Markov Equilibrium between High Frequency Traders

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Abstract

We model an optimal behaviour of a finite number of (perhaps high frequency) traders at a limit order market where a financial instrument possibly paying dividends is traded. The traders are assumed to trade continuously and to maximize their discounted consumption while keeping the probability of near-bankruptcy states at a prescribed level. The latency times, i.e.., the delays between the order submissions and the corresponding order books' changes, are taken into account. We show that the process describing such a market is Markov given the largest among information sets of the agents.

Keywords

limit order market, Markov property, optimal trading

JEL Classification: C51,G10

1 Introduction

As the most of the trading with financial instruments is done by means of limit order markets nowadays, the importance of mathematical modelling of those markets grows. There exist a large number of models assuming a rational behavior of agents involved (see e.g., [1], [3] or [4] and the references therein); the majority of those models, however, make too many simplifying assumptions to be realistic. A notable exception in that respect is the model formulated by [4], combining the problem of optimal trading with a portfolio choice problem with; its authors, however, formulate no theoretical results concerning the model.

We formulate a model similar to that of [4]. For simplicity, we do not assume multiple financial instruments; however, contrary to [4], the times of the agents' actions are implicit in our settings. In addition, we take latency times between decisions and their effects into account.

Due to the complexity of the problem, we are unable to get analytical results, either. Our main result, however, has something to say about the stochastic behaviour of the market. In particular, we show that, in equilibrium, all the traders potentially use the same information set, which implies that the process describing the market is Markov with respect to this set.

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2 The Setting

We consider a limit order market with discrete positive prices by means of which a financial instrument possibly paying dividends is traded; without loss of generality, we assume that prices take values from $\{1, 2, \dots, P\}$, $P \in \mathbb{N}$. For simplicity, we assume that the limit orders are satisfied by proportional priority rule, that is, once limit orders with amounts $q_1, q_2, \dots, q_K$ is offered to buy/sell for the price p, an eventual sell/buy order of amount $q_0 \le \sum_i q_i$ is satisfied by amounts $q_1/q_0, q_2/q_0, \dots, q_K/q_0$ from the individual buy/sell orders.

There is N traders at the market, the i-th one disposing of $M_{i,0}$ units of cash and $N_{i,0}$ units of the instrument at time zero.

We describe the state of the order profile of the i-th agent at the time $t \in \mathbb{R}_0^+$ by

$$x_{i,t} = (x_{i,t}^1, \dots, x_{i,t}^P), \qquad x_{i,t}^j \in \mathbb{R}$$

where the positive/negative components of the vector stand for sell/buy orders. In particular, the fact that $x = x_{t,i}^p$ is positive/negative means that the agent offers $|x|$ units to sell/buy for the price at least/most p. We naturally assume the collection profiles to be *consistent*; i.e.., that, for any i, j, the indices of positive components in the i-th profile are always greater than those of negative values in the j-th profile. Further, we require the profiles to be *affordable* in sense that an agent has to hold a sufficient amount of both the money and the traded instrument to be able to meet obligations arising from his orders. In particular, it has to be

$$\mathbf{P}(x_{i,t}^+) \le M_{i,t}, \qquad \mathbf{V}(x_{i,t}^-) \le N_{i,t}$$

where $M_{i,t}$ and $N_{i,t}$ are the current amount of the cash, the instrument, respectively, held by the agent, and

$$\mathbf{V}(y) = \sum_{j=1}^{P} y^j, \qquad \mathbf{P}(y) = \sum_{j=1}^{P} j y^j, \qquad y \in \mathbb{R}^P.$$

The agents may observe an actual state of the market and react immediately; however, there is a latency time between his reaction and a corresponding change of the order profile (this delay may be due to network communication, for instance); when a new request is made by the agent before the previous one took effect, the previous request is cancelled.

Formally, if $0 = t_{i,0} < t_{i,j} < \dots$ are the request times of the i-th agent, then the actual changes of the i-th profile happen at times

$$T_i = \{\tau_{i,j} : \tau_{i,j} = t_{i,j} + \theta_{i,j}, t_{i,j+1} > t_{i,j} + \theta_{i,j}\}$$

where $\theta_{i,j}$ the latency time; it is assumed that all $\theta_\bullet$ are i.i.d. exponential with a (very high) intensity κ.

