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On Stolarsky inequality for Sugeno and Choquet integrals

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ABSTRACT

Recently Flores-Franulič, Román-Flores and Chalco-Cano proved the Stolarsky type inequality for Sugeno integral with respect to the Lebesgue measure λ. The present paper is devoted to generalize this result by relaxing some of its requirements. Moreover, Stolarsky inequality for Choquet integral is added, too.

1. Introduction

Non-additive measures and corresponding integrals can be used for modelling problems in non-additive environment. Since Sugeno [23] initiated research on fuzzy measure and fuzzy integral (known as Sugeno integral), this area has been widely developed and a wide variety of topics have been investigated (see, e.g., [3,7,19,21,25] and references therein).

Integral inequalities are an important aspect of the classical mathematical analysis [4,22]. Recently, Román-Flores and his collaborators generalized several classical integral inequalities to Sugeno integral (cf. [6,8]). Flores-Franulič and Román-Flores [6] provided a Chebyshev type inequality for Lebesgue measure-based Sugeno integral of continuous and strictly monotone functions. This inequality was generalized to arbitrary fuzzy measure-based Sugeno integral of monotone functions by Ouyang et al. [15]. Later, Mesiar, Ouyang and Li further generalized this inequality to a rather general form [12,16–18]. Jensen inequality was generalized in [20]. Some other inequalities are proved in [1,2]. In [8] Flores-Franulič et al. proved a Stolarsky type inequality for Lebesgue measure-based Sugeno integral and a continuous and strictly monotone function $f : [0,1] \to [0,1]$. In this contribution, we generalize this inequality to fuzzy measure-based Sugeno integral and a general monotone function *f*.

After recalling some basic concepts and known results in the next section, Section 3 presents our main results, as generalization of Stolarsky inequality for Sugeno integral obtained in [8], including illustrative examples. In Section 4, Stolarsky theorem for Choquet integral is shown. Finally, in Section 5, some concluding remarks are added.

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2. Preliminaries

In this section we recall some basic definitions and previous results which will be used in the sequel. Let $(X, \mathcal{A})$ be a measurable space, i.e., X is a non-void set and $\mathcal{A}$ a σ-algebra of subsets of X. Throughout this paper, all considered subsets are supposed to belong to $\mathcal{A}$. Let $\mathcal{F}_+(X)$ denote the set of all measurable functions $f : X \to [0,1]$ with respect to $\mathcal{A}$. For $f \in \mathcal{F}_+(X)$, we will denote by F_α the set $\{x \in X | f(x) \geqslant \alpha\}$ for $\alpha \geqslant 0$. Clearly, F_α is nonincreasing with respect to α, i.e., $\alpha \leqslant \beta$ implies $F_\alpha \supset F_\beta$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$, and we will often exploit the notation $f(a_+)$ or $f(a_-)$ for a function f defined on $[0,1]$ and $a \in]0,1[$, given by $f(a_+) = \lim_{\varepsilon \to 0^+} f(a+\varepsilon)$ and $f(a_-) = \lim_{\varepsilon \to 0^+} f(a-\varepsilon)$.

Definition 2.1 [23]. A set function $m : \mathcal{A} \to [0,1]$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $m(\varnothing) = 0, m(X) = 1$;
(FM2) $A \subset B$ implies $m(A) \leqslant m(B)$.

When m is a fuzzy measure, the triple $(X, \mathcal{A}, m)$ is called a fuzzy measure space.

Definition 2.2. [19,23,25]. Let $(X, \mathcal{A}, m)$ be a fuzzy measure space and $A \in \mathcal{A}$, the Sugeno integral of f on A, with respect to the fuzzy measure m, is defined by

$$(S)\int_A f\, dm = \bigvee_{\alpha \in [0,1]} (\alpha \wedge m(A \cap F_\alpha)).$$

When $A = X$, then

$$(S)\int_X f\, dm = (S)\int f\, dm = \bigvee_{\alpha \in [0,1]} (\alpha \wedge m(F_\alpha)).$$

The following theorem collects some basic properties of Sugeno integral, which can be verified directly.

