Information Sciences 257 (2014) 183–192

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Atoms of weakly null-additive monotone measures and integrals

Jun Li [a,*], Radko Mesiar [b,c], Endre Pap [d,e,f]

[a] School of Sciences, Communication University of China, Beijing 100024, China
[b] Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia
[c] UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
[d] Faculty of Sciences, University of Novi Sad, 21 000 Novi Sad, Serbia
[e] Óbuda University, H-1034 Budapest, Becsi út 96/B, Hungary
[f] Educons University, 21208 Sremska Kamenica, Serbia

ARTICLE INFO

Article history:
Received 18 January 2013
Received in revised form 29 July 2013
Accepted 3 September 2013
Available online 11 September 2013

Keywords:
Monotone measure
Atom
Weak null-additivity
Regularity
Sugeno integral
Choquet integral

ABSTRACT

In this paper, we prove some properties of atoms of weakly null-additive monotone measures. By using the regularity and weak null-additivity, a sin-gleton characterization of atoms of monotone measures on a metric space is shown. It is a generalization of previous results obtained by Pap. The calculation of the Sugeno integral and the Choquet integral over an atom is also presented, respectively. Similar results for recently introduced universal integral are also given. Following these results, it is shown that the Sugeno integral and the Choquet integral over an atom of monotone measure is maxitive linear and standard linear, respectively. Convergence theorems for the Sugeno integral and the Choquet integral over an atom of a monotone measure are also shown.

1. Introduction

An atom of a measure is an important concept in the classical measure theory [6] and probability theory. This concept was generalized in non-additive measure theory. The atoms for submeasures on locally compact Hausdorff spaces were discussed by Dobrakov [4]. In 1991, Suzuki [24] first introduced the concept of an atom of fuzzy measures (non-negative monotone set functions with continuity from below and above and vanishing at $\emptyset$), and investigated some analytical properties of atoms of fuzzy measures. Further research on this matter was made by Pap [19–22], Jiang and Suzuki [7,8], Li et al. [15], Wu and Sun [26], and Kawabe [9,11]. In [21] Pap showed a singleton characterization of atoms of regular null-additive monotone set functions, i.e., if a non-negative monotone set function μ is regular and null-additive, then every atom of μ has an outstanding property that all the mass of the atom is concentrated on a single point in the atom.

In this paper, we shall further investigate some properties of atoms of weakly null-additive monotone measures on metric spaces. We shall show that if a regular monotone measure is weakly null-additive, then the previous results obtained by Pap [21] remain valid. This fact makes easy the calculation of the Sugeno integral and the Choquet integral over an atom of a monotone measure which is regular and countably weakly null-additive. Following these results, it is shown that the Sugeno integral and the Choquet integral over an atom of a monotone measure is maxitive linear and standard linear (cf.[17]),

* Corresponding author. Tel.: +86 10 65783583.
E-mail addresses: lijun@cuc.edu.cn (J. Li), mesiar@math.sk (R. Mesiar), pap@dmi.uns.ac.rs (E. Pap).

0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2013.09.013

respectively. The convergence theorems for the Sugeno integral and the Choquet integral over an atom of a monotone measure are shown, too.

2. Preliminaries

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X, and $\mathbb{N}$ denote the set of all positive integers. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

A set function $\mu : \mathcal{F} \to [0,\infty]$ is said to be *continuous from below* [3], if $\lim_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$; *continuous from above* [3], if $\lim_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $A_n \searrow A$ and there exists n_0 with $\mu(A_{n_0}) < \infty$; *continuous*, if μ is continuous from below and above; *order continuous* [21], if $\lim_{n\to\infty}\mu(A_n) = 0$ whenever $A_n \searrow \emptyset$; *exhaustive* [4], if $\lim_{n\to\infty}\mu(E_n) = 0$ for any infinite disjoint sequence $\{E_n\}_{n\in N}$; *strongly order continuous* [13], if $\lim_{n\to\infty}\mu(A_n) = 0$ whenever $A_n \searrow A$ and $\mu(A) = 0$.

A set function μ is called *finite*, if $\mu(X) < \infty$; *σ-finite* [21], if there exists a sequence $\{X_n\} \subset \mathcal{F}$ such that

$$X_1 \subset X_2 \subset \cdots, \quad X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad \mu(X_n) < \infty \ (n = 1,2,\ldots).$$

Definition 2.1 [27]. A *monotone measure* on $\mathcal{F}$ is an extended real valued set function $\mu : \mathcal{F} \to [0,\infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$; (vanishing at $\emptyset$).
(2) $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$. (monotonicity).

When μ is a monotone measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space [21,27].

In this paper, we always assume that μ is a monotone measure on $\mathcal{F}$.

Definition 2.2 [27]. μ is called weakly null-additive, if for any $E,\ F \in \mathcal{F}$,

$$\mu(E) = \mu(F) = 0 \Rightarrow \mu(E \cup F) = 0.$$

In the following we recall several concepts related to the weak null-additivity of non-negative set functions.

