{RV}_t^{-1/2}$, and the right side of the table presents the results for the original S&P 500 stock market returns r_t.

	(a) Estimates using $r_t \widehat{RV}_t^{-1/2}$				(b) Estimates using r_t			
	Cusp		Linear	Logistic	Cusp		Linear	Logistic
	Unrestricted	Restricted			Unrestricted	Restricted		
α_0	−2.378***	1.333***			5.008***	5.000***		
α_1	4.974***	3.369***			4.426***	2.730***		
α_2	0.062***				0.771***			
α_3	0.300***	−0.064***			−0.463***	−0.534***		
β_0	−4.654***	−1.334***			−5.011***	−5.000***		
β_1	−5.011***					−1.201***		
β_2	0.139***	−0.054***			−1.138***	−0.208***		
β_3	0.422***	0.299***			0.644***	0.683***		
ω_0	−0.700***	0.329***			0.786***	0.826***		
ω_1	0.492***	0.905***			0.407***	0.402***		
R^2	0.800	0.769	0.385	0.823	0.637	0.530	0.405	0.687
LL	−4786.161	−4733.071	−7634.756	−5885.983	−7728.981	−8174.635	−7811.134	−5648.799
AIC	9592.321	9482.142	15 279.512	11 789.970	15 477.960	16 365.270	15 632.270	11 311.600
BIC	9660.209	9533.682	15 313.456	11 851.070	15 545.820	16 419.800	15 666.350	11 359.310

***Significance level of 0%.
**Significance level of 0.001%.
*Significance level of 0.01%.
'.' Significance level of 0.05%.
' 'Significance level of 0.1%.

of the market and can be used as a good proxy for fundamental investors. Therefore, the ratio of advancing and declining stock volume should mainly contribute to the asymmetry side of the model. Conversely, the activity of market speculators and technical traders should be well captured by the measures of sentiment, precisely the OEX put/call ratio, which is the ratio of daily put and daily call option volume with the underlying S&P 500 index. Financial options are widely used and are the most popular instruments for speculative purposes. Therefore, they serve as a good measure of speculative money in capital markets (see e.g. Bates (1991), Finucane (1991), or Wang *et al.* (2006)) because they represent the data about extraordinary premiums and excessive greed or fear on the market. Thus, they should represent the internal forces that lead the market to bifurcation within the cusp catastrophe model. Overall, we assume the OEX put/call option ratio mainly contributes to the bifurcation side of the model. Moreover, we use a third control variable, the daily change in total trading volume, as a driver for both the fundamental and speculative money in the market. The daily change in the total volume indicator is generally related to continuous fundamental trading activity, but it may also reflect elevated speculative activity on the market as well. Therefore, we expect this variable to help the regression not only on the asymmetry side but also on the bifurcation side. The time span for all of these data matches the time span of the S&P 500 returns, i.e. 24 February 1984 to 17 November 2010. The descriptive statistics for all of the data are in table A1 in the appendix.

5.2. *Full sample static estimates*

In the estimation, we primarily aim to test whether the cusp catastrophe model is able to describe the stock market data in the time-varying volatility environment and therefore that stock markets show signs of bifurcations. In doing so, we follow the statistical testing described earlier in the text.

Table 2 shows the estimates of the cusp fits. Let us concentrate on the left side of table 2(a), where we fit the cusp catastrophe model to the standardized returns $r_t \widehat{RV}_t^{-1/2}$. First, we do not make any restrictions, and we use all three control variables; thus, $\alpha_x = \alpha_0 + \alpha_1 x_{1,t} + \alpha_2 x_{2,t} + \alpha_3 x_{3,t}$ and $\beta_x = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t}$, where x_1 is the ratio of advancing and declining stock volume, x_2 is the OEX put/call option ratio and x_3 is the rate of change of the total volume. The state variable is $\tilde{r}_t$, and the returns are normalized with estimated realized volatility. In terms of log likelihood, the cusp model describes the data much better than the linear regression model. The ω_1 coefficient is far away from zero, although some degree of transformation of the data is needed. All of the other coefficients are strongly significant at the 99% level. Most importantly, when the cusp fit is compared with the logistic fit in terms of AIC and BIC, we can see that the cusp model strongly outperforms the logistic model.