Theorem 2.3. *Let $(X, \mathcal{A}, m)$ be a fuzzy measure space, then*

(*i*) $m(A \cap F_{\alpha-}) \geqslant \alpha \iff (S)\int_A f\, dm \geqslant \alpha$, *where* $m(A \cap F_{\alpha-}) = \lim_{\varepsilon \to 0} m(A \cap F_{\alpha-\varepsilon})$;
(*ii*) $m(A \cap F_{\alpha+}) \leqslant \alpha \iff (S)\int_A f\, dm \leqslant \alpha$, *where* $m(A \cap F_{\alpha+}) = \lim_{\varepsilon \to 0} m(A \cap F_{\alpha+\varepsilon})$;
(*iii*) $(S)\int_A f\, dm < \alpha \iff$ *there exists* $\gamma < \alpha$ *such that* $m(A \cap F_\gamma) < \alpha$;
(*iv*) $(S)\int_A f\, dm > \alpha \iff$ *there exists* $\gamma > \alpha$ *such that* $m(A \cap F_\gamma) > \alpha$;
(*v*) $(S)\int_A 1\, dm = m(A)$;
(*vi*) $A \subset B \to (S)\int_A f\, dm \leqslant (S)\int_B f\, dm$;
(*vii*) $f \leqslant g \to (S)\int_A f\, dm \leqslant (S)\int_A g\, dm$.

Note that by (i) and (ii) of the above theorem we infer that $(S)\int_A f\, dm = \alpha$ if and only if $m(A \cap F_{\alpha-}) \geqslant \alpha \geqslant m(A \cap F_{\alpha+})$. Recall that two functions $f, g : X \to \mathbb{R}$ are said to be comonotone if for all $(x, y) \in X^2$ we have

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0.$$

In [12] we proved a general Chebyshev inequality for Sugeno integral on abstract spaces. Although the fuzzy measure m over there is assumed to possess continuity, the result remains true if we abandon this assumption, that is, the following theorem holds.

Theorem 2.4. *Let $f, g \in \mathcal{F}_+(X)$ and m be an arbitrary fuzzy measure. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$(S)\int_A f \star g\, dm \geqslant \left((S)\int_A f\, dm\right) \star \left((S)\int_A g\, dm\right)$$

holds.

For a nondecreasing function $f : [0,1] \to [0,1]$, its pseudo-inverse $f^{(-1)} : [0,1] \to [0,1]$ is given by $f^{(-1)}(t) = \sup\{x \in [0,1] | f(x) > t\}$, see [9,10,24].

3. Main results

Our main aim is to generalize Stolarsky-type inequality for Sugeno integral which was proven under special constraints in [8]. Our next result considers the case on nonincreasing functions. Let $\mathcal{B}([0,1])$ be the Borel σ-algebra over $[0,1]$.

Theorem 3.1. *Let $f : [0,1] \to [0,1]$ be a nonincreasing function, $([0,1], \mathcal{B}([0,1]), m)$ a fuzzy measure space, and define $h : [0,1] \to [0,1]$ by $h(a) = m([0,a])$ for $a \in [0,1]$. Let β, γ be automorphisms on $[0,1]$ (i.e., $\beta, \gamma : [0,1] \to [0,1]$ are increasing bijections) and $\alpha = (\beta^{-1} \star \gamma^{-1})^{-1}$. If $\star : [0,1]^2 \to [0,1]$ is a continuous aggregation function which is jointly strictly increasing and bounded from above by min, and which is dominated by h, i.e., for all x, y from $[0,1]$ it holds*

$$h(x \star y) \geqslant h(x) \star h(y),$$

then

$$(S) \int_0^1 f(\alpha)\, dm \geqslant (S) \int_0^1 f(\beta)\, dm \star (S) \int_0^1 f(\gamma)\, dm, \tag{3.1}$$

where $f(\alpha)$ means the composite function defined on $[0,1]$ and given by $f(\alpha)(x) = f(\alpha(x))$.

