Fuzzy Optim Decis Making (2014) 13:273–286
DOI 10.1007/s10700-014-9182-4

Universal integrals based on copulas

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Published online: 7 February 2014

Abstract A hierarchical family of integrals based on a fixed copula is introduced and discussed. The extremal members of this family correspond to the inner and outer extension of integrals of basic functions, the copula under consideration being the corresponding multiplication. The limits of the members of the family are just copula-based universal integrals as recently introduced in Klement et al. (IEEE Trans Fuzzy Syst 18:178–187, 2010). For the product copula, the family of integrals considered here contains the Choquet and the Shilkret integral, and it belongs to the class of decomposition integrals proposed in Even and Lehrer (Econ Theory, 2013) as well as to the class of superdecomposition integrals introduced in Mesiar et al. (Superdecomposition integral, 2013). For the upper Fréchet-Hoeffding bound, the corresponding

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hierarchical family contains only two elements: all but the greatest element coincide with the Sugeno integral.

Keywords Capacity · Copula · Universal integral · Choquet integral · Sugeno integral · Shilkret integral

1 Introduction

Integrals aggregate the knowledge contained in a measure (for example, describing the weights of criteria sets) and in a function (for example, a score vector) into one representative value. Our aim is to discuss copula-based integrals, where copulas model the relationship between the values of the functions, measures and integrals under consideration.

Copulas are tools describing the stochastic dependence structure of random vectors (Joe 1997; Nelsen 2006; Sklar 1959). They were also applied in several decision problems, such as the description of joint distribution functions for Dempster-Shafer belief structures (Yager 2013), or in connection with the transitivity of fuzzy preference relations (Díaz et al. 2010).

Our framework for the integrals is the concept of universal integrals in the sense of Klement et al. (2010) which can be defined for arbitrary measurable spaces, arbitrary monotone measures and arbitrary measurable functions, and which generalize well-known integrals such as the Choquet (1954), Shilkret (1971) and Sugeno (1974) integral.

We consider *measurable spaces* $(X, \mathscr{A})$, where $\mathscr{A}$ is a σ-algebra of subsets of the universe X, and denote by $\mathscr{S}$ the class of all measurable spaces. For a given measurable space $(X, \mathscr{A})$, $\mathscr{F}_{(X,\mathscr{A})}$ denotes the set of all $\mathscr{A}$-measurable functions $f : X \to [0, 1]$. We also consider the set $\mathscr{M}_{(X,\mathscr{A})}$ of all *monotone measures* (*capacities*), i.e., set functions $m : \mathscr{A} \to [0, 1]$ satisfying $m(A) \leq m(B)$ whenever $A \subseteq B$, and the boundary conditions $m(\emptyset) = 0$ and $m(X) = 1$.

Given a measurable space $(X, \mathscr{A})$, for each $c \in [0, 1]$ and each $A \in \mathscr{A}$, the *basic function* $c \cdot \mathbf{1}_A : X \to [0, 1]$, where $\mathbf{1}_A : X \to \{0, 1\}$ is the *characteristic function* of the subset A of X, is an element of $\mathscr{F}_{(X,\mathscr{A})}$. These basic functions (see also Benvenuti et al. 2002) play a fundamental role in the characterization and decomposition of measurable functions: for each $f \in \mathscr{F}_{(X,\mathscr{A})}$ we have

$$f = \lim_{n\to\infty}\Big(\sum_{i=1}^{n} \tfrac{i}{n} \cdot \mathbf{1}_{\{\frac{i}{n} \leq f < \frac{i+1}{n}\}}\Big) = \lim_{n\to\infty}\Big(\sum_{i=1}^{n} \tfrac{1}{n} \cdot \mathbf{1}_{\{f \geq \frac{i}{n}\}}\Big).$$

Therefore, a natural way to construct an integral with respect to a given monotone measure is to define first the integral of the basic functions and then to construct the integral of arbitrary measurable functions.

We adopt here the philosophy of *universal integrals* recently introduced and studied in Klement et al. (2010), where one of the axioms requires that for all $m \in \mathscr{M}_{(X,\mathscr{A})}$, $c \in [0, 1]$ and $A \in \mathscr{A}$,

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A)$$

for some fixed *pseudo-multiplication* $\otimes : [0,1]^2 \to [0,1]$ (see Klement et al. 2010), a binary operation which is monotone increasing in each coordinate, has 0 as annihilator and a neutral element different from 0. Note that the monotonicity of $\otimes$ in both coordinates is an immediate consequence if we require the integral to be monotone both with respect to the underlying measures and with respect to the integrand. If we also suppose that for each $c \in [0,1]$ and for each $A \in \mathscr{A}$ we have $\mathbf{I}(m, 1_A) = m(A)$ and $\mathbf{I}(m, c \cdot \mathbf{1}_X) = c$, then it follows that 1 is a neutral element of $\otimes$, i.e., $\otimes$ is a semicopula (see Bassan and Spizzichino 2005; Durante and Sempi 2005).

In this paper, we will restrict ourselves to a special class of semicopulas, namely, the class of all (binary) copulas (Sklar 1959). Recall that $C : [0,1]^2 \to [0,1]$ is a copula whenever it is a supermodular semicopula, i.e., a semicopula satisfying, for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y}).$$

There is a one-to-one correspondence between binary copulas and probability measures $P : \mathscr{B}([0,1]^2) \to [0,1]$ on the Borel subsets $\mathscr{B}([0,1]^2)$ of the unit square $[0,1]^2$ with uniformly distributed margins, i.e., satisfying for all $A \in \mathscr{B}([0,1])$

$$P(A \times [0,1]) = P([0,1] \times A) = \lambda(A),$$

where $\lambda : \mathscr{B}([0,1]) \to [0,1]$ is the standard Lebesgue measure on the Borel subsets $\mathscr{B}([0,1])$ of $[0,1]$. The relationship between a copula C and the corresponding probability measure P_C is given by

$$P_C([0,x] \times [0,y]) = C(x,y)$$

for each $(x,y) \in [0,1]^2$. As an immediate consequence the probability measures P_{C_1} and P_{C_2} are different (and, therefore, incomparable) whenever the copulas C_1 and C_2 are different. For more details on copulas we recommend the monographs (Joe 1997; Nelsen 2006).

