A note on the use of copulas in chance-constrained programming

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Abstract. In this paper we are concentrated on a problem of linear chance-constrained programming where the constraint matrix is considered random with a known distribution of the matrix rows. The rows are not considered to be independent; instead, we make use of the copula notion to describe the dependence of the matrix rows. In particular, the distribution of the rows is driven by so-called Archimedean class of copulas. We provide a review of very basic properties of Archimedean copulas and describe how they can be used to transform the stochastic programming problem into a deterministic problem of second-order cone programming. Also the question of convexity of the problem is explored and importance of the selected class of copulas is commented. At the end of the paper, we provide a simple example to illustrate the concept used.

Keywords: Chance-constrained optimization, Archimedean copulas, Convexity, Second-order cone programming.

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AMS classification: 90C15

1 Introduction

In many economic decision problems we are faced to the problem how to deal with uncertainty of the data. In this contribution we introduce an extension to the probabilistic programming approach based on the description of probabilistic properties of the data through the notion of copulas.

1.1 Problem Formulation

Consider an *uncertain linear optimization problem* in the form

$$\min c^T x \qquad \text{subject to} \qquad \Xi x \leq h,\ x \in X \tag{1}$$

where $\Xi \subset \mathbb{R}^n$ is a deterministic closed convex set, $c \in \mathbb{R}^n$, $h = (h_1, \ldots, h_K) \in \mathbb{R}^K$ are deterministic vectors, and $\Xi \in \mathbb{R}^K \times \mathbb{R}^n$ is an uncertain (unknown) matrix. We denote $\Xi_1^T, \ldots, \Xi_K^T$ the rows of the matrix Ξ. It is easy to see that if X is polyhedral, and a realization of the data element Ξ is known in advance, (1) is an instance of the classical linear optimization problem. As already G. Dantzig noticed in his early work [4], this is rarely the case. Instead, we have to consider *uncertainty of the data* as a natural element of the optimization model.

Several approaches can be drawn to deal with uncertainty of the data. Among well known ones we cite (ex-post) sensitivity analysis of the linear programming, parametric programming (in which data element is considered to be parameter), or robust approach (the constraints of the problem are required to be satisfied for all realizations from some predefined uncertainty set). In our paper, we concentrate on the stochastic programming approach: we suppose that Ξ is a random vector with a known distribution, and the constraints are required to be satisfied with a prescribed, sufficiently high probability. More specifically, for a fixed $p \in [0;1]$, we consider a *chance-constrained linear optimization problem* of the form

$$\min c^T x \qquad \text{subject to} \qquad \Pr\{\Xi x \leq h\} \geq p,\ x \in X. \tag{2}$$

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For easier referencing, we drop the deterministic constraint $x \in X$ from our future consideration (without losing the generality of our conclusions), and denote the feasible set of problem (2) as

$$X(p) := \{x \in \mathbb{R}^n \mid \Pr\{\Xi x \le h\} \ge p\}. \tag{3}$$

Chance-constrained programming problems were treated many times in the literature, starting with the work [2], and summarized by the classical book [14] and recent chapters [15] and [5]. It were realized that problem (2), especially in the presented form with random data matrix, is not easy to analyze — from the theoretical as well as from the practical point of view. It is worth to declare that two main difficulties were revealed: first, even with its inner linear structure, the feasible set $X(p)$ has not to be convex; and second, the presence of multidimensional integrals prevents the probability in (2) to be easily calculated. We concentrate on the first issue here. Convexity of the set $X(p)$ was treated several times in the literature; apart from mentioned references we mention recent results by [6], [7], and [16]. In our paper, we will use an approach developed by [11] and provide conditions under which the problem happen to be convex, together with a simple application illustrating the concept used.

2 Dependence — Notion of Copulas

Convexity results mentioned shortly in the introduction of the paper (except the last one) are based on the assumption of *row independence*, that is, the random vectors $\Xi_1^T, \dots, \Xi_K^T$ are supposed to be pairwise independent. Although it can be seen as an obstacle, we have a handy tool to generalize this assumption: a copula notion. This is well known from the field of probability theory and mathematical statistics but rarely used in stochastic optimization, especially it was not used to describe structural properties of chance-constrained problems. Apart of already mentioned reference we can cite [8] and [1] but these works do not directly concern the problem of the form (2). In this section we provide necessary definitions and theorems needed to describe dependence properties of problem (2); we refer the reader to the book [13] for details.

Definition 1. A *copula* is the distribution function $C : [0;1]^K \to [0;1]$ of some K-dimensional random vector whose marginals are uniformly distributed on [0;1].