Proof. Let $(S)\int_0^1 f(\alpha)\, dm = a$, $(S)\int_0^1 f(\beta)\, dm = b$ and $(S)\int_0^1 f(\gamma)\, dm = c$. Since the Sugeno integral w.r.t. a fuzzy measure $(m(X) = 1)$ is idempotent, we know that $a, b, c \in [f(1), f(0)]$. Since

$$(S) \int_0^1 f(\alpha)\, dm = \bigvee_{t \in [f(1), f(0)]} (t \wedge m(\{x | f(\alpha(x)) \geqslant t\})) = a,$$

we have

$$m(\{x | f(\alpha(x)) \geqslant a_-\}) \geqslant a \geqslant m(\{x | f(\alpha(x)) \geqslant a_+\}),$$

that is

$$m(\{x | x \leqslant \alpha^{-1}(f^{(-1)}(a_-))\}) \geqslant a \geqslant m(\{x | x \leqslant \alpha^{-1}(f^{(-1)}(a_+))\}),$$

i.e.,

$$h(\alpha^{-1}(f^{(-1)}(a_-))) \geqslant a \geqslant h(\alpha^{-1}(f^{(-1)}(a_+))),$$

where $f^{(-1)}$ stands for the pseudo-inverse of f. In the same way, we can prove that

$$h(\beta^{-1}(f^{(-1)}(b_-))) \geqslant b \geqslant h(\beta^{-1}(f^{(-1)}(b_+))),$$

and

$$h(\gamma^{-1}(f^{(-1)}(c_-))) \geqslant c \geqslant h(\gamma^{-1}(f^{(-1)}(c_+))).$$

On the other hand, by the fact that $\star$ is bounded from above by min we have

$$\alpha^{-1} = \beta^{-1} \star \gamma^{-1} \leqslant \beta^{-1} \wedge \gamma^{-1},$$

which implies that $\alpha \geqslant \beta, \alpha \geqslant \gamma$. Thus we have $f(\alpha) \leqslant f(\beta)$ and $f(\alpha) \leqslant f(\gamma)$. Moreover, by the monotonicity of Sugeno integral, we conclude that $a \leqslant b$ and $a \leqslant c$. If $a = b \wedge c$ then $a \geqslant b \star c$ and (3.1) holds. So without any loss of generality we can assume that $a < b \wedge c$. Hence it holds

$$\begin{aligned}(S) \int_0^1 f(\alpha)\, dm = a &\geqslant h(\alpha^{-1}(f^{(-1)}(a_+))) \geqslant h(\alpha^{-1}(f^{(-1)}((b \wedge c)_-))) = h(\beta^{-1}(f^{(-1)}((b \wedge c)_-)) \star \gamma^{-1}(f^{(-1)}((b \wedge c)_-))) \\ &\geqslant h(\beta^{-1}(f^{(-1)}((b \wedge c)_-))) \star h(\gamma^{-1}(f^{(-1)}((b \wedge c)_-))) \geqslant h(\beta^{-1}(f^{(-1)}(b_-))) \star h(\gamma^{-1}(f^{(-1)}(c_-))) \geqslant b \star c \\ &= (S) \int_0^1 f(\beta)\, dm \star (S) \int_0^1 f(\gamma)\, dm. \qquad \square\end{aligned}$$

Remark 3.2. If m is the Lebesgue measure λ then we have $h = id$ (the identity mapping) and $h(x \star y) \geqslant h(x) \star h(y)$ for any operator $\star$. For an arbitrary fuzzy measure m, the corresponding function h satisfies $h(x \star y) \geqslant h(x) \star h(y)$ considering the aggregation function $\star = \min$, i.e., then the constraints of Theorem 3.1 are satisfied.

The following example shows that the condition $\star \leqslant \min$ in Theorem 3.1 is necessary.

Example 3.3. Let $\star : [0,1]^2 \to [0,1]$ be defined as $x \star y = x + y - xy$ (i.e., $\star$ is the probabilistic sum S_P [10]) and $\beta = \gamma = id$. Then $\alpha(x) = 1 - \sqrt{1 - x}$. Let the fuzzy measure be defined by $m = \lambda^2$, where λ is the Lebesgue measure, then $h(x) = x^2$ and thus $h(x \star y) = (x + y - xy)^2 \geqslant x^2 + y^2 - x^2y^2 = h(x) \star h(y)$. If we take $f(x) = 1 - x$ for all $x \in [0,1]$ then we obtain

$$(S)\int_0^1 f(\beta)\,dm = (S)\int_0^1 f(\gamma)\,dm = \bigvee_{t\in[0,1]} (t\wedge(1-t)^2) = \frac{3-\sqrt{5}}{2},$$
$$(S)\int_0^1 f(\alpha)\,dm = (S)\int_0^1 \sqrt{1-x}\,dm = \bigvee_{t\in[0,1]} (t\wedge(1-t^2)^2).$$

Since $\left(1-\left(\frac{\sqrt{5}-1}{2}\right)^2\right)^2 = \left(\frac{\sqrt{5}-1}{2}\right)^2 < \frac{\sqrt{5}-1}{2}$, we conclude that

$$(S)\int_0^1 f(\alpha)\,dm < \frac{\sqrt{5}-1}{2} = (S)\int_0^1 f(\beta)\,dm \star (S)\int_0^1 f(\gamma)\,dm,$$

which contradicts with inequality (3.1).