Coming back to integrals, our aim is to compute them, independently of the underlying measurable space $(X, \mathscr{A})$ and the monotone measure m. Observe, that the classical Riemann integral acts on subsets of real line $\mathbb{R}$ (or of $\mathbb{R}^n$) only, while the Lebesgue integral requires the σ-additivity of the underlying measure. However, the Choquet (1954), the Sugeno (1974) and the Shilkret (1971) integral fit into our approach.

Our aim is to propose integrals knowing their values on basic functions, considering a fixed copula C as the underlying semicopula. Note that we already have presented some preliminary steps into this direction (Klement et al. 2013).

The paper is organized as follows. In the following section, we present a hierarchical family of copula-based integrals. In Sect. 3, some relationships of these integrals with integrals known from the literature are given. In Sect. 4 we examine the strict monotonicity of these families and the equality of two central members thereof. Finally, we indicate some possible applications of these copula-based integrals.

2 Hierarchical families of copula-based integrals

The three basic copulas are given by the lower and upper Fréchet-Hoeffding bounds W and M, and by the product copula Π, given by, respectively,

$$\begin{aligned} W(x, y) &= \max(0, x + y - 1), \\ M(x, y) &= \min(x, y), \\ \Pi(x, y) &= x \cdot y. \end{aligned}$$

Another example of a copula is the Hamacher product (Ali-Mikhail-Haq copula with parameter 0) (Klement et al. 2000) $H : [0, 1]^2 \to [0, 1]$ given by

$$H(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

Obviously, for each copula $C : [0, 1]^2 \to [0, 1]$ we have $W \le C \le M$ and, in particular, $W < \Pi < H < M$.

For arbitrary copulas $C_1, \ldots, C_n$ and $(c_1, \ldots, c_n) \in [0, 1]^n$ with $\sum_{i=1}^n c_i = 1$, also the convex combination $\sum_{i=1}^n c_i \cdot C_i$ is a copula, and we have

$$P_{\sum\limits_{i=1}^{n} c_i \cdot C_i} = \sum_{i=1}^{n} c_i \cdot P_{C_i}. \tag{1}$$

In Klement et al. (2010), for a given copula $C : [0, 1]^2 \to [0, 1]$, the two universal integrals $\mathbf{I}_{(C)}$ and $\mathbf{I}_C$ on $[0, 1]$ were introduced and studied, and in Klement et al. (2013) the universal integral $\mathbf{I}^C$ was considered (where it was denoted $\mathbf{K}_C$):

$$\mathbf{I}_{(C)}, \mathbf{I}_C, \mathbf{I}^C : \bigcup_{(X,\mathscr{A})\in\mathscr{S}} \left(\mathscr{M}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}\right) \to [0, 1]$$

are given by, respectively,

$$\begin{aligned} \mathbf{I}_{(C)}(m, f) &= \sup\{C(t, m(\{f \ge t\})) \mid t \in [0, 1]\}, \\ \mathbf{I}_C(m, f) &= P_C(\{(x, y) \in [0, 1]^2 \mid y < m(\{f \ge x\})\}), \\ \mathbf{I}^C(m, f) &= P_C(\{(x, y) \in [0, 1]^2 \mid y \le m(\{f \ge x\})\}). \end{aligned}$$

Note that $\mathbf{I}_{(C)}$ is the smallest universal integral having C as the underlying semicopula, while $\mathbf{I}_\Pi = \mathbf{I}^\Pi$ and $\mathbf{I}_M = \mathbf{I}^M$ coincide with the *Choquet* and *Sugeno integral*, respectively (see also Sect. 3).

Inspired by a special hierarchical family of integrals introduced in Mesiar and Stupňanová (2013) which is based on the product copula Π (observe that these integrals are universal integrals as introduced in Klement et al. 2010 as well as decomposition integrals as introduced in Even and Lehrer 2013), we propose the following family of copula-based integrals.

Definition 1 For each copula $C : [0,1]^2 \to [0,1]$ and each $n \in \mathbb{N}$ the *lower* (C, n)-*universal integral* (on [0, 1])

$$\mathbf{I}_C^{(n)} : \bigcup_{(X,\mathscr{A})\in\mathscr{S}} (\mathscr{M}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}) \to [0,1]$$

is defined by

$$\mathbf{I}_C^{(n)}(m, f) = \sup\Big\{\sum_{i=1}^{n} \big(C\big(a_i, m(\{f \geq a_i\})\big) - C\big(a_{i-1}, m(\{f \geq a_i\})\big)\big) \Big| \\ 0 = a_0 \leq a_1 \leq \cdots \leq a_n \leq 1\Big\}. \qquad (2)$$

Obviously, if $n = 1$, then for an arbitrary semicopula $S : [0,1]^2 \to [0,1]$ we have $\mathbf{I}_S^{(1)} = \mathbf{I}_{(S)}$, i.e., $\mathbf{I}_S^{(1)}$ is the smallest universal integral linked to the semicopula S.

However, in the case $n = 2$, for an arbitrary semicopula $S : [0,1]^2 \to [0,1]$ the functional $\mathbf{I}_S^{(2)}$ constructed via (2) is not monotone, in general (see Example 1 below), i.e., not a universal integral. To ensure that $\mathbf{I}_S^{(2)}$ is a universal integral, S has be supermodular, i.e., a copula.