Our hypothesis is that the ratio of advancing and declining stock volume only contributes to the asymmetry side, and the OEX put/call ratio, representing the measure of speculative money in the market, contributes to the bifurcation side of the model. To test this hypothesis, we set the parameters $\alpha_2 = \beta_1 = 0$ and refer to it as a restricted model. From table 2(a), we can see that the log likelihood of the restricted model improves in comparison with the unrestricted model. Additionally, in terms of the information criteria, the fit further improves. All of the parameters are again strongly significant, and we can see that they change considerably. This result can be attributed to the fact that $x_{1,t}$ seems to contribute strongly to both sides of the market in the unrestricted model. Although the β_2 coefficient representing the speculative money is quite small in comparison with the other coefficients, it is still strongly significant. Because this coefficient is the key for the model in driving the stock market to bifurcation, we further investigate its impact in the following sections. It is interesting to note that the ω_1 parameter increases very close to one in the restricted model. This result means that the observed data are close to the state variable.

When moving to the right (b) side of table 2, we repeat the same analysis, but this time, we use the original r_t returns as the state variable. We wish to compare the cusp catastrophe fit to the data with strongly varying volatility. In using the data's very long time span where the volatility varies considerably, we expect the model to deteriorate. Although the application of the cusp catastrophe model to the non-stationary data can be questioned, we provide these estimates to compare them with our modelling approach. We see an important result. While the linear and logistic models provide very similar fits in terms of the log likelihoods, the information criteria and R^2 deteriorate in both the unrestricted and restricted cusp models. The ω_1 coefficient, together with all of the other coefficients, is still strongly different from zero, but the important result is that the logistic model not only describes the data better, but also the presence of bifurcations in the raw return data cannot be claimed.

To conclude this section, the results suggest strong evidence that over the long period of almost 27 years, the stock markets are better described by the cusp catastrophe model. Using our two-step modelling approach, we have shown that the cusp model fits the data well and the fundamental and bifurcation sides are controlled by the indicators for the fundamental and speculative money, respectively. In contrast, when the cusp is fit to the original data with a strong variation in volatility, the model deteriorates. We should note that these results resemble the results from the simulation; thus, the simulation also strongly supports our modelling approach.

5.3. Examples of the 1987 and 2008 crashes

While the results from the previous section are supportive of the cusp catastrophe model, the sample period of almost 27 years may contain many structural changes. Thus, we wish to further investigate how the model performs over time. Therefore, we use the two very distinct crashes of 1987 and 2008 and compare them to the localized cusp fits. There are several reasons to study these particular periods. These crashes were distinct in time, as there were 21 years between them, so they offer us an opportunity to determine how the data describe the periods. On the one hand, the stock market crash of 1987 has not yet been explained, and many analysts believe it was an endogenous crash. Therefore, it constitutes a perfect candidate for the cusp model. On the other hand, the 2008 period covered a much deeper recession, so it was very different from 1987. Finally, the two periods contain all of the largest one-day drops, which occurred on 19 October 1987, 26 October 1987, 29 September 2008, 9 October 2008, 15 October 2008 and 1 December 2008, recording declines of 20.47, 8.28, 8.79, 7.62, 9.03 and 8.93%, respectively. In the following estimations, we restrict ourselves to our newly proposed two-step approach for the cusp catastrophe fitting procedure, and we utilize samples covering one-half year.