It is Sklar's Theorem (given below) that justifies usefulness of copula notion to describe dependence properties of a problem. Before stating it, we recall the definition of quantile function which plays a fundamental role in the sequel of the paper.

Definition 2. Given a one-dimensional distribution function $F : \mathbb{R} \to [0;1]$, its *quantile function* is defined as

$$F^{(-1)}(\pi) := \inf\{t \in \mathbb{R} \mid F(t) \ge \pi\}.$$

Proposition 1 (Sklar's Theorem)**.** *For any K-dimensional distribution function $F : \mathbb{R}^K \to [0;1]$ with marginals $F_1, \dots, F_K$, there exists a copula C such that*

$$\forall z \in \mathbb{R}^K \quad F(z) = C(F_1(z_1), \dots, F_K(z_K)). \tag{4}$$

If, moreover, F_k are continuous, then C is uniquely given by

$$C(u) = F(F_1^{-1}(u_1), \dots, F_K^{-1}(u_K)). \tag{5}$$

Otherwise, C is uniquely determined on $\operatorname{range} F_1 \times \dots \times \operatorname{range} F_K$.

Through Sklar's Theorem, an arbitrary dependence structure can be described by an appropriately chosen (unique) copula. A leading example is so-called *independent (product) copula*

$$C_\Pi(u) = \prod_k u_k \tag{6}$$

providing us the well known independence formula for distributions functions — if the marginals $F_1, \dots, F_K$ are independent then their joint distribution function F is given by

$$F(z) = C_\Pi\big(F_1(z_1), \dots, F_K(z_K)\big) = \prod_{k=1}^{K} F_k(z_k).$$

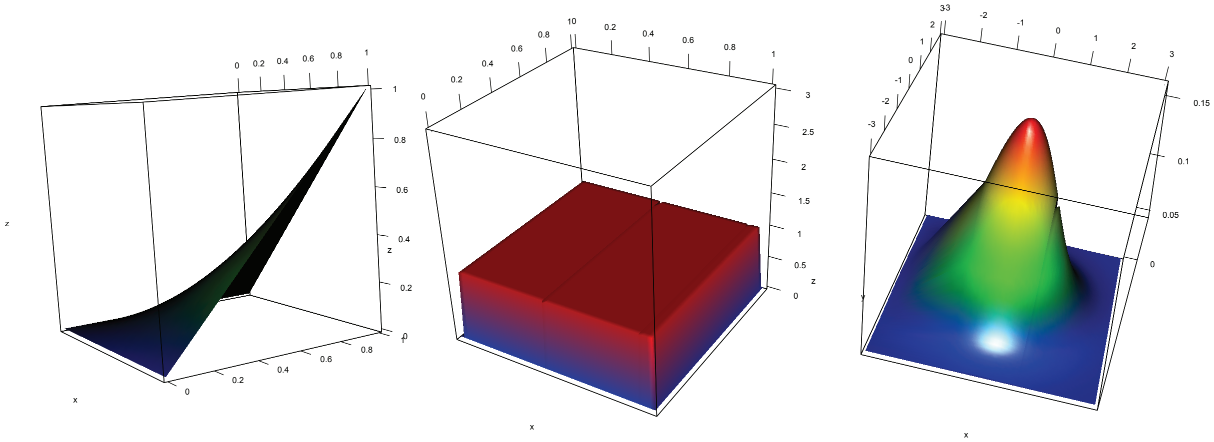

Figure 1 Independent copula: distribution, density, and density with standard normal marginals.

For reader convenience we provide a graphical representation of the independence copula in Figure 1. The third graph of the figure is the well-known two-dimensional density of the normal distribution with independent marginals. The contours of this density are in form of "regular" ellipsoids (circles in this special case).

To an easy treatment of problem (2) we need an algebraically convenient class of copulas. This is provided by the Archimedean copulas.

Definition 3. A copula C is called *Archimedean* if there exists a continuous strictly decreasing function $\psi : [0;1] \to [0;+\infty]$, called *generator of* C, such that $\psi(1) = 0$ and

$$C(u) = \psi^{(-1)}\left(\sum_{i=1}^{n} \psi(u_i)\right). \tag{7}$$

If $\lim_{t\to 0} \psi(t) = +\infty$ then C is called a *strict Archimedean copula* and ψ is called a *strict generator*.

A necessary and sufficient condition for a continuous, strictly decreasing function to be an Archimedean generator is not easy task and was provided by [12]. We need only a sufficient condition for strict Archimedean generators.