Example 1 Let $X = [0,1]$, $\mathscr{A} = \mathscr{B}([0,1])$, $\lambda : \mathscr{B}([0,1]) \to [0,1]$ be the standard Lebesgue measure, and let $f, g : X \to [0,1]$ be given by

$$g = \tfrac{1}{2} \cdot \mathbf{1}_{\left[0,\frac{1}{2}\right]},$$
$$f = \tfrac{3}{4} \cdot \mathbf{1}_{\left[0,\frac{1}{4}\right]} + \tfrac{1}{2} \cdot \mathbf{1}_{\left]\frac{1}{4},\frac{1}{2}\right]}.$$

Then we obtain

$$\mathbf{I}_S^{(2)}(\lambda, g) = S(\tfrac{1}{2}, \tfrac{1}{2}),$$
$$\mathbf{I}_S^{(2)}(\lambda, f) = S(\tfrac{1}{2}, \tfrac{1}{4}) + \sup\{S(a, \tfrac{1}{2}) - S(a, \tfrac{1}{4}) \mid a \in \big[0, \tfrac{1}{4}\big]\}.$$

Consider now the semicopula $S : [0,1]^2 \to [0,1]$ (which is not a copula) given by

$$S(x, y) = \begin{cases} 0 & \text{if } x + y < \frac{3}{4}, \\ \frac{1}{4} & \text{if } (x, y) \in \big[0, \frac{1}{2}\big]^2 \text{ and } x + y \geq \frac{3}{4}, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then we have $f \leq g$ but $\mathbf{I}_S^{(2)}(\lambda, f) = \frac{1}{2} > \frac{1}{4} = \mathbf{I}_S^{(2)}(\lambda, g)$, i.e, the functional $\mathbf{I}_S^{(2)}$ is not monotone.

Proposition 1 *If* $C : [0,1]^2 \to [0,1]$ *is a copula then we have:*

(i) $\mathbf{I}_{(C)} = \mathbf{I}_C^{(1)} \leq \mathbf{I}_C^{(2)} \leq \cdots \leq \mathbf{I}_C^{(n)} \leq \cdots \leq \mathbf{I}_C$,
(ii) $\sup\{\mathbf{I}_C^{(n)} \mid n \in \mathbb{N}\} = \lim_{n\to\infty} \mathbf{I}_C^{(n)} = \mathbf{I}_C$.

Proof For each copula C and each $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\mathbf{I}_C^{(n+1)}(m, f) \\
&\quad = \sup\Big\{\sum_{i=1}^{n+1} \big(C\big(a_i, m(\{f \geq a_i\})\big) - C\big(a_{i-1}, m(\{f \geq a_i\})\big)\big) \mid \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 = a_0 \leq a_1 \leq \cdots \leq a_{n+1} \leq 1\Big\} \\
&\quad \geq \sup\Big\{\sum_{i=1}^{n} \big(C\big(a_i, m(\{f \geq a_i\})\big) - C\big(a_{i-1}, m(\{f \geq a_i\})\big)\big) \mid \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 = a_0 \leq a_1 \leq \cdots \leq a_n \leq 1\Big\} \\
&\quad = \mathbf{I}_C^{(n)}(m, f),
\end{aligned}$$

independently of $(X, \mathscr{A}) \in \mathscr{S}$, $m \in \mathscr{M}_{(X,\mathscr{A})}$ and $f \in \mathscr{F}_{(X,\mathscr{A})}$, i.e, the family $(\mathbf{I}_C^{(n)})_{n\in\mathbb{N}}$ is monotone, implying $\sup\{\mathbf{I}_C^{(n)} \mid n \in \mathbb{N}\} = \lim_{n\to\infty} \mathbf{I}_C^{(n)}$.

Moreover, for each copula C and each set $E \in \mathscr{B}([0, 1]^2)$ with a countable second projection $\{y \in [0, 1] \mid (x, y) \in E \text{ for some } x \in [0, 1]\}$ we have $P_C(E) = 0$. Therefore $\mathbf{I}_C^{(n)} \leq \mathbf{I}_C$ for each $n \in \mathbb{N}$ and, subsequently, $\lim_{n\to\infty} \mathbf{I}_C^{(n)} \leq \mathbf{I}_C$.

Conversely, if $0 = a_0 \leq a_1 \leq \cdots \leq a_n \leq 1$ then the formula

$$\sum_{i=1}^{n} \big(C\big(a_i, m(\{f \geq a_i\})\big) - C\big(a_{i-1}, m(\{f \geq a_i\})\big)\big)$$

is equivalent to

$$P_C\left(\bigcup_{i=1}^{n} (\big[a_{i-1}, a_i\big] \times [0, m(\{f \geq a_i\})])\right).$$

If, for $n \in \mathbb{N}$ and $i \in \{0, 1, \ldots, n\}$, we put $a_i = \frac{i}{n}$ then

$$\bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} \big([\tfrac{i-1}{n}, \tfrac{i}{n}] \times [0, m(\{f \geq \tfrac{i}{n}\})]\big)\right) \supseteq \{(x, y) \in]0, 1]^2 \mid y < m(\{f \geq x\}),$$

and, therefore, $\lim_{n\to\infty} \mathbf{I}_C^{(n)} \geq \mathbf{I}_C$ because of the monotonicity of P_C. □

The sequence of integrals $(\mathbf{I}_C^{(n)})_{n\in\mathbb{N}}$ converges from below to the integral $\mathbf{I}_C$. In a similar way, we consider a monotone non-increasing sequence of universal integrals.

Definition 2 For each copula $C : [0, 1]^2 \to [0, 1]$ and for each $n \in \mathbb{N}$ the *upper* (C, n)*-universal integral* (on [0, 1])

$$\mathbf{I}_{(n)}^C : \bigcup_{(X,\mathscr{A})\in\mathscr{S}} (\mathscr{M}_{(X,\mathscr{A})} \times \mathscr{F}_{(X,\mathscr{A})}) \to [0, 1]$$

is defined by

$$\mathbf{I}^C_{(n)}(m, f) = \inf\Big\{\sum_{i=1}^{n}\Big(C\big(a_i, m(\{f > a_{i-1}\})\big) - C\big(a_{i-1}, m(\{f > a_{i-1}\})\big)\Big) \;\Big|\; 0 = a_0 \le a_1 \le \cdots \le a_n \le 1\Big\}. \quad (3)$$

It is obvious that for each copula C and for $n = 1$ we obtain

$$\mathbf{I}^C_{(1)}(m, f) = C(\sup f, m(\{f > 0\})).$$

Proposition 2 *If* $C : [0, 1]^2 \to [0, 1]$ *is a copula then we have:*

(i) $\mathbf{I}^C_{(1)} \ge \mathbf{I}^C_{(2)} \ge \cdots \ge \mathbf{I}^C_{(n)} \ge \cdots \ge \mathbf{I}^C$,
(ii) $\inf\{\mathbf{I}^C_{(n)} \mid n \in \mathbb{N}\} = \lim_{n\to\infty} \mathbf{I}^C_{(n)} = \mathbf{I}^C$.