The following example shows that the condition $h(x\star y) \geqslant h(x)\star h(y)$ cannot be omitted.

Example 3.4. Let $([0,1],\mathcal{B}[0,1],m)$ be a fuzzy measure space such that

$$m(A) = \min\left(2\lambda(A), \frac{\lambda(A)+1}{2}\right),$$

where λ is the Lebesgue measure. Then $h(x) = \min\left(2x, \frac{x+1}{2}\right)$. If $\star$ is the usual product, then we have

$$h\left(\frac{1}{4}\right) = \frac{1}{2} < 1 = h\left(\frac{1}{2}\right)h\left(\frac{1}{2}\right).$$

Let $\beta(x) = \gamma(x) = x^2$ for all $x\in[0,1]$, then $\alpha(x) = x$. Put $f(x) = \max(1-20x,0)$, then we have

$$(S)\int_0^1 f(\alpha)\,dm = (S)\int_0^1 \max(1-20x,0)\,dm = \frac{1}{11},$$
$$(S)\int_0^1 f(\beta)\,dm = (S)\int_0^1 f(\gamma)\,dm = (S)\int_0^1 \max(1-20x^2,0)\,dm = \frac{\sqrt{21}-1}{10}.$$

Therefore

$$(S)\int_0^1 f(\alpha)\,dm < \left((S)\int_0^1 f(\beta)\,dm\right)\left((S)\int_0^1 f(\gamma)\,dm\right),$$

which violates inequality (3.1).

Till now, only nonincreasing functions f were considered. In the next result, we generalize the Stolarsky-like theorem for Sugeno integral shown in [8] for nondecreasing functions.

Theorem 3.5. *Let $f:[0,1]\to[0,1]$ be a nondecreasing function and $([0,1],\mathcal{B}([0,1]),m)$ a fuzzy measure space. Let β,γ be automorphisms on $[0,1]$ (i.e., $\beta,\gamma:[0,1]\to[0,1]$ are increasing bijections) and $\alpha = (\beta^{-1}\star\gamma^{-1})^{-1}$. If $\star:[0,1]^2\to[0,1]$ is any continuous aggregation function which is jointly strictly increasing and bounded from above by min, then we have*

$$(S)\int_0^1 f(\alpha)\,dm \geqslant (S)\int_0^1 f(\beta)\,dm \star (S)\int_0^1 f(\gamma)\,dm.$$

Proof. Since the aggregation function $\star$ is bounded from above by min, $\alpha^{-1} = \beta^{-1}\star\gamma^{-1} \leqslant \beta^{-1}\wedge\gamma^{-1}$ and thus $\alpha\geqslant\beta, \alpha\geqslant\gamma$. Since f is nondecreasing, we have $f(\alpha)\geqslant f(\beta)\wedge f(\gamma)\geqslant f(\beta)\star f(\gamma)$. Note that all the three functions $f(\alpha), f(\beta)$ and $f(\gamma)$ are nondecreasing. Applying the Chebyshev type inequality for Sugeno integral (Theorem 2.4), by the monotonicity of Sugeno integral there holds

$$(S)\int_0^1 f(\alpha)\,dm \geqslant (S)\int_0^1 f(\beta)\star f(\gamma)\,dm \geqslant (S)\int_0^1 f(\beta)\,dm \star (S)\int_0^1 f(\gamma)\,dm. \qquad \square$$

Let $\beta = x^{\frac{1}{a}}, \gamma = x^{\frac{1}{b}}$ and $\star$ be the usual multiplication. Thus $\alpha = x^{\frac{1}{a+b}}$. If m is the Lebesgue measure λ, then we have the following result which generalize the main result of [8].