The cumulative dividend (or possibly interest) payment is described by an increasing pure jump type process (i.e.., with isolated jumps) d_t. The information flow, available to the i-th agent, is modelled by a pure jump process $\eta_{i,t}$ taking values in some measurable space; we assume that the dividend flow is always observable, i.e.., $\Delta d_t \in \sigma(\eta_{i,t})$, $t \in \mathbb{R}^+$.

Thanks to the independence and the absolute continuity of $\theta's$,

$$\text{the points of } T_i \text{ do coincide neither with points of } \sum_{j \ne i} T_j \text{ nor with the jumps of } d \qquad (1)$$

almost sure; so, without a change of the distributions, we may assume that (1) holds for each elementary event.

2.1 Dynamics of the Market

Denoting

$$x_t = \sum_i x_{i,t},$$

(the cumulative state of the order books),

$$a_t = a(x_t), \qquad a(x) = \min(0, \min(k : x_k > 0))$$
$$b_t = b(x_t), \qquad b(x) = \max(0, \max(k : x_k < 0))$$

(the best ask, best bid, respectively), and

$$T = \sum_j T_j$$

(the set of the agents' actions' times), the dynamics of the limit order market may be described by a jump right continuous process

$$\xi_t = (x_{1,t}, \dots, x_{N,t})$$

which is constant outside T and, once $\tau = \tau_{i,j}$ for some i, j, it fulfils

$$x_{i,\tau} = v_{i,j} - q(v_{i,j}, x^\star_{i,\tau-}),$$
$$x_{k,\tau} = x_{k,\tau-} + q_{k,\tau}, \qquad k \neq i,$$

where $v_{i,j} \in \mathbb{R}^P$ is the request taking effect at $t_{i,j}$ and where we put

$$x^\star_{i,t} = \sum_{k \neq i} x_{k,t},$$

$$q_{i,\tau} = q(x^\star_{i,\tau-}, v_{i,j}),$$
$$q_{k,\tau} = -\frac{x_{k,\tau-}}{x^\star_{i,\tau-}} q_{i,\tau}, \qquad k \neq i,$$

and

$$q(x,v)^k = \begin{cases} \left.\begin{cases} \min(x^k, v^k) & \text{if } a(x) = k \\ \min(\sum_{j=a(x)+1}^{k} (x^j - q(x,v)^{j-1}), v^k) & \text{if } a(x) < k \leq b(v) \\ 0 & \text{otherwise} \end{cases}\right\} & \text{if } a(x) \leq b(v) \\ \\ \left.\begin{cases} \max(x^k, v^k) & \text{if } b(x) = k \\ \max(\sum_{j=b(x)-1}^{k} (x^j - q(x,v)^{j+1}), v^k) & \text{if } a(v) \leq k < b(x) \\ 0 & \text{otherwise} \end{cases}\right\} & \text{if } b(x) \geq a(v) \\ \\ 0 & \text{otherwise} \end{cases}$$

(in words, $q(x, v)$ is the vector of amounts traded for individual prices given that a request v meets order book x).

If we assume the agents to consume parts of their money earnings (for simplicity, we allow this to happen solely only at time points from T) then the evolution of the instrument holding and the money account of the i-th agent is naturally described as

$$\Delta N_{i,t} = \mathbf{V}(q_{i,t}) \Delta T_t, \qquad t \in \mathbb{R}^+,$$
$$\Delta M_{i,t} = -\mathbf{P}(q_{i,t}) \Delta T_t - \Delta c_{i,t} + N_{i,t-} \Delta d_t \qquad t \in \mathbb{R}^+,$$

where $c_{i,t}$ the consumption process.

2.2 Strategies

Without loss of generality, we may assume

$$t_{i,j} = t_{i,j-1} + \theta_{i,j-1};$$

(if there is a delay between $t_{i,j-1} + \theta_{i,j-1}$ then we may put $v_{i,j} = x_{t_{i,j}}$) so that we may describe the control of the i-th agent by a process

$$u_{i,t} \in \mathbb{R}^p, \qquad u_{i,t} = \sum_{i=1}^{\infty} \mathbf{1}_{[t_{i,j-1},t_{i,j})}(t) v_{i,j}.$$

We say that the request $u_{i,t}$ is *effective* at t if $q(x_{t-}, u_{i,t}) \neq 0$ or $x_{t-} \neq u_{i,t} - q(u_{i,t}, x_{t-})$ (i.e. the request would change at least one of the profiles if it is fulfilled at t). It is easy to see that