Proof We can use similar arguments as in the proof of Proposition 1. To show that $\mathbf{I}^C_{(n+1)} \ge \mathbf{I}^C_{(n)}$, it suffices to consider $(n+1)$-tuples $(a_0, a_1, \ldots, a_{n+1})$ satisfying $0 = a_0 \le a_1 \le \ldots a_{n+1} = \sup f$. The inequality $\mathbf{I}^C_{(n)} \ge \mathbf{I}^C$ follows from the monotonicity of P_C and the already exploited fact that the marginal probabilities of P_C have no atoms (implying $P_C(E) = 0$ whenever the second projection of E, $\{y \mid (x, y) \in E$ for some $x \in [0, 1]\}$, is countable). Therefore $\lim_{n\to\infty} \mathbf{I}^C_{(n)} \ge \mathbf{I}^C$. Since for the set

$$F = \bigcap_{n=1}^{\infty}\Big(\bigcup_{i=1}^{n}([\tfrac{i-1}{n}, \tfrac{i}{n}] \times [0, m(\{f > \tfrac{i-1}{n}\})])\Big) \setminus \{(x, y) \in [0, 1]^2 \mid y \le m(\{f \ge x\})\}$$

we have $P_C(F) = 0$, the opposite inequality $\mathbf{I}^C \ge \lim_{n\to\infty} \mathbf{I}^C_{(n)}$ follows. □

Combining Propositions 1 and 2 we obtain the following:

Corollary 1 *For each copula* $C : [0, 1]^2 \to [0, 1]$ *we have*

$$\mathbf{I}^{(1)}_C \le \mathbf{I}^{(2)}_C \le \cdots \le \mathbf{I}^{(n)}_C \le \cdots \le \mathbf{I}_C \le \mathbf{I}^C \le \cdots \le \mathbf{I}^C_{(n)} \le \cdots \le \mathbf{I}^C_{(2)} \le \mathbf{I}^C_{(1)}.$$

3 Distinguished cases of copula-based integrals

Recall that all copula-based integrals considered so far are universal integrals in the sense of Klement et al. (2010). For the product copula Π, all integrals $\mathbf{I}^{(n)}_{\Pi}$ and the integral $\mathbf{I}_{\Pi}$ are special *decomposition integrals* in the sense of Even and Lehrer (2013), while the integrals $\mathbf{I}^{\Pi}_{(n)}$ and $\mathbf{I}^{\Pi}$ are *superdecomposition integrals* as introduced in Mesiar et al. (2013). Moreover, $\mathbf{I}^{(1)}_{\Pi}$ is the *Shilkret integral* **Sh** (see Shilkret 1971), and $I_{\Pi} = I^{\Pi}$ coincides with the Choquet integral **Ch** (see Choquet 1954), which are given by, respectively,

$$\mathbf{Ch}(m, f) = \int_0^1 m(\{f \geq t\})\,dt,$$

$$\mathbf{Sh}(m, f) = \sup\{t \cdot m(\{f \geq t\}) \mid t \in [0, 1]\}.$$

The following example provides an illustration of Π-based and W-based upper and lower universal integrals.

Example 2 Let $\lambda : \mathscr{B}([0, 1]) \to [0, 1]$ be the standard Lebesgue measure on the Borel subsets $\mathscr{B}([0, 1])$ of $[0, 1]$, and consider the identity function $\mathrm{id}_{[0,1]}$ on $[0, 1]$.

(i) For the product copula Π we get (see Fig. 1)

$$\mathbf{Ch}(\lambda, \mathrm{id}_{[0,1]}) = \int_0^1 (1-t)\,dt = \frac{1}{2},$$

$$\mathbf{Sh}(\lambda, \mathrm{id}_{[0,1]}) = \mathbf{I}_{\Pi}^{(1)}(\lambda, \mathrm{id}_{[0,1]}) = \sup\{t(1-t) \mid t \in [0, 1]\} = \frac{1}{4},$$