Definition 2.3.

(i) μ is said to be *null-additive*, if $\mu(E \cup F) = \mu(E)$ whenever $E, F \in \mathcal{F}$ and $\mu(F) = 0$, see [27].
(ii) μ is said to be *weakly asymptotic null-additive*, if $\mu(E_n \cup F_n) \searrow 0$ whenever $\{E_n\}$ and $\{F_n\}$ are decreasing sequences with $\mu(E_n) \searrow 0$ and $\mu(F_n) \searrow 0$, see [10].
(iii) μ is said to have *pseudometric generating property* (for short, (p.g.p.)), if $\mu(E_n \cup F_n) \to 0$ whenever the sequences $\{E_n\} \subset \mathcal{F}$ and $\{F_n\} \subset \mathcal{F}$ with $\mu(E_n) \to 0$ and $\mu(F_n) \to 0$, see [5].

Obviously, the null-additivity of μ implies weak null-additivity. The pseudometric generating property implies weak asymptotic null-additivity, and the latter implies weak null-additivity.

Definition 2.4 [14]. μ is called *countably weakly null-additive*, if for any $\{A_n\}_{n\in N} \subset \mathcal{F}$,

$$\mu(A_n) = 0, \quad \forall n \geqslant 1 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0.$$

Definition 2.5 [1]. μ is called *null-continuous*, if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ for every increasing sequence $\{A_n\}_{n\in N} \subset \mathcal{F}$ such that $\mu(A_n) = 0,\ n = 1,2,\ldots$.

We give some relationships among the above introduced properties.

Proposition 2.6. *μ is countably weakly null-additive if and only if μ is both weakly null-additive and null-continuous.*

Proposition 2.7 [1]. *If μ is weakly null-additive and strongly order continuous, then it is null-continuous.*

Proposition 2.8. *μ is countably weakly null-additive if at least one of the following conditions holds:*

(i) *μ is weakly null-additive and strongly order continuous,*
(ii) *μ is weakly asymptotic null-additive and order continuous,*
(iii) *μ has (p.g.p.) and it is order continuous.*

Note 2.9. A weakly null-additive monotone measure may not be null-additive.

Example 2.10. Let $X = \{a, b\}$ and $\mathcal{F} = \mathcal{P}(X)$. Put

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ \frac{1}{2} & \text{if } E = \{b\}, \\ 0 & \text{if } E = \{a\} \text{ or } E = \emptyset. \end{cases}$$

Then obviously μ is a weakly null-additive monotone measure, but it is not null-additive.

Note 2.11. A null-continuous monotone measure may not be countably weakly null-additive.

Example 2.12. Let $X = \mathbb{N}$ and $\mathcal{F} = \mathcal{P}(\mathbb{N})$. Put

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \text{ or } E = \{n\} \ (n \in \mathbb{N}), \\ 1 & \text{otherwise}. \end{cases}$$

Then monotone measure μ satisfies continuity from below. Thus μ is also null-continuous. On the other hand, μ is not weakly null-additive, and hence it is neither null-additive nor countably weakly null-additive.

Note 2.13. A weakly null-additive monotone measure may not be countably weakly null-additive.

Example 2.14. Let $X = \mathbb{N}$ and $\mathcal{F} = \mathcal{P}(\mathbb{N})$. Put

$$\mu(E) = \begin{cases} 0 & \text{if } |E| < \infty, \\ 1 & \text{if } |E| = \infty, \end{cases}$$

where $|E|$ stands for the number of elements of E.

Obviously, μ is monotone measure and null-additive, and hence it is weakly null-additive. However μ is neither countably weakly null-additive nor null-continuous.

Observe that μ has (p.g.p.), but it is not order continuous.

Note 2.15. The weak null-additivity and strong order continuity are independent of each other.

Example 2.16. Let $X = \mathbb{N}$ and $\mathcal{F} = \mathcal{P}(\mathbb{N})$. Put

$$\mu(E) = \begin{cases} 0 & \text{if } |E| < \infty, \\ \sum_{i \in E} \frac{1}{2^i} & \text{if } |E| = \infty, \end{cases}$$

where $|E|$ stands for the number of elements of E.

Then μ is monotone measure and weakly null-additive. However μ is not strongly order continuous. Observe that μ is order continuous, but it has not (p.g.p.).

Example 2.17. Let $X = \{a, b\}$ and $\mathcal{F} = \mathcal{P}(X)$. Put

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \{a\}, \{b\}, \\ 1 & \text{if } E = X. \end{cases}$$

Then obviously μ is not weakly null-additive monotone measure, but it is strongly order continuous, and null-continuous.

Note 2.18. The order continuity and (p.g.p.) are independent of each other (see Examples 2.14 and 2.17 above).

Remark 2.19. In [21] Pap introduced the concept of σ-null-additivity for general set functions (see Definition 2.7 in [21]). When μ is a monotone measure on $\mathcal{F}$, the σ-null-additivity of μ implies countable weak null-additivity. A countably weakly null-additive monotone measure may not be σ-null-additive (see Example 2.10 above). From Proposition 2.3 in [21], we can obtain the following result: if μ is a monotone measure on $\mathcal{F}$, then μ is σ-null-additive if and only if μ is null-additive and countably weakly null-additive.