Corollary 3.6. *Let $f:[0,1]\to[0,1]$ be a monotone function and λ be the Lebesgue measure on $\mathbb{R}$. Then the inequality*

$$(S)\int_0^1 f(x^{\frac{1}{a+b}})\,d\lambda \geqslant \left((S)\int_0^1 f\left(x^{\frac{1}{a}}\right)d\lambda\right)\left((S)\int_0^1 f\left(x^{\frac{1}{b}}\right)d\lambda\right)$$

holds, where $a,b>0$.

4. Stolarsky inequality for Choquet integral

Recall that Choquet integral is defined on a fuzzy measure space $([0,1],\mathcal{B}([0,1]),m)$ for measurable function $f:[0,1]\to[0,1]$ by

$$(Ch)\int f\,dm=\int_0^1 m(f\geqslant t)\,dt,$$

where the right-hand side is the Riemann integral, for more details see [5,7,19]. In the case of lower-semicontinuous fuzzy measures, for any f and m there is a classical σ-additive measure m_f such that

$$(Ch)\int f\,dm=\int_0^1 f\,dm_f,$$

where the right-hand side is the Lebesgue integral. Moreover, if functions f and g are comonotone, then we can consider $m_f=m_g$. For more details see [13,14]. Now we are ready to introduce Stolarsky theorem for Choquet integral.

Theorem 4.1. *Let $([0,1],\mathcal{B}([0,1]),m)$ be a fuzzy measure space such that m is a lower-semicontinuous fuzzy measure absolutely continuous with respect to the Lebesgue measure λ, i.e., if $\lambda(E)=0$ for some $E\in\mathcal{B}([0,1])$ then also $m(E)=0$. Then the Stolarsky inequality holds for the corresponding Choquet integral, i.e., for any nonincreasing function $f:[0,1]\to[0,1]$ it holds*

$$(Ch)\int f(x^{\frac{1}{a+b}})\,dm\geqslant\left((Ch)\int f\left(x^{\frac{1}{a}}\right)\,dm\right)\left((Ch)\int f(x^{\frac{1}{b}})\,dm\right).$$

Proof. Observe that for a nonincreasing function $f:[0,1]\to[0,1]$, also the functions given by $f\left(x^{\frac{1}{a+b}}\right),f\left(x^{\frac{1}{a}}\right)$ and $f\left(x^{\frac{1}{b}}\right)$ are nonincreasing and they are comonotone for arbitrary $a>0$ and $b>0$. Then there is a σ-additive measure (in fact, a probability measure) m_f such that

$$(Ch)\int f(x^{\frac{1}{a+b}})\,dm=\int_0^1 f(x^{\frac{1}{a+b}})\,dm_f,$$
$$(Ch)\int f(x^{\frac{1}{a}})\,dm=\int_0^1 f(x^{\frac{1}{a}})\,dm_f,$$

and

$$(Ch)\int f(x^{\frac{1}{b}})\,dm=\int_0^1 f(x^{\frac{1}{b}})\,dm_f.$$

Moreover, m_f is absolutely continuous with respect to λ, and thus there is a Radon–Nikodym derivative $w=\frac{dm_f}{d\lambda}$ such that

$$(Ch)\int f(x^{\frac{1}{a+b}})\,dm=\int_0^1 f(x^{\frac{1}{a+b}})\,dm_f=\int_0^1 w(x)f(x^{\frac{1}{a+b}})\,d\lambda,$$
$$(Ch)\int f(x^{\frac{1}{a}})\,dm=\int_0^1 f(x^{\frac{1}{a}})\,dm_f=\int_0^1 w(x)f(x^{\frac{1}{a}})\,d\lambda,$$

and

$$(Ch)\int f(x^{\frac{1}{b}})\,dm=\int_0^1 f(x^{\frac{1}{b}})\,dm_f=\int_0^1 w(x)f(x^{\frac{1}{b}})\,d\lambda.$$

Now, it is enough to apply the Stolarsky inequality with general weights shown by Maligranda et al. in [11]. □

5. Conclusion

We have proved a Stolarsky type inequality for Sugeno integral on a fuzzy measure space $([0,1],\mathcal{B}([0,1]),m)$ based on a product-like operation ★. It generalizes the results of [8]. Moreover, we have introduced Stolarsky inequality also for Choquet integral acting on $X=[0,1]$. We believe that our results will contribute to approximation and estimation theory in information sciences systems when considering Sugeno or Choquet integral.

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