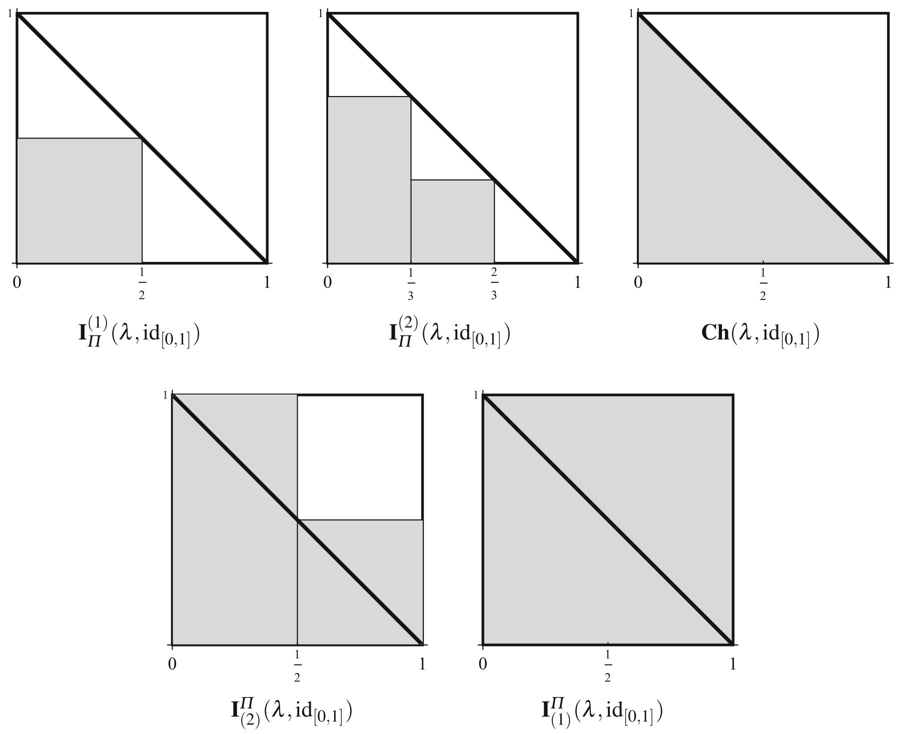

Fig. 1 The surfaces of the grey areas correspond to special Π-based integrals as considered in Example 2(i)

$$\mathbf{I}_{\Pi}^{(2)}(\lambda, \mathrm{id}_{[0,1]}) = \sup\{a(1-a) + b(1-a-b) \mid a, b, a+b \in [0,1]\} = \frac{1}{3},$$
$$\mathbf{I}_{\Pi}^{(n)}(\lambda, \mathrm{id}_{[0,1]}) = \frac{n}{2(n+1)},$$
$$\mathbf{I}_{(1)}^{\Pi}(\lambda, \mathrm{id}_{[0,1]}) = 1,$$
$$\mathbf{I}_{(2)}^{\Pi}(\lambda, \mathrm{id}_{[0,1]}) = \inf\{a + (1-a)^2 | a \in [0,1]\} = \frac{3}{4},$$
$$\mathbf{I}_{(n)}^{\Pi}(\lambda, \mathrm{id}_{[0,1]}) = \frac{n+1}{2n}.$$

Note that $\lim_{n\to\infty} \mathbf{I}_{\Pi}^{(n)}(\lambda, \mathrm{id}_{[0,1]}) = \lim_{n\to\infty} \mathbf{I}_{(n)}^{\Pi}(\lambda, \mathrm{id}_{[0,1]}) = \mathbf{Ch}(\lambda, \mathrm{id}_{[0,1]})$.

(ii) For the lower Fréchet-Hoeffding bound W we get

$$\mathbf{I}_{W}^{(1)}(\lambda, \mathrm{id}_{[0,1]}) = \cdots = \mathbf{I}_{W}^{(n)}(\lambda, \mathrm{id}_{[0,1]}) = \cdots = \mathbf{I}_{W}(\lambda, \mathrm{id}_{[0,1]}) = 0,$$
$$\mathbf{I}_{(1)}^{W}(\lambda, \mathrm{id}_{[0,1]}) = \cdots = \mathbf{I}_{(n)}^{W}(\lambda, \mathrm{id}_{[0,1]}) = \cdots = \mathbf{I}^{W}(\lambda, \mathrm{id}_{[0,1]}) = 1.$$

Example 3 If we consider the upper Fréchet-Hoeffding bound M then we get (independently of the measurable space $(X, \mathscr{A})$, the monotone measure $m \in \mathscr{M}_{(X,\mathscr{A})}$ and the measurable function $f \in \mathscr{F}_{(X,\mathscr{A})}$)

$$\mathbf{I}_{M}^{(1)} = \mathbf{I}_{M}^{(2)} = \cdots = \mathbf{I}_{M}^{(n)} = \cdots = \mathbf{I}_{M} = \mathbf{I}^{M} = \cdots = \mathbf{I}_{(n)}^{M} = \cdots = \mathbf{I}_{(2)}^{M} = \mathbf{Su},$$

where the *Sugeno integral* **Su** (see Sugeno 1974) is given by

$$\mathbf{Su}(m, f) = \sup\{\min\{t, m(\{f \geq t\})\} \mid t \in [0,1]\}.$$

However, we have $\mathbf{I}_{(1)}^{M}(m, f) = \min\{\sup f, m(\{f > 0\})\} \neq \mathbf{Su}(m, f)$.

Example 4 Let $\lambda : \mathscr{B}([0,1]) \to [0,1]$ be the standard Lebesgue measure on the Borel subsets $\mathscr{B}([0,1])$ of $[0,1]$, and consider the identity function $\mathrm{id}_{[0,1]}$ on $[0,1]$. Then for the Hamacher product (Ali-Mikhail-Haq copula with parameter 0) H we obtain the following (see Table 1)

$$\mathbf{I}_{H}^{(1)}(\lambda, \mathrm{id}_{[0,1]}) = \frac{1}{3},$$
$$\mathbf{I}_{H}^{(n)}(\lambda, \mathrm{id}_{[0,1]}) = \sup\{\sum_{i=1}^{n}(H(b_i, 1-b_i) - H(b_{i-1}, 1-b_i)) \mid 0 = b_0 \leq \cdots \leq b_n \leq 1\},$$

Table 1 Values of some integrals $\mathbf{I}_{H}^{(n)}(\lambda, \mathrm{id}_{[0,1]})$ and $\mathbf{I}_{(n)}^{H}(\lambda, \mathrm{id}_{[0,1]})$

n	1	2	3	4	5	6	7	…	$n \to \infty$
$\mathbf{I}_{H}^{(n)}(\lambda, \mathrm{id}_{[0,1]})$	0.3333	0.3821	0.4048	0.4182	0.4271	0.4335	0.4383	…	0.4728
$\mathbf{I}_{(n)}^{H}(\lambda, \mathrm{id}_{[0,1]})$	1.0000	0.6667	0.5918	0.5585	0.5396	0.5275	0.5191	…	0.4728

$$\mathbf{I}^H_{(n)}(\lambda, \mathrm{id}_{[0,1]}) = \inf\{\sum_{i=1}^{n}(H(b_i, 1-b_{i-1}) - H(b_{i-1}, 1-b_{i-1})) \mid 0 = b_0 \leq \cdots \leq b_n = 1\},$$

$$\mathbf{I}_H(\lambda, \mathrm{id}_{[0,1]}) = \mathbf{I}^H(\lambda, \mathrm{id}_{[0,1]}) = \int_0^1 \int_0^{1-x} \frac{2xy}{(x+y-xy)^3}\, dy\, dx = \frac{4\pi\sqrt{3}-9}{27}.$$

4 Equalities in the families of lower and upper (C, n)-universal integrals

For each copula $C : [0,1]^2 \to [0,1]$, the lower $(C,1)$-universal integral $\mathbf{I}^{(1)}_C$ is the smallest universal integral based on the (semi-)copula C.