Let $\mathbf{F}_+$ denote the class of all nonnegative real-valued measurable functions on X. Let $f \in \mathbf{F}_+$, $E \in \mathcal{F}$. The *Sugeno (fuzzy) integral* of f on E with respect to μ, denoted by $(S)\int_E f\,d\mu$, is defined by

$$(S)\int_E f\,d\mu = \sup_{0 \leqslant \alpha < +\infty} [\alpha \wedge \mu(\{x : f(x) \geqslant \alpha\} \cap E)].$$

The *Choquet integral* of f on E with respect to μ, denoted by $(C)\int_E f\,d\mu$, is defined by

$$(C)\int_E f\,d\mu = \int_0^\infty \mu(\{x : f(x) \geqslant t\} \cap E)\,dt,$$

where the right side integral is Riemann integral.

3. Atoms of monotone measures and integrals on them

Definition 3.1 [24]. Let μ be a monotone measure on $\mathcal{F}$. A set $A \in \mathcal{F}$ is called an atom of μ if $\mu(A) > 0$ and for every $B \subset A$ from $\mathcal{F}$ holds either

(i) $\mu(B) = 0$, or

(ii) $\mu(A) = \mu(B)$ and $\mu(A - B) = 0$.

Proposition 3.2 [24]. *Every subset B of an atom A of μ is also an atom of μ if $\mu(B) > 0$.*

Proposition 3.3. *If μ is σ-finite and countably weakly null-additive, then for every atom A of μ, $\mu(A) < \infty$ always holds.*

Proof. Since μ is σ-finite, then there exists a sequence $\{X_n\} \subset \mathcal{F}$ such that $X_1 \subset X_2 \subset \cdots$ and

$$A = \bigcup_{n=1}^{\infty} (A \cap X_n) \quad \text{and} \quad \mu(A \cap X_n) < \infty \ (n = 1, 2, \ldots).$$

If we suppose that for every $n = 1, 2, \ldots, \mu(A \cap X_n) = 0$, then by the countable weak null-additivity of μ, we obtain $\mu(\bigcup_{n=1}^{\infty}(A \cap X_n)) = 0$, i.e., $\mu(A) = 0$. This is in the contradiction with $\mu(A) > 0$. Therefore, there exists n_0 such that $\mu(A \cap X_{n_0}) > 0$. Since A is atom of μ, we obtain by (ii) in Definition 3.1 that $\mu(A) = \mu(A \cap X_{n_0}) < \infty$. □

Proposition 3.4. *Let f be a nonnegative real-valued measurable function on $(X, \mathcal{F})$. If μ is countably weakly null-additive, then for any atom A of μ, there is a real number $\alpha^* = \alpha^*(f, A)$, such that $\mu(\{f = \alpha^*\} \cap A) = \mu(A)$. Furthermore, we have*

$$(S)\int_A f\,d\mu = \alpha^* \wedge \mu(A)$$

and

$$(C)\int_A f\,d\mu = \alpha^* \cdot \mu(A).$$

Proof. Put $R(A) = \{\alpha \geqslant 0 : \mu(\{f < \alpha\} \cap A) = 0\}$, and $\alpha^* = \sup R(A)$, i.e.,

$$\alpha^* = \sup\{\alpha \geqslant 0 : \mu(\{f < \alpha\} \cap A) = 0\}.$$

Observe that $0 \in R(A)$, i.e., $R(A) \neq \emptyset$. Suppose that $R(A)$ is unbounded. Then there exist a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subset R(A)$ such that $\alpha_n \nearrow \infty (n \to \infty)$. Then,

$$\{f < \alpha_n\} \cap A \nearrow \bigcup_{n=1}^{\infty}(\{f < \alpha_n\} \cap A) = A.$$

Therefore, noting that $\alpha_n \in R(A)$, we have

$$\mu(\{f < \alpha_n\} \cap A) = 0, \quad n = 1, 2, \ldots.$$

Since μ is countably weakly null-additive, then

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty}(\{f < \alpha_n\} \cap A)\right) = 0.$$

This is in contradiction with $\mu(A) > 0$. So $R(A)$ is bounded and hence $\alpha^* \in [0, \infty)$.