Definition 4. A real function $f : \mathbb{R} \to \mathbb{R}$ is called *completely monotonic* on an open interval $I \subseteq \mathbb{R}$ if it is nondecreasing, differentiable for each order k, and its derivatives alternate in sign, i. e.

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) \geq 0 \;\forall k = 0, 1, \ldots, \text{ and } \forall t \in I. \tag{8}$$

Proposition 2. *Let* $\psi : [0;1] \to \mathbb{R}_+$ *be a strictly decreasing function with* $\psi(1) = 0$ *and* $\lim_{t\to 0} \psi(t) = +\infty$. *Then it is a generator of a strict Archimedean copula for each dimension* $K \geq 2$ *if and only if* $\psi^{(-1)}$ *is completely monotonic on* $(0;+\infty)$.

It is easy to see that the independent copula is Archimedean with the strict generator $\psi(t) = -\ln t$. Another popular example of an Archimedean copula is the *Gumbel–Hougaard copula* defined by $\psi(t) = (-\ln t)^\theta$ where $\theta \geq 1$ can be seen as a parameter of the dependence. The Gumbel–Hougaard copula is presented in Figure 2. Compared to the independent case, the density (the third subfigure) is "pointed" and the ellipsoidal contours of the independent copula take form of drops. For other illustrations of Archimedean copulas (Clayton, Joe) see [11]; an exhaustive listing of 22 families of Archimedean copulas treated by the literature is given by Table 4.1 in [13].

3 Convex Reformulation

The Archimedean copulas can be used for an equivalent convex reformulation of the feasible set $X(p)$.

Theorem 3 ([11], Theorem 3.1)**.** *Suppose, in* (3), *that the rows* Ξ_k^T *have* n*-variate normal distribution with means* μ_k *and positive definite covariance matrices* Σ_k. *Then the feasible set of problem* (2) *can be*

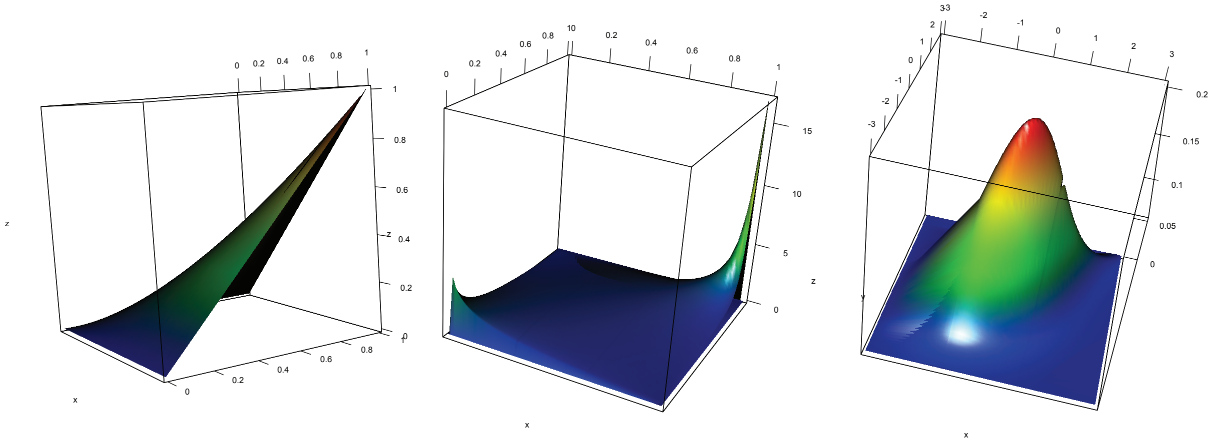

Figure 2 Gumbel-Hougaard copula ($\theta = 1.6$): distribution, density, and density with standard normal marginals.

equivalently written as

$$X(p) = \left\{ x \mid \exists y_k \geq 0 : \sum_k y_k = 1,\ {\mu_k}^T x + \Phi^{(-1)} \left(\psi^{(-1)}(y_k \psi(p)) \right) \sqrt{x^T \Sigma_k x} \leq h_k\ \forall k \right\} \tag{9}$$

where Φ *is the distribution function of a standard normal distribution and* ψ *is the generator of an Archimedean copula describing the dependence properties of the rows of the matrix* Ξ*. Moreover, if* $\psi^{(-1)}$ *is completely monotonic, and* $p > p^* := \Phi\left(\max\{\sqrt{3}, 4\lambda^{(k)}_{max}[\lambda^{(k)}_{min}]^{-3/2}\|\mu_k\|\}\right)$*, where* $\lambda^{(k)}_{max}, \lambda^{(k)}_{min}$ *are the largest and lowest eigenvalues of the matrices* Σ_k*, then the problem is convex.*