However, the upper $(C,1)$-universal integral $\mathbf{I}^C_{(1)}$ is not necessarily the greatest universal integral based on C: in general, we may have $\mathbf{I}^C_{(1)} < \mathbf{I}^C_{(0)}$, where $\mathbf{I}^C_{(0)}$ is the greatest universal integral based on C given by

$$\mathbf{I}^C_{(0)}(m, f) = C(\sup\{t \in [0,1] \mid m(\{f \geq t\}) > 0\}, m(\{f > 0\})).$$

For each copula C we have $\mathbf{I}^{(1)}_C < \mathbf{I}^C_{(1)}$, i.e., we always can find $(X, \mathscr{A}) \in \mathscr{S}, m \in \mathscr{M}_{(X,\mathscr{A})}$ and $f \in \mathscr{F}_{(X,\mathscr{A})}$ such that $\mathbf{I}^{(1)}_C(m, f) < \mathbf{I}^C_{(1)}(m, f)$.

As already mentioned, for the upper Fréchet-Hoeffding bound M we obtain $\mathbf{I}^{(1)}_M = \mathbf{I}^M_{(2)}$ (and all lower and upper (C, n)-universal integrals up to $I^M_{(1)}$ coincide with **Su**). This is not true for any copula $C \neq M$.

Proposition 3 *Let $C : [0,1]^2 \to [0,1]$ be a copula such that $\mathbf{I}^{(1)}_C = \mathbf{I}^C_{(2)}$. Then necessarily $C = M$.*

Proof Consider the standard Lebesgue measure $\lambda : \mathscr{B}([0,1]) \to [0,1]$ be on the Borel subsets $\mathscr{B}([0,1])$ of $[0,1]$.

For $c \in [0,2]$ define $f_c : [0,1] \to [0,1]$ by $f_c(x) = \min(1, \max(0, c-x))$. Due to the continuity of C, the function $h : [0,2] \to [0,1]$ given by $h(c) = \mathbf{I}^{(1)}_C(\lambda, f_c)$ is a continuous, monotone non-decreasing function satisfying $h(0) = 0$ and $h(2) = 1$. Moreover, there exists a continuous, monotone non-decreasing function $g : [0,2] \to [0,1]$ which satisfies $g(0) = 0, g(2) = 1, c - g(c) = m(\{f_c \geq g(c)\})$ and $h(c) = C(g(c), c - g(c))$ for all $c \in [0,2]$. The equality $\mathbf{I}^{(1)}_C(\lambda, f_c) = \mathbf{I}^C_{(2)}(\lambda, f_c)$ holding for all $c \in [0,2]$ implies

$$P_C\left(\left\{(x,y) \in [0,1]^2 \mid y \leq m(\{f_c \geq x\})\right\} \setminus [0, g(c)] \times [0, c - g(c)]\right) = 0.$$

From

$$\bigcup_{c \in [0,2]} \left(\left\{(x,y) \in [0,1]^2 \mid y \leq m(\{f_c \geq x\})\right\} \setminus [0, g(c)] \times [0, c - g(c)]\right)$$
$$= [0,1]^2 \setminus \{(g(c), c - g(c)) \mid c \in [0,2]\}$$

we deduce that the support of copula C is a subset of the curve $\{(g(c), c - g(c)) \mid c \in [0, 2]\}$. Because of the monotonicity of g this implies $C = M$ (and, therefore, $h(c) = g(c) = \frac{c}{2}$ for all $c \in [0, 2]$). □

Obviously, if we consider a finite space X with cardinality n, then for any copula C we obtain $\mathbf{I}_C^{(n)} = \mathbf{I}_{(n)}^C$ (i.e., $\mathbf{I}_C^{(n)} = \mathbf{I}_C^{(n+1)} = \cdots = \mathbf{I}_C = \mathbf{I}^C = \cdots = \mathbf{I}_{(n+1)}^C = \mathbf{I}_{(n)}^C$).

If X and $\mathscr{A}$ are both infinite, then for each absolutely continuous copula C we have

$$I_C^{(1)} < \mathbf{I}_C^{(2)} < \cdots < \mathbf{I}_C^{(n)} < \cdots < \mathbf{I}_C \le \mathbf{I}^C < \cdots < \mathbf{I}_{(n)}^C < \cdots < \mathbf{I}_{(2)}^C < \mathbf{I}_{(1)}^C,$$

i.e., for each $n \in \mathbb{N}$ there always exist $(X, \mathscr{A}) \in \mathscr{S}, m \in \mathscr{M}_{(X,\mathscr{A})}$ and $f \in \mathscr{F}_{(X,\mathscr{A})}$ such that $\mathbf{I}_C^{(n)}(m, f) < \mathbf{I}_C^{(n+1)}(m, f)$ or $\mathbf{I}_{(n)}^C(m, f) > \mathbf{I}_{(n+1)}^C(m, f)$.

For singular copulas there is an interesting result: if the support of a singular copula $C : [0, 1]^2 \to [0, 1]$ consists of the graph of n monotone increasing functions, then we have $\mathbf{I}_C^{(n)} = \mathbf{I}_C^{(n+1)} = \cdots = \mathbf{I}_C = \mathbf{I}^C = \cdots = \mathbf{I}_{(n+1)}^C$.