Lemma 1. *The intensity of the changes of ξ_t by the i-th agent at t is κ, if u_t is effective and it is zero otherwise.*

If we assume that, besides his information flow, the i-th agent observes the actual state of the market, i.e.., all the information available to him is

$$\Xi_{i,t} = (\xi_t, \eta_{i,t}),$$

then the overall strategy of the i-th agent may be described by

$$U_{i,t} = (u_{i,t}, \Delta c_{i,t}), \qquad U_{i,t} \in \mathcal{F}_{i,t},$$

where $\mathcal{F}_{i,t}$ is the filtration generated by $\Xi_{i,t}$.

2.3 The Optimization Problem

We suppose that the $i-$th agent maximizes his discounted consumption while trying to protect against its low values; in particular, he solves and optimization problem

$$\begin{array}{llr} \max\limits_{U} & \mathbb{E}(\sum_{\tau\in T} \exp\{-\rho_i \tau\} \Delta c_\tau | \mathcal{F}_{i,0}) & (2) \\ & \mathbb{P}(M_{\tau_t} \leq \mu_i | \mathcal{F}_{i,t}) \leq \alpha_i & PM(t) \\ & \mathbb{P}(N_{\tau_t} \leq \nu_i | \mathcal{F}_{i,t}) \leq \beta_i & PN(t) \\ & \mathbf{P}(u_t^+) \leq M_{i,t} & AM(t) \\ & \mathbf{V}(u_t^-) \leq N_{i,t} & AN(t) \\ & \text{for each } t \geq 0 & \end{array}$$

for some constants $\rho_i, \mu_i, \nu_i, \alpha_i$ and β_i where τ_t is the first jump of T after t.

3 Main Result

Proposition 1. *Let $\mathcal{F}_t$ be a filtration generated by*

$$\Xi_t = (\xi_t, \eta_t), \qquad \eta_t = (\eta_{1,t}, \eta_{2,t}, \dots, \eta_{N,t}).$$

Let $1 \leq i \leq N$. Let $t \in \mathbb{R}$, and there exists a random element I_t be such that

$$(\xi_t, M_{1,t}, N_{1,t}, M_{2,t}, N_{2,t}, \dots, M_{N,t}, N_{N,t}) \in \sigma(I_t),$$

$$I_t \in \mathcal{F}_{i,t},$$
$$\eta^{(t,\infty)} \perp\!\!\!\perp_{I_t} \mathcal{F}_t \tag{3}$$

where $z^I = (z_t)_{t \in I}$ for any process z. Let the strategies of the agents be $U_1, U_2, \ldots, U_N$ be such that, for each $j \neq i$,

$$U_j^{[t,\infty)} = f_j(I_t, \xi_t^{[t,\infty)}, \eta_j^{[t,\infty)}), \qquad j \neq i, \tag{4}$$

for some mappings f_j. Let U_i be optimal with respect to (2). Then
(i) (4) holds for $j = i$.
(ii) $U_1^{[t,\infty)}, \ldots, U_N^{[t,\infty)} \perp\!\!\!\perp_{I_t} \mathcal{F}_t$

Proof. Note first that, by [2], Proosition 6.13, (3) implies an existence of $E \perp\!\!\!\perp \mathcal{F}_t$ such that

$$\eta^{[t,\infty)} = f(I_t, E) \tag{5}$$

for some f.

Let $K_1, \ldots, K_N$ be independent Poisson processes all with intensity κ. By Lemma 1, we may assume the jumps of ξ caused by the i-th agent happen exactly in points of the process $L_{i,t}$ given by

$$\Delta L_{i,t} = \begin{cases} \Delta K_{i,t} & \text{if } u_{i,t} \text{ is effective} \\ 0 & \text{otherwise} \end{cases}$$

from which and Section 2.1 it is clear that, for any s,

$$(M^{[t,s]}, N^{[t,s]}, \xi^{[t,s]}, \eta^{[t,s]}) = F_s(\xi_t, K_\bullet^{[t,s]}, U_\bullet^{[t,s]}, \eta^{[t,\infty)})$$
$$\overset{(4)}{=} G_s(U^\star, \xi_t, K_\bullet^{[t,s]}, \eta^{[t,s]}, I_t) \overset{(5)}{=} H_s(U^\star, \xi_t, V, I_t) \tag{6}$$

where $U_s^\star = U_i^{[t,s]}$ and $V = (K_\bullet^{[t,\infty)}, E)$; note that $V \perp\!\!\!\perp \mathcal{F}_t$, so, by Proposition 6.13 again,

$$M^{[t,s]}, N^{[t,s]}, \xi^{[t,s]}, \eta^{[t,s]} \perp\!\!\!\perp_{I_t, U_s^\star} \mathcal{F}_t,$$