Proof. See [10]. The threshold p^* was provided for independent case by [7] using properties of r-concave functions (see also [9]) but it have not to be changed under our dependency assumptions. □

Theorem 3 offers a way to replace multidimensional integral in (3) by a set of one-dimensional (separable) constraints of the form (9). Reference [11] (based on the work [3]) also provides a numerical approximation of (9) by two second-order cone programming (SOCP) problems, using the fact that the function $y \mapsto H(y) := \Phi^{(-1)} \left(\psi^{(-1)}(y_k \psi(p))\right)$ is convex. For example, a piecewise-convex approximation of this functions leads to the following SOCP problem giving a lower bound for the optimal value of the original probabilistic problem

$$\begin{aligned} \min c^T x \qquad & \text{subject to} \\ & {\mu_k}^T x + \sqrt{z^{kT} \Sigma_k z^k} \leq h_k,\ z^k \geq a_{kj} x + b_{kj} w^k\ (\forall k, \forall j) \\ & \sum_k w^k = x,\ w^k \geq 0,\ z^k \geq 0\ (\forall k), \end{aligned} \tag{10}$$

where

$$a_{kj} := H(y_{kj}) - b_{kj} y_{kj}, \qquad\qquad b_{kj} := \frac{\psi(p)}{\phi(H(y_{kj}))\psi'\left(\psi^{(-1)}(y_{kj}\psi(p))\right)},$$

y_{kj} are given partition points of the interval $(0; 1]$, and ϕ is the standard normal density. In [11], also an inner approximation is provided for the problem.

4 Application to the Model with EEA Indicators

To present a simple example of using this approach, we pursue the direction presented already in [9] and previous works referenced therein. We briefly summarize the model and its description; the details can be found in the mentioned reference [9]. We did not include numerical results (based on simulations) in the current paper as they are matter of our current and still not finished research.

Consider vector mappings $i_k(x;\xi) := x^T A^k \xi$, representing a functional dependence of the EEA indicators (of ecological stability) i_k on decision vector x (possible ecological arrangements), and uncertainty factor ξ. The indicators of ecological stability monitor basic properties of the environment, e.g. air and water quality, pollution, noise, etc. Possible arrangements include desulphurisation of plants, water cleaners, soundproofing of roads, limits provided by law, and many others.

To convene our assumption we suppose this dependence to be linear. Denote L_k required threshold values for the indicators i_k; apart x, L_j are also considered to be decision variables of the problem in our formulation. Our aim is to maximize the ecological limits in order to meet them at least with the required fixed probability p (i. e., to find the optimal critical values for ecological indicators achievable with high probability). The problems can be then formulated as

$$\max \sum_k L_k \qquad \text{subject to} \qquad \Pr\left\{x^T A^k \xi \geq L_k, k \in \mathcal{K}\right\} \geq p,\ x \in X_0; \tag{11}$$

X_0 contains budget and other deterministic constraints.

Problem (11) falls into the frame of problem (2). Suppose that the joint distribution of the constraints rows $\mathcal{J}$ is driven by an Archimedean (for example, by the Gumbel–Hougaard) copula. If ξ is normally distributed random vector, then so is the product $A^k\xi$, and the problem is convex by Theorem 3 for sufficiently high value of p. The outer approximating SOCP problem of the form (10) is straightforward, and lower bound for the optimal value can be easily found as the side effect of the problem. The parameters μ_k, Σ_k are provided by probabilistic properties of $A^k\xi$. Using the exact form of the generator ψ of the Gumbel–Hougaard copula, we can provide the coefficients a_{kj}, b_{kj} in the following explicit form

$$a_{kj} := \Phi^{(-1)}\left(p^{y_{kj}^{1/\theta}}\right) - b_{kj} y_{kj}, \qquad b_{kj} := \frac{(-\ln p)^\theta p^{y_{kj}^{1/\theta}}}{\theta \phi\left(\Phi^{(-1)}\left(p^{y_{kj}^{1/\theta}}\right)\right)\left(\ln p^{y_{kj}^{1/\theta}}\right)^{\theta-1}}.$$

5 Conclusion

In this work we have provided a methodological note on a stochastic programming problem with chance constraints. In particular, a use of a specific class of Archimedean copulas having completely monotonic inverse generators on a problem with normally distributed rows was given. The problem is reformulated appropriately and the convexity of the feasible set is stated for a sufficiently high probability level. We also describe how the problem can be approximated by an second-order cone programming problem which is solvable by traditional optimization procedures. The accompanying (but still artificial) example on indicators of ecological stability offers an insight into a possible application of the methodology described.

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