Example 5 Let $C : [0, 1]^2 \to [0, 1]$ be given by

$$C(x, y) = \begin{cases} x & \text{if } y \ge 2x \\ \frac{y}{2} & \text{if } x \in \left[0, \frac{1}{2}\right] \text{ and } y < 2x \\ y & \text{if } y \le 2x - 1 \\ x + \frac{y-1}{2} & \text{otherwise.} \end{cases}$$

Observe that C is a singular copula whose support consists of the graphs of the functions $\varphi : \left[0, \frac{1}{2}\right] \to [0, 1]$ given by $\varphi(x) = 2x$, and $\psi : \left[\frac{1}{2}, 1\right] \to [0, 1]$ given by $\psi(x) = 2x - 1$. For this copula we get $\mathbf{I}_C^{(2)} = \mathbf{I}_C = \mathbf{I}^C = \mathbf{I}_{(3)}^C$, see Fig. 2, where all the integrals mentioned there have the value $a + b - \frac{1}{2}$ (independently of the measure m and the function f). Note that, in general, the function $h^{(m,f)}[0, 1] \to [0, 1]$ given by $h^{(m,f)}(t) = m(\{f \ge t\})$ is always monotone non-increasing, but not necessarily continuous; however, the constants a and b are uniquely determined by m and f, and they satisfy

$$\lim_{t \to a^+} h^{(m,f)}(t) \le \varphi(a) \le \lim_{t \to a^-} h^{(m,f)}(t),$$
$$\lim_{t \to b^+} h^{(m,f)}(t) \le \psi(b) \le \lim_{t \to b^-} h^{(m,f)}(t)).$$

As we have seen in Example 4, the equality $\mathbf{I}_C = \mathbf{I}^C$ does not hold, in general. In such a case there exist $(X, \mathscr{A}) \in \mathscr{S}, m \in \mathscr{M}_{(X,\mathscr{A})}$ and $f \in \mathscr{F}_{(X,\mathscr{A})}$ such that $\mathbf{I}^C(m, f) > \mathbf{I}_C(m, f)$, i.e., $P_C(\{(x, m(\{f \ge x\})) \mid x \in [0, 1]\}) > 0$. This means that the copula C has a singular part whose support contains a graph of some monotone decreasing function defined on a subinterval of $[0, 1]$.

Proposition 4 *For a copula $C : [0, 1]^2 \to [0, 1]$ we have $\mathbf{I}_C = \mathbf{I}^C$ if and only if either C is absolutely continuous or the support of its singular part consists of graphs of monotone increasing functions only.*

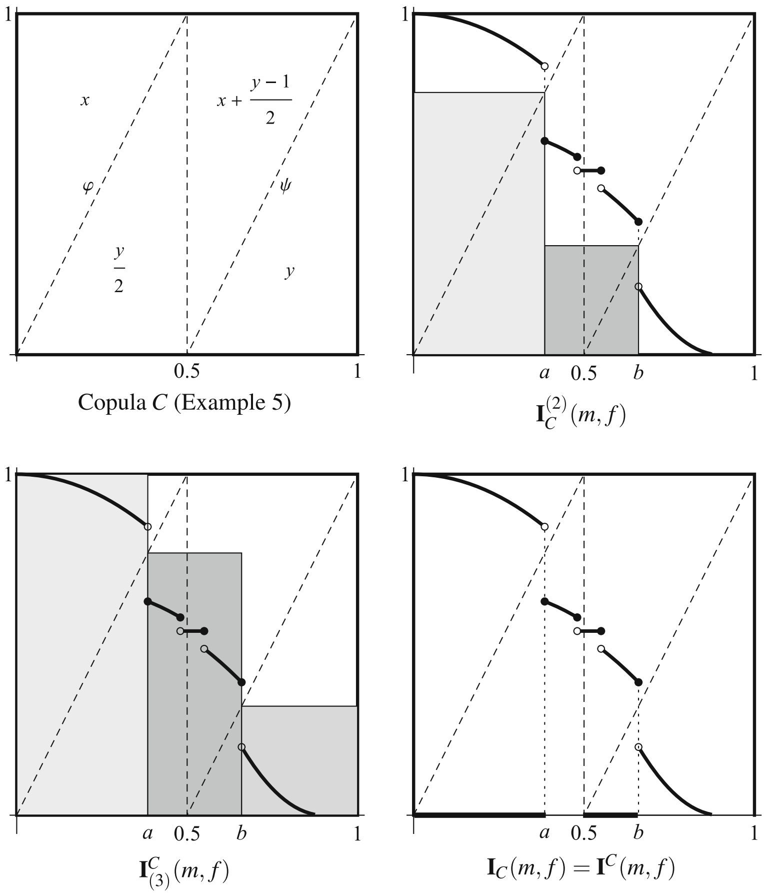

Fig. 2 The copula C given in Example 5, and the integrals $\mathbf{I}_C^{(2)}(m,f)$, $\mathbf{I}_{(3)}^C(m,f)$ and $\mathbf{I}_C(m,f) = \mathbf{I}^C(m,f)$; the curve is the graph of the (non-continuous) function $h^{(m,f)}$

Proof The sufficiency was already shown. The necessity follows from Proposition 2.7 in Klement et al. (2010), proving, for each monotone non-decreasing function $h : [0,1] \to [0,1]$ satisfying $h(0) = 1$, the existence of $(X, \mathscr{A}) \in \mathscr{S}$, $m \in \mathscr{M}_{(X,\mathscr{A})}$ and $f \in \mathscr{F}_{(X,\mathscr{A})}$ such that $h(t) = m(\{f \geq t\})$ for all $t \in [0,1]$. □