Now, denote

$$J_{[s,t)} = \sum_{t=0}^{\infty} \mathbf{1}_{[s,t)}(\tau_k) \exp\{-\rho_i \tau_k\} c_{i,\tau_k} = \int_{[s,t)} e^{-\rho_i t} dc_{i,t},$$

and note that (2) may be reformulated as

$$\begin{aligned} &\max_{U^{[0,t)}} \mathbb{E}\left(J_{[0,t)} e^{-\rho_i t} V_t | \mathcal{F}_{i,0}\right) \\ &\quad PM(\tau), PN(\tau), \\ &\quad AM(\tau), AN(\tau), \\ &\quad \tau \in [0,t) \end{aligned}$$

where

$$\begin{aligned} V_t \;=\; &\max_{U^\star} \mathbb{E}(J_{[t,\infty)} | \mathcal{F}_{i,t}) \\ &\quad PM(\tau), PN(\tau), \\ &\quad AM(\tau), AN(\tau), \\ &\quad \tau \in [t,\infty) \end{aligned} \tag{7}$$

It is obvious that $J_{[t,\infty)}$ is a function of $U^\star_\infty$, and that $AM(\tau), AN(\tau)$ are functions of ξ_τ and $U^\star_\tau$. As to $PM(\tau), PN(\tau)$, note that, by [2] Proposition 6.8., for any $s \leq \tau$,

$$M_\tau, N_\tau \perp\!\!\!\perp_{I_t, U^\star_s, \xi^{[t,s]}, \eta^{[t,s]}} \mathcal{F}_t$$

so

$$\begin{aligned}\mathbb{P}(M_{\tau_s} \leq \mu_i | \mathcal{F}_{i,s}) = \mathbb{E}(\mathbb{P}(M_{\tau_s} \leq \mu_i | \xi^{[t,s]}, \eta_i^{[t,s]}, \mathcal{F}_t) | \mathcal{F}_{i,s})\\ = \mathbb{E}(\mathbb{P}(M_{\tau_s} \leq \mu_i | \xi^{[t,s]}, \eta_i^{[t,s]}, I_t, U^\star_s)) | \mathcal{F}_{i,s})\\ = \mathbb{P}(M_{\tau_s} \leq \mu_i | \xi^{[t,s]}, \eta_i^{[t,s]}, I_t, U^\star_s) \perp\!\!\!\perp_{I_t, U^\star_s} \mathcal{F}_t\end{aligned}$$

because conditional probabilities are measurable with respect to their conditions. Similarly we show that

$$\mathbb{P}(M_{\tau_s} \leq \mu_i | \mathcal{F}_{i,s}) \perp\!\!\!\perp_{I_t, U^\star_s} \mathcal{F}_t$$

Summarized, out of all $\mathcal{F}_t$, the parameters of problem (7), which generally depends on all the variables generating $\mathcal{F}_t$, in fact depends solely on I_t so its optimal solution has to fulfil (4) too and, consequently, (ii) has to hold. □

Corollary 1. *If the assumptions of the Proposition hold for each $t \in \mathbb{R}^+$ then $(I_t)_{t \geq 0}$ is Markov.*

Proof. For any $t < t_1 < t_2 < \cdots < t_K$,

$$(I_{t_1}, I_{t_v}, \ldots, I_{t_K}) \perp\!\!\!\perp_{I_t} \mathcal{F}_t$$

which is because the random element on the RHS is a function of $\Xi^{[t_1, t_K]}$ which, as it was shown in the proof of the Proposition, is a function of I_t and a variable independent of $\mathcal{F}_t$. □

4 Conclusion

In words, our result says that, no matter how rational the traders are, they might have to adjust their trading to other agents. For instance, when they are "trendists" at the market, even the completely rational traders have to take trends into account.

References

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