Choose a sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset R(A)$ such that $\beta_n \nearrow \alpha^*(n \to \infty)$. Then

$$\{f < \beta_n\} \cap A \nearrow \{f < \alpha^*\} \cap A \quad (n \to \infty),$$

i.e.,

$$\{f < \alpha^*\} \cap A = \bigcup_{n=1}^{\infty}(\{f < \beta_n\} \cap A).$$

Since $\beta_n \in R(A)$,

$$\mu(\{f < \beta_n\} \cap A) = 0, \quad n = 1, 2, \ldots,$$

and by using the countably weak null-additivity of μ, then

$$\mu(\{f < \alpha^*\} \cap A) = \mu\left(\bigcup_{n=1}^{\infty}(\{f < \beta_n\} \cap A)\right) = 0.$$

Thus we can get

$$\mu(\{f \geqslant \alpha^*\} \cap A) = \mu(A).$$

Indeed, if $\mu(\{f \geqslant \alpha^*\} \cap A) = 0$, noting that $\mu(\{f < \alpha^*\} \cap A) = 0$ and μ is weakly null-additive, then we have

$$\mu(A) = \mu((\{f \geqslant \alpha^*\} \cap A) \cup (\{f < \alpha^*\} \cap A)) = 0.$$

This is impossible. Thus we have $\mu(\{f \geqslant \alpha^*\} \cap A) > 0$. Since A is an atom of μ, we get $\mu(\{f \geqslant \alpha^*\} \cap A) = \mu(A)$.

On the other hand, for any $\alpha > \alpha^*$, we have

$$\mu(\{f \geqslant \alpha\} \cap A) = 0.$$

In fact, from the definition of α^*, $\mu(\{f < \alpha\} \cap A) > 0$. Since A is atom, then $\mu(\{f < \alpha\} \cap A) = \mu(A)$ and $\mu(A - \{f < \alpha\} \cap A) = 0$, i.e., $\mu(\{f \geqslant \alpha\} \cap A) = 0$.

Take a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$, $\gamma_n > \alpha^*$ $(n = 1, 2, \ldots)$ such that $\gamma_n \searrow \alpha^*$ $(n \to \infty)$, then we have

$$\{f > \alpha^*\} \cap A = \bigcup_{n=1}^{\infty}(\{f \geqslant \gamma_n\} \cap A).$$

Since $\mu(\{f \geqslant \gamma_n\} \cap A) = 0$ $n = 1, 2, \ldots$, and by using the countably weak null-additivity, we have $\mu(\{f > \alpha^*\} \cap A) = 0$.

If we suppose that $\mu(\{f = \alpha^*\} \cap A) = 0$, then, by applying the weak null-additivity of μ and

$$\{f \geqslant \alpha^*\} \cap A = (\{f > \alpha^*\} \cap A) \cup (\{f = \alpha^*\} \cap A),$$

we have $\mu(\{f \geqslant \alpha^*\} \cap A) = 0$. This is in contradiction with

$$\mu(\{f \geqslant \alpha^*\} \cap A) = \mu(A).$$

Therefore $\mu(\{f = \alpha^*\} \cap A) > 0$, and hence

$$\mu(\{f = \alpha^*\} \cap A) = \mu(A).$$

Now we show the second part of conclusions.

Using that $\mu(\{f \geqslant \alpha\} \cap A) = 0$ $(\alpha > \alpha^*)$ and $\mu(\{f \geqslant \alpha\} \cap A) = \mu(A)$ $(\alpha \leqslant \alpha^*)$, we obtain

$$(S)\int_A f\,d\mu = \sup\{\alpha \wedge \mu(\{f \geqslant \alpha\}\cap A\}) : 0 \leqslant \alpha \leqslant \alpha^*\} = \alpha^* \wedge \mu(A)$$

and

$$\begin{aligned}(C)\int_A f\,d\mu &= \int_0^\infty \mu(\{x : f(x) \geqslant t\}\cap A)\,dt \\ &= \int_0^{\alpha^*} \mu(\{x : f(x) \geqslant t\}\cap A)\,dt \\ &+ \int_{\alpha^*}^\infty \mu(\{x : f(x) \geqslant t\}\cap A)\,dt \\ &= \int_0^{\alpha^*} \mu(\{x : f(x) \geqslant t\}\cap A)\,dt = \alpha^* \cdot \mu(A). \quad \square\end{aligned}$$

Remark 3.5. Recently, Klement et al. [12] have introduced the concept of universal integral based on a pseudo-multiplication $\otimes : [0,\infty]^2 \to [0,\infty]$ (nondecreasing in each component, 0 is annihilator, there exists neutral element e different from 0) satisfying two basic properties:

(i) $(U)\int_E c\,d\mu = (U)\int_X c \cdot 1_E\,d\mu = c \otimes \mu(E)$ for any monotone measure μ on $(X,\mathcal{F})$ and, any real constant $c \in [0,\infty]$ and any $E \in \mathcal{F}$.

(ii) $(U) - \int_X f\,d\mu = (U)\int_X g\,dv$ for any $f, g \in \mathbf{F}_+$ and any monotone measures μ, v on $(X,\mathcal{F})$ such that

$$\mu(\{x : f(x) \geqslant t\}) = v(\{x : g(x) \geqslant t\})$$

for all $t \in]0,\infty[$.

(For more details see [12]).