When focusing on the estimation results for the 1987 crash, we can see that both the restricted and the unrestricted models fit the data much better than the linear regression. The ω_1 coefficients are significantly different from zero, and when the cusp models are compared with the logistic model, they seem to provide much better fits. Thus, the cusp catastrophe model explains the data very well, and we can conclude that the stock market crash of 1987 was led by internal forces. This result confirms our previous findings in Barunik and Vosvrda (2009), although a comparison cannot be made directly because we used a different sample length in our previous study. When comparing the fits of the unrestricted and restricted models, we can see that they do not differ significantly within the log likelihoods, AIC and BIC. In addition, the coefficient estimates are close to each other. The reason for this result is that the unrestricted model estimates the α_2 coefficient, which cannot be distinguished from zero, and the β_1 coefficient is significant only at a 90% level of significance. Thus, x_1 is proven to drive the fundamentals, and x_2 drives the speculators. Note that the β_2 coefficient is much larger in magnitude than on the fit for the full sample in the previous section. Interestingly, x_3 seems to drive the speculative money in the 1987 crash, but it does not help to explain the 2008 behaviour.

The data from the 2008 period present different results. While the cusp fits are much better in comparison with the linear regression, they cannot be statistically distinguished from the logistic model. Therefore, there is very weak evidence of discontinuities in this period. This result is interesting because it may suggest that the large drops in 2008 were not driven endogenously by stock market participants but exogenously by the burst of the housing market bubble.

5.4. Rolling regression estimates

While the 1987 data are explained by the cusp catastrophe model very well and the 2008 data are not, we would like to further investigate how the cusp catastrophe fit changes over time. With almost 27 years of data needed for our two-step method of estimation, we estimate the cusp catastrophe model on one-half year rolling samples with a step of one month. The one-half year period is reasonable because it represents enough data for a sound statistical fit, but it is not a very long period, so we can uncover any structural breaks in the data.† In the estimation, we again restrict ourselves to our two-step estimation procedure. To keep the results under control, we use the final restricted model, where we assume that x_1 controls the asymmetry side of the model, and x_2 controls the bifurcation side solely, while x_3 contributes to both sides. Thus, $\alpha_2 = \beta_1 = 0$.

Before we proceed to interpreting the rolling regression results, let us discuss the bimodality of the rolling samples. Stochastic catastrophe corresponds to a transition from a uni-

†Various combinations of rolling sample lengths and steps had been used in the preliminary analysis without affecting the overall aggregated results, e.g. comparing one day and one month steps. The outcomes of the preliminary analysis are available from authors upon request.

Table 3. Estimation results for the two distinct periods of the S&P 500 normalized stock market returns, $r_t \widehat{RV}_t^{-1/2}$.

	1987				2008			
	Cusp		Linear	Logistic	Cusp		Linear	Logistic
	Unrestricted	Restricted			Unrestricted	Restricted		
α_0	−0.970*	−0.535***			−3.938***	−0.598***		
α_1	1.792***	1.794***			1.793***	1.798***		
α_2	−0.073				−1.322			
α_3	0.191	0.029			−0.814		0.005***	
β_0	0.562*	0.322*			−0.542***	−1.142*		
β_1	−0.395*				−1.803***			
β_2	−1.255**	−0.731*			−0.990**	−0.031***		
β_3	1.648***	1.547***			−0.803	0.282		
ω_0	0.453***	0.771***			−0.637***	0.680***		
ω_1	0.561***	0.602***			0.476***	0.641***		
R^2	0.855	0.827	0.454	0.895	0.816	0.858	0.483	0.884
LL	−85.971	−90.659	−208.557	−104.045	−88.054	−103.437	−185.150	−90.148
AIC	191.943	197.317	427.114	226.089	196.108	222.875	380.300	198.296
BIC	220.385	220.071	441.335	251.687	224.550	245.628	394.521	223.894

***Significance level of 0%.
**Significance level of 0.001%.
*Significance level of 0.01%.
'.'Significance level of 0.05%.
' 'Significance level of 0.1%.