5 Concluding remarks

For each copula $C \neq M$, we have introduced an infinite hierarchical family of C-based universal integrals,

$$\mathbf{I}_C^{(1)} \leq \mathbf{I}_C^{(2)} \leq \cdots \leq \mathbf{I}_C^{(n)} \leq \cdots \leq \mathbf{I}_C \leq \mathbf{I}^C \leq \cdots \leq \mathbf{I}_{(n)}^C \leq \cdots \leq \mathbf{I}_{(2)}^C \leq \mathbf{I}_{(1)}^C \leq \mathbf{I}_{(0)}^C,$$

where the extremal cases $\mathbf{I}_C^{(1)}$ and $\mathbf{I}_{(0)}^C$ are extremal C-based universal integrals introduced in Klement et al. (2010), and the limits $\mathbf{I}_C$ and $\mathbf{I}^C$ are P_C-based integrals proposed in Klement et al. (2010) (see also Klement et al. 2004). We have discussed these families of integrals for some particular copulas. Especially important is the case when the product copula Π is considered. Note that then the integrals $\mathbf{I}_\Pi^{(n)}$ are also decomposition integrals in the sense of Even and Lehrer (2013). The two extremal cases $\mathbf{I}_\Pi^{(1)}$ and $\mathbf{I}_\Pi = \sup\{\mathbf{I}_\Pi^{(n)} \mid n \in \mathbb{N}\}$ coincide with the Shilkret (1971) and the Choquet (1954) integral , respectively. Note also that for the comonotonicity copula M all M-based universal integrals (up to the greatest M-based universal integral $\mathbf{I}_{(0)}^M$) coincide with the Sugeno (1974) integral . Taking into account the fact that binary copulas model the stochastic dependence of two random variables, our approach reflects possible dependencies of the function values and the values of the underlying capacities.

There are several open problems, especially effective algorithms to compute the integrals are still missing in many cases. One problem is the choice of a suitable copula as the background of integration when modeling real data problems from different branches of economics and engineering (observe that in economics mostly the product copula Π is applied, reflecting the independence between the values of the function and the capacity, with a distinguished role played by the Choquet integral, while in several decision making areas dealing with ordinal scales the Sugeno integral plays an important role, i.e., here the comonotonicity copula M is applied).

Therefore, we expect applications in multicriteria decision making and related areas. Our integrals also have to do with optimization in economics: $\mathbf{I}_C^{(n)}$ maximizes finite lower integral sums, $\mathbf{I}_{(n)}^C$ minimizes finite upper integral sums. Another application is in the field of bibliometrics, especially when proposing new citation indices (Beliakov and James 2011; Gagolewski and Mesiar 2014; Torra and Narukawa 2008).

Acknowledgments The work on this paper was supported by the "Technologie-Transfer-Förderung" Wi-2013-168051/WWin/Kai of the Upper Austrian Government and by the grants APVV-0073-10, GAČR P402/11/0378 and VEGA 1/0143/11.

References

Bassan, B., & Spizzichino, F. (2005). Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. *Journal of Multivariate Analysis*, *93*, 313–339.

Beliakov, G., & James, S. (2011). Citation-based journal ranks: The use of fuzzy measures. *Fuzzy Sets and Systems*, *167*, 101–119.

Benvenuti, P., Mesiar, R., & Vivona, D. (2002). Monotone set functions-based integrals. In E. Pap (Ed.), *Handbook of measure theory*, chapter 33 (pp. 1329–1379). Amsterdam: Elsevier Science.

Choquet, G. (1954). Theory of capacities. *Annales de l'Institut Fourier (Grenoble)*, *5*, 131–295.

Díaz, S., De Baets, B., & Montes, S. (2010). General results on the decomposition of transitive fuzzy relations. *Fuzzy Optimization and Decision Making*, *9*, 1–29.

Durante, F., & Sempi, C. (2005). Semicopulæ. *Kybernetika (Prague)*, *41*, 315–328.

Even, Y., Lehrer, E. (2013). Decomposition-integral: Unifying Choquet and the concave integrals. *Economic Theory* doi:10.1007/s00199-013-0780-0.

Gagolewski, M., & Mesiar, R. (2014). Monotone measures and universal integrals in a uniform framework for the scientific impact assessment problem. *Information Sciences*. doi:10.1016/j.ins.2013.12.004.

Joe, H. (1997). *Multivariate models and dependence concepts*. London: Chapman & Hall.

Klement, E. P., Kolesárová, A., Mesiar, R., & Stupňanová, A. (2013). A generalization of universal integrals by means of level dependent capacities. *Knowledge-Based Systems*, *38*, 14–18.

Klement, E. P., Mesiar, R., & Pap, E. (2000). *Triangular norms*. Dordrecht: Kluwer.

Klement, E. P., Mesiar, R., & Pap, E. (2004). Measure-based aggregation operators. *Fuzzy Sets and Systems*, *142*, 3–14.

Klement, E. P., Mesiar, R., & Pap, E. (2010). A universal integral as common frame for Choquet and Sugeno integral. *IEEE Transactions on Fuzzy Systems*, *18*, 178–187.

Klement, E. P., Spizzichino, F., Mesiar, R., & Stupňanová, A. (2013). Copula-based universal integrals. In *Proceedings 2013 IEEE international conference on fuzzy systems* (pp. 1065–1069). Hyderabad: IEEE.

Mesiar, R., Li, J., & Pap, E. (2013). Superdecomposition integral (submitted for publication).

Mesiar, R., & Stupňanová, A. (2013). Decomposition integrals. *Internat. J. Approx. Reason.*, *54*, 1252–1259.

Nelsen, R. B. (2006). *An introduction to copulas* (2nd ed., Vol. 139) Lecture Notes in Statistics. New York: Springer.

Shilkret, N. (1971). Maxitive measure and integration. *Indagationes Mathematicae*, *33*, 109–116.

Sklar, A. (1959). Fonctions de répartition à *n* dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, *8*, 229–231.

Sugeno, M. (1974). Theory of fuzzy integrals and its applications. PhD thesis, Tokyo Institute of Technology.

Torra, V., & Narukawa, Y. (2008). The *h*-index and the number of citations: Two fuzzy integrals. *IEEE Transactions on Fuzzy Systems*, *16*, 795–797.

Yager, R. R. (2013). Joint cumulative distribution functions for Dempster-Shafer belief structures using copulas. *Fuzzy Optimization and Decision Making*, *12*, 393–414.