Now it is evident that, under the constraint of Proposition 3.4, it holds

$$\mu(\{x : f \cdot 1_A(x) \geqslant t\}) = \mu(\{x : \alpha^* \cdot 1_A(x) \geqslant t\}),$$

$t \in]0,\infty[$, and hence

$$(U)\int_A f\,d\mu = (U)\int_X f \cdot 1_A\,d\mu = (U)\int_X \alpha^* \cdot 1_A\,d\mu = (U)\int_A \alpha^*\,d\mu = \alpha^* \otimes \mu(A).$$

As a corollary of this general result, the second part of Proposition 3.4 is obtained, noting that $\otimes = \wedge$ in the case of the Sugeno integral, while $\otimes = \cdot$ when considering the Choquet integral, both mentioned integrals being particular universal integrals.

Observe that the weak null-additivity of μ in Proposition 3.4 is a sufficient but not necessary condition.

Example 3.6. Consider a finite set X and $\mathcal{F} = \mathcal{P}(X)$, and a finite monotone measure $\mu : \mathcal{P}(X) \to [0,\infty[$ with unique atom A. Then necessarily μ is given by $\mu(E) \geqslant d$ if $E \supseteq A$ and $\mu(E) = 0$ otherwise, with $d = \mu(A) \in]0,\infty[$. For any universal integral related to a pseudo-multiplication $\otimes : [0,\infty]^2 \to [0,\infty]$, it holds, for any $f \in \mathbf{F}_+$,

$$(U)\int_A f\,d\mu = \alpha^* \otimes d,$$

where $\alpha^* = \min\{f(x) : x \in A\}$. Note that μ is weakly null-additive only if A is a singleton (compare Theorem 4.6, see also Example 2.10 with $A = \{b\}$). Hence if card $(A) > 1$, we see that the conclusion of Proposition 3.4 may be satisfied also for monotone measures which are not weakly null-additive.

4. Atoms of regular monotone measures and integrals on them

In this section, we suppose that X is a metric space, and that $\mathcal{O}$ and $\mathcal{K}$ are the classes of all open and compact subsets in X, respectively. $\mathcal{F}$ denotes Borel σ-algebra on X, i.e., it is the smallest σ-algebra containing $\mathcal{O}$.

Definition 4.1. [18,21,25]A monotone measure μ is called regular, if for each $A \in \mathcal{F}$ and each $\epsilon > 0$, there exist a compact set $K_\epsilon \in \mathcal{K}$ and an open set $G_\epsilon \in \mathcal{O}$ such that

$$K_\epsilon \subset A \subset G_\epsilon \quad \text{and} \quad \mu(G_\epsilon - K_\epsilon) < \epsilon.$$

Obviously, if μ is regular, then for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exists a compact set $K_\epsilon \in \mathcal{K}$ such that

$$K_\epsilon \subset A \quad \text{and} \quad \mu(A - K_\epsilon) < \epsilon.$$

From the definitions of atom and regularity, we can obtain the following properties.

Proposition 4.2. *Let μ be a regular monotone measure. If A is an atom of μ, then there exists a compact set $K \in \mathcal{K}$ such that $K \subset A$ and $\mu(A - K) = 0$.*

Proposition 4.3. *Let μ be a weakly null-additive monotone measure and A be an atom of μ. If $B \subset A$ and $\mu(A - B) = 0$, then $\mu(B) > 0$ and $\mu(B) = \mu(A)$.*

Proposition 4.4. *Let μ be a weakly null-additive regular monotone measure and A be an atom of μ. If B is a subset of A with $\mu(B) > 0$, then there exists compact subset $K \subset B$ such that $\mu(K) > 0$ and $\mu(K) = \mu(B) = \mu(A)$.*

Remark 4.5. Propositions 4.2, 4.3 and 4.4 hold for any Hausdorff space (cf.[23]).

The following result gives a singleton characterization of atoms of regular weakly null-additive monotone measures.

Theorem 4.6. *Let μ be a weakly null-additive regular monotone measure. If A is an atom of μ, then there exists a unique point $a \in A$ such that*

$$\mu(A) = \mu(\{a\}).$$

Proof. Let $A \in \mathcal{F}$ be an arbitrarily fixed atom of μ. We denote by $\mathcal{K}_1$ the family of all compact sets $K \subset A$ such that $\mu(A - K) = 0$, i.e.,

$$\mathcal{K}_1 = \{K \in \mathcal{K} : K \subset A, \mu(A - K) = 0\}.$$

From Propositions 4.2 and 4.3 we know that $\mathcal{K}_1$ is non-empty and for any $K \in \mathcal{K}_1$, $\mu(K) > 0$, and hence K is an atom of μ and $\mu(K) = \mu(A)$.

If $K_1, K_2 \in \mathcal{K}_1$, by the weak null-additivity of μ, we have

$$\mu(A - K_1 \cap K_2) = \mu((A - K_1) \cup (A - K_2)) = 0$$

and hence $K_1 \cap K_2 \in \mathcal{K}_1$. This indicates that $\mathcal{K}_1$ possesses finite intersection property, i.e., for any finite sets from $\mathcal{K}_1$, intersection of these finite sets is non-empty.

We take

$$K_0 = \bigcap_{K \in \mathcal{K}_1} K,$$

then K_0 is a non-empty compact set.