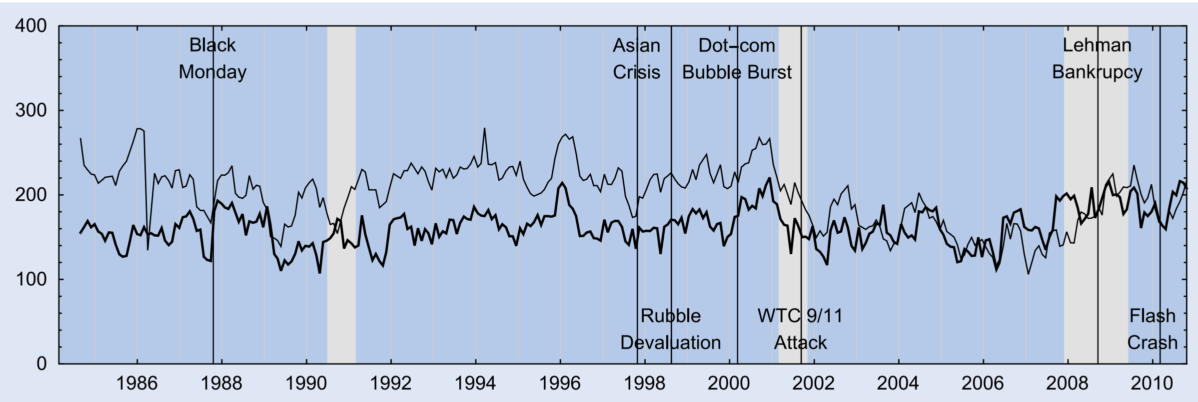

Figure 5. Rolling values of BIC information criteria for the cusp catastrophe (in bold black) and the logistic (in black) models.

modal to a bimodal distribution. Thus, we first need to test for bimodality to be able to draw any conclusions from our analysis. To do this, we use the dip test of unimodality developed by Hartigan (1985) and Hartigan and Hartigan (1985). The dip statistic is the maximum difference between the empirical distribution function and the unimodal distribution function, and it measures the departure of the sample from unimodality. Asymptotically, the dip statistics for samples from a unimodal distribution approach zero, and for samples from any multimodal distribution, the dip statistics approach a positive constant. We use bootstrapped critical values for the small rolling sample sizes to assess the unimodality. Figure A1 shows the histogram of all of the dip statistics, together with its bootstrapped critical value 0.0406 at the 90% significance level. The results suggest that unimodality is rejected at the 90% significance level for several periods, but for most of the periods, unimodality cannot be rejected. Thus, we observe a transition from unimodal to bimodal (or possibly multimodal) distributions several times during the studied period.

Encouraged by the knowledge that the bifurcations could be present in our data-set, we move to the rolling cusp results. Figure A2 shows the rolling coefficient estimates together with their significances. The ω_1 is significantly different from zero in all periods, and the α_1 coefficient is strongly significant over the whole period, although it becomes lower in magnitude during the latest years. Thus, the ratio of advancing and declining volume is a good measure for fundamentalists driving the asymmetry portion of the model. Much more important, however, is the β_2 parameter, which drives the bifurcations in the model. We can observe that until 1996, β_2 was significant, and its value changed considerably over time, but after 1996, it cannot be distinguished statistically from zero (except for some periods). This result is very interesting because it shows that the OEX put/call ratio was a good measure of speculative money in the market, and it controlled the bifurcation side of the model. During the first period, the OEX put/call ratio drove the stock market into bifurcations, but in the second period, the market was rather stable under the model. The parameter only started to play a role in the model again in the last few years and during the recent 2008 recession. However, its contribution was still relatively small.

This result is also confirmed when we compare the cusp model to the logistic model. Figure 5 compares the Bayesian Information Criteria of the two models, and we can see that the cusp catastrophe model was a much better fit for the data up to 2003, while for roughly 2003–2009, the cusp catastrophe model cannot be distinguished from the logistic model or logistic model strongly outperforms the cusp model. In the last period after 2009 and before the Flash Crash, the cusp again explains the data better, but the difference is not as strong as in the pre-2003 period. This result shows that before 2003, the stock markets showed signs of bifurcation behaviour according to the cusp model, but after 2003, in the period of stable growth when participants believed that stock markets were stable, the markets no longer showed signs of bifurcation behaviour.

To conclude this section, we find that despite the fact that we modelled volatility in the first step, the stock markets showed signs of bistability over several crisis periods.

6. Conclusion

In this paper, we contribute to the literature on the modelling of stock market crashes and the quantitative application of the stochastic cusp catastrophe theory. We develop a two-step estimation procedure and estimate the cusp catastrophe model under time-varying stock market volatility. This approach allows us to test Zeeman's (1974) qualitative hypotheses on cusp catastrophe and bring new empirical results to our previous work in this area (Barunik and Vosvrda 2009).

In the empirical testing, we use unique, high frequency and sentiment data on the US stock market covering almost 27 years. The results suggest that over a long period, stock markets are well described by the stochastic cusp catastrophe model. Using our two-step modelling approach, we show that the cusp model fits the data well and that the fundamental and bifurcation sides are controlled by the indicators for fundamental and speculative money, respectively. In contrast, when the cusp model is fit to the original data with strong variations in volatility, the model deteriorates. We should note that these results are similar to the results from a Monte Carlo study that we ran; thus, our simulation strongly supports our analysis. Furthermore, we develop a rolling estimation, and we find that until 2003, the cusp catastrophe model explains the data well, but this result changes during the period of stable growth from 2003–2008.

In conclusion, we find that despite the fact that we modelled volatility in the first step, the stock markets showed signs of bistability during several crisis periods. An interesting venue of future research will be to translate these results to a probability of the crash occurrence and its possible prediction.

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Appendix

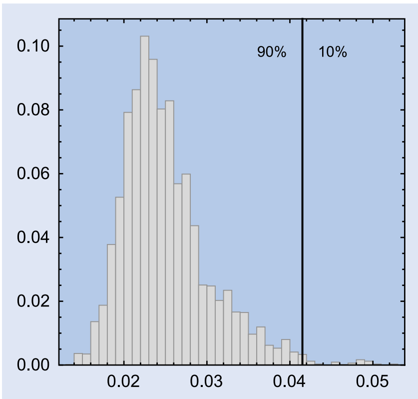

Figure A1. Histogram of the dip statistics for bimodality computed for all of the rolling window periods, together with the bootstrapped critical value 0.0406 for the 90% significance level plotted in bold black.

Table A1. Descriptive Statistics of the data. The sample period extends from 24 February 1984 to 17 November 2010. The S&P 500 stock market returns r_t, realized volatility RV_t, daily returns normalized by the realized volatility $r_t RV_t^{-1/2}$ and data for the independent variables $\{x_1, x_2, x_3\}$ are the ratio of advancing and declining stock volume, the OEX put/call options and the change in total volume, respectively.

	r_t	RV_t	$r_t RV_t^{-1/2}$	x_1	x_2	x_3
Mean	0.00030	0.00697	0.15535	1.67051	1.16636	0.02029
Median	0.00059	0.00571	0.11005	1.12730	1.11000	0.00301
Std. Dev.	0.01175	0.00478	1.42534	2.21562	0.36245	0.21405
Skewness	−1.32728	3.55338	0.13660	6.71651	1.37368	2.38477
Ex. Kurtosis	29.44420	23.96040	−0.02713	76.35370	4.87145	17.36620
Minimum	−0.22887	0.00122	−5.21961	0.00187	0.30000	−0.76040
Maximum	0.10957	0.07605	5.45738	44.06780	4.56000	2.19614