We shall show that $K_0 \in \mathcal{K}_1$, i.e., $\mu(A - K_0) = 0$. If that is not true, i.e., $\mu(A - K_0) > 0$. Then, from Proposition 4.4, there exists $\widehat{K}_1 \in \mathcal{K}_1$ such that $\widehat{K}_1 \subset A - K_0$ and $\mu(\widehat{K}_1) > 0$. Since A is an atom of μ, we have $\mu(A) = \mu(\widehat{K}_1)$ and $\mu(A - \widehat{K}_1) = 0$. Thus $\widehat{K}_1 \in \mathcal{K}_1$, and hence $K_0 \subset \widehat{K}_1$. This is in contradiction with the fact that $K_0 \cap \widehat{K}_1 = \emptyset$. Therefore $\mu(A - K_0) = 0$. This shows $K_0 \in \mathcal{K}_1$.

Now we shall show that K_0 is a singleton set. Suppose that it is not, i.e., there exist two elements $t_1, t_2 \in K_0, t_1 \neq t_2$. Let U be an open neighborhood of t_1 such that $\overline{U}$ does not contain t_2. Then

$$K_0 = (K_0 - U) \cup (K_0 \cap \overline{U}),$$

and $K_0 \neq (K_0 - U)$, $K_0 \neq (K_0 \cap \overline{U})$. Since μ is weakly null-additive, we have $\mu(K_0 - U) > 0$ or $\mu(K_0 \cap \overline{U}) > 0$ (otherwise, by the weak null-additivity of μ, we get $\mu(K_0) = 0$ which is impossible, since $K_0 \in \mathcal{K}_1, \mu(K_0) > 0$). Thus, one of the sets $K_0 - U$ or $K_0 \cap \overline{U}$ must belong to $\mathcal{K}_1$, but K_0 is the least element from $\mathcal{K}_1$, we obtain a contradiction. Therefore, there exists $a \in A$ such that $K_0 = \{a\}$ and, noting that $\mu(A) = \mu(K_0)$, we have

$$\mu(A) = \mu(\{a\}).$$

Now we show that the point a is unique. Suppose that there exist two points $a_1, a_2 \in A$, $a_1 \neq a_2$, such that

$$\mu(A) = \mu(\{a_1\}) = \mu(\{a_2\}).$$

Since $\mu(A - \{a_1\}) = 0$, and $\{a_2\} \subset A - \{a_1\}$, thus we get $\mu(\{a_2\}) = 0$ which contradicts the fact that $\mu(\{a_2\}) = \mu(A) > 0$.

The proof is now complete. □

Since null-additivity implies weak null-additivity, as a direct result of Theorem 4.6, we get the following corollary, which is a generalization of the result obtained by Pap [21].

Corollary 4.7 (*Pap[16, Theorem 9.6]*). *Let μ be a regular null-additive monotone measure. If A is an atom of μ, then there exists a point $a \in A$ such that*

$$\mu(A) = \mu(\{a\}).$$

Corollary 4.8. *Let μ be a regular and weakly asymptotic null-additive (or has (p.g.p.)). If A is an atom of μ, then the conclusion of* Theorem 4.6 *holds.*

From Proposition 3.4 and Theorem 4.6, we can obtain the following result.

Corollary 4.9. *Let μ be a regular and countably weakly null-additive monotone measure. If A is an atom of μ, then there exists a unique point $a \in A$ such that*

$$(S)\int_A f\,d\mu = f(a) \wedge \mu(\{a\})$$

and

$$(C)\int_A f\,d\mu = f(a) \cdot \mu(\{a\})$$

for any non-negative measurable function *f*.

Corollary 4.10. *Under the conditions of* Corollary 4.9, *if A is an atom of μ, then for any non-negative measurable functions f and g on A and $\alpha \geqslant 0$,*

$$(S)\int_A (\alpha \wedge f)\,d\mu = \alpha \wedge (S)\int_A f\,d\mu$$

and

$$(S)\int_A (f \vee g)\,d\mu = (S)\int_A f\,d\mu \vee (S)\int_A g\,d\mu.$$

Proof. Applying Corollary 4.9 to the measurable function $\alpha \wedge f$, there exists a unique point $a \in A$ such that

$$(S)\int_A (\alpha \wedge f)d\mu = (\alpha \wedge f)(a) \wedge \mu(\{a\}),$$

i.e.,

$$(S)\int_A (\alpha \wedge f)\,d\mu = (\alpha \wedge f(a)) \wedge \mu(\{a\}) = \alpha \wedge [f(a) \wedge \mu(\{a\})].$$