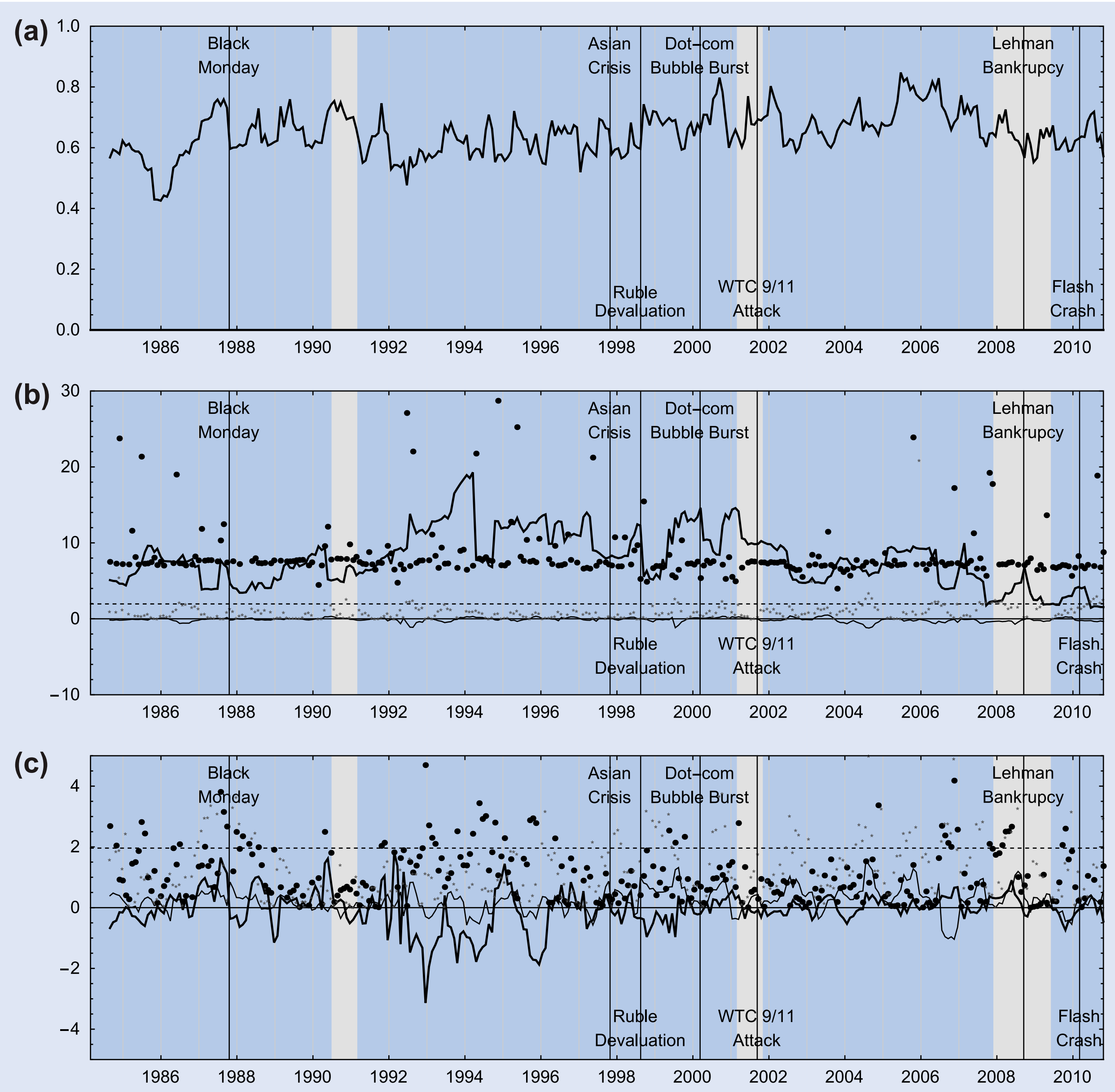

Figure A2. Rolling coefficients with their |z-values|. (a) Estimated ω_1 coefficient values. (b) Estimated values of asymmetry coefficients, α_1 in bold black, α_3 in black. |z-values| related to both coefficient estimates are depicted as • and $*$, respectively. (c) Estimated values of bifurcation coefficients, β_2 in bold black, β_3 in black. |z-values| related to both coefficient estimates are as • and $*$, respectively. Plots (b) and (c) also contain the 95% reference z-value as a dashed black line.