Since the point a in Corollary 4.9 is unique, we have

$$(S)\int_A f\,d\mu = f(a) \wedge \mu(\{a\}),$$

and therefore

$$(S)\int_A (\alpha \wedge f)\,d\mu = \alpha \wedge (S)\int_A f\,d\mu.$$

The proof of the rest is similar. □

Corollary 4.11. *Under the conditions of* Corollary 4.9, *if A is an atom of μ, then for any non-negative measurable functions f and g on A and $\alpha \geqslant 0$,*

$$(C)\int_A (\alpha \cdot f)\,d\mu = \alpha \cdot (C)\int_A f\,d\mu$$

and

$$(C)\int_A (f + g)\,d\mu = (C)\int_A f\,d\mu + (C)\int_A g\,d\mu.$$

Proof. It is similar to the proof of Corollary 4.10. □

Definition 4.12. Let $f \in \mathbf{F}_+$ and $\{f_n\} \subset \mathbf{F}_+$. $\{f_n\}$ is said to *converge to f almost everywhere* on A, in symbols $f_n \xrightarrow[A]{\text{a.e.}} f$, if there is a subset $N \subset A$ such that $\mu(N) = 0$ and f_n converges to f on $A - N$.

Corollary 4.13. *Under the conditions of* Corollary 4.9, *if A is an atom of μ, then for any $f \in \mathbf{F}_+$ and $\{f_n\} \subset \mathbf{F}_+$, $f_n \xrightarrow[A]{\text{a.e.}} f$ $(n \to \infty)$ imply*

$$\lim_{n\to\infty} (S)\int_A f_n\,d\mu = (S)\int_A f\,d\mu$$

and

$$\lim_{n\to\infty}(C)\int_A f_n\,d\mu = (C)\int_A f\,d\mu.$$

Proof. Since $f_n \xrightarrow[A]{\text{a.e.}} f$, there is a subset $N \subset A$ such that $\mu(N) = 0$ and f_n converges to f on $A - N$. From Theorem 4.6, there exists a unique point $a \in A$ such that

$$\mu(A) = \mu(\{a\})$$

Since A is atom, $\mu(A) = \mu(\{a\}) > 0$, so the point $a \in A - N$. Thus, as $n \to \infty$, $f_n(a) \to f(a)$. Therefore, as $n \to \infty$,

$$f_n(a) \wedge \mu(\{a\}) \to f(a) \wedge \mu(\{a\}),$$

and

$$f_n(a)\cdot\mu(\{a\}) \to f(a)\cdot\mu(\{a\}).$$

By using Corollary 4.9, we have

$$\lim_{n\to\infty}(S)\int_A f_n\,d\mu = (S)\int_A f\,d\mu.$$

and

$$\lim_{n\to\infty}(C)\int_A f_n\,d\mu = (C)\int_A f\,d\mu. \quad \square$$

Similarly, we can get the following corollary.

Corollary 4.14. *Under the conditions of Corollary 4.9, if A is an atom of μ, and $\{f_n\} \subset \mathbf{F}_+$ and $\sum_{n=1}^{\infty} f_n < \infty$, then*

$$(C)\int_A \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty}(C)\int_A f_n\,d\mu.$$

From Proposition 2.6, 2.8, we can obtain directly the following corollary.

Corollary 4.15. *Let μ be a regular monotone measure, and let μ satisfy one of the following conditions:*

(i) *μ is weakly null-additive and null-continuous;*
(ii) *μ is weakly null-additive and strongly order continuous;*
(iii) *μ is weakly asymptotic null-additive and order continuous;*
(iv) *μ has (p.g.p.) and, is order continuous.*

If A is an atom of μ, then the conclusions of Corollaries (4.9–4.14) *hold.*

In [9] Kawabe discussed the regularity of continuous monotone measure on a complete and separable metric space. The following result was established in a fairly general setting in [9].

Proposition 4.16. *Let μ be finite continuous monotone measure. If μ is weakly null-additive, then for each $A \in \mathcal{F}$ and each $\epsilon > 0$, there exist a compact set $K_\epsilon \in \mathcal{K}$ and an open set $G_\epsilon \in \mathcal{O}$ such that*

$$K_\epsilon \subset A \subset G_\epsilon \quad \text{and} \quad \mu(G_\epsilon - K_\epsilon) < \epsilon.$$

When X is a complete and separable metric space, from Corollary 4.15 and Proposition 4.16, we can obtain the following result:

Theorem 4.17. *Let μ be finite continuous monotone measure, and let μ satisfies at least one of the following conditions:*

(i) *μ is weakly null-additive;*
(ii) *μ is null-additive;*
(iii) *μ is weakly asymptotic null-additive;*
(iv) *μ has (p.g.p.).*

If A is an atom of μ, then the conclusions of Theorem 4.6, Corollaries 4.9, 4.10, 4.11, 4.13 and 4.14 *hold.*

Remark 4.18. When X is a regular space (cf.[23]), we can easily know that Theorem 4.6, Corollaries 4.7–4.15 remain valid.

5. Concluding remarks

We have introduced and discussed some properties of atoms of weakly null-additive monotone measures and their impact on Sugeno and Choquet integrals. Our results from Section 3 and 4 can be extended for any universal integral [12] in the following sense: for any n-ary operation $H : [0,\infty]^n \to [0,\infty]$, non-decreasing in each component, left distributive over a pseudo-multiplication $\otimes$ (i.e., $H(a_1,\dots,a_n)\otimes b = H(a_1\otimes b,\dots,a_n\otimes b)$) and any universal integral linked to $\otimes$, it holds

$$(U)\int_A H(f_1,\dots,f_n)\,d\mu = H\left((U)\int_A f_1\,d\mu,\dots,(U)\int_A f_n\,d\mu\right),$$

where $f_1,\dots,f_n \in \mathbf{F}_+$ (note that f_i can be a constant function, too). In particular, Benvenuti integral introduced and discussed in [2] satisfies the left distributivity of the applied pseudo-addition $\oplus : [0,\infty]^2 \to [0,\infty]$ over the corresponding pseudo-multiplication $\otimes$ and thus this integral is pseudo-linear.

Acknowledgement

The research summarized in this paper was supported by the NSFC Grant No. 11371332, and the grants VEGA 1/0171/12, GAČR P402/11/0378, and MNPRS 174009.

References

[1] S. Asahina, K. Uchino, T. Murofushi, Relationship among continuity conditions and null-additivity conditions in non-additive measure theory, Fuzzy Sets and Systems 157 (2006) 691–698.
[2] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals, in: E. Pap (Ed.), Handbook of Measure Theory, vol. II, Elsevier Science, 2002, pp. 1329–1379.
[3] D. Denneberg, Non-Additive Measure and Integral, The Netherlands, Kluwer, Dordrecht, 1994.
[4] I. Dobrakov, On submeasures I, Dissertations Mathematica 112 (1974) 1–35.
[5] I. Dobrakov, J. Farkova, On submeasures II, Mathematica Slovaca 30 (1980) 65–81.
[6] P.R. Halmos, Measure Theory, Van Nostrand, New York, 1968.
[7] Q. Jiang, H. Suzuki, Lebesgue and Saks decomposition of σ-finite fuzzy measures, Fuzzy Sets and Systems 75 (1995) 373–385.
[8] Q. Jiang, H. Suzuki, Fuzzy measures on metric spaces, Fuzzy Sets and Systems 83 (1996) 99–106.
[9] J. Kawabe, The Alexandroff theorem for Riesz space-valued non-additive measures, Fuzzy Sets and Systems 158 (2007) 2413–2421.
[10] J. Kawabe, Continuity and compactness of the indirect product of two non-additive measures, Fuzzy Sets and Systems 160 (2009) 1327–1333.
[11] J. Kawabe, Regularities of Riesz space-valued non-additive measures with applications to convergence theorems for Choquet integrals, Fuzzy Sets and Systems 161 (2010) 642–650.
[12] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Transactions on Fuzzy Systems 18 (2010) 178–187.
[13] J. Li, Order continuity of monotone set function and convergence of measurable functions sequence, Applied Mathematics and Computation 135 (2003) 211–218.
[14] J. Li, R. Mesiar, Lusin's theorem on monotone measure spaces, Fuzzy Sets and Systems 175 (2011) 75–86.
[15] J. Li, M. Yasuda, J. Song, Regularity properties of null-additive fuzzy measures on metric spaces, in: V. Torra, Y. Narukawa, Y. Miyamoto (Eds.), MDAI 2005, LNCS(LNAI), vol. 3558, Springer, Heidelberg, 2005, pp. 59–66.
[16] R. Mesiar, Choquet-like integrals, Journal of Mathematical Analysis and Applications 194 (1995) 477–488.
[17] R. Mesiar, Fuzzy integrals and linearity, International Journal of Approximation Reasoning 47 (2008) 352–358.
[18] Y. Narukawa, T. Murofushi, Regular null-additive measure and Choquet integral, Fuzzy Sets and Systems 143 (2004) 487–492.
[19] E. Pap, Regular Borel t-decomposition measures, Univ. u Novom Sadu. Ser. Mat. 22 (2) (1990) 113–120.
[20] E. Pap, The range of null-additive fuzzy and non-fuzzy measure, Fuzzy Sets and Systems 65 (1994) 105–115.
[21] E. Pap, Null-Additive Set Functions, Kluwer, Dordrecht, 1995.
[22] E. Pap, Regular null additive monotone set functions, Univ. u Novom Sadu Zb. rad Prirod. -Mat. Fak. Ser. mat. 25 (2) (1995) 93–101.
[23] H.L. Royden, Real Analysis, 3rd ed., Macmillan, New York, 1988.
[24] H. Suzuki, Atoms of fuzzy measures and fuzzy integrals, Fuzzy Sets and Systems 41 (1991) 329–342.
[25] C. Wu, M. Ha, On the regularity of the fuzzy measure on metric fuzzy measure spaces, Fuzzy Sets and Systems 66 (1994) 373–379.
[26] C. Wu, B. Sun, Pseudo-atoms of fuzzy and non-fuzzy measures, Fuzzy Sets and Systems 158 (2007) 373–379.
[27] Z. Wang, G.J. Klir, Generalized Measure Theory, Springer, 2009.