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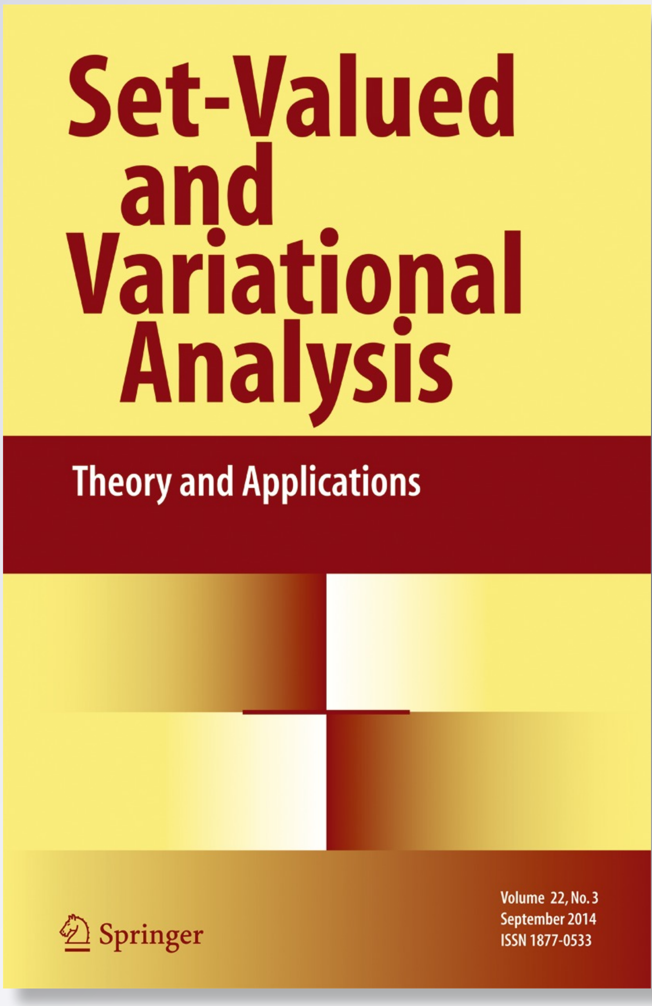
**Set-Valued and Variational Analysis**
Theory and Applications

ISSN 1877-0533
Volume 22
Number 3

Set-Valued Var. Anal (2014) 22:575-595
DOI 10.1007/s11228-014-0278-3




Springer

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# On Stability of M-stationary Points in MPCCs

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Received: 11 October 2013 / Accepted: 27 March 2014 / Published online: 27 April 2014


**Abstract** We consider parameterized Mathematical Programs with Complementarity Constraints arising, e.g., in modeling of deregulated electricity markets. Using the standard rules of the generalized differential calculus we analyze qualitative stability of solutions to the respective M-stationarity conditions. In particular, we provide characterizations and criteria for the isolated calmness and the Aubin properties of the stationarity map. To this end, we introduce the second-order limiting coderivative of mappings and provide formulas for this notion and for the graphical derivative of the limiting coderivative in the case of the normal cone mapping to $\mathbb{R}^n_+$.

**Keywords** Parameterized mathematical programs with complementarity constraints · M-stationarity · Sensitivity analysis · Isolated calmness · Aubin property

**Mathematics Subject Classifications (2010)** 90C31 · 90C33 · 49K40

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## 1 Introduction

In [18], the authors have coined the currently used terminology for various stationarity concepts arising in mathematical programs with complementarity constraints (MPCCs) and studied their mutual relationships. This issue has then been taken up in [4], where the notions of M(Mordukhovich)- and S(Strong)-stationarity have been extended to and analyzed in the setting of general mathematical programs, and the results have then been applied to optimization problems with a disjunctive constraint structure. Nowadays these stationarity concepts are widely used in the MPCC and MPEC literature and have found a lot of applications not only in optimality/stationarity conditions but also in numerics, cf. e.g., [5].

All types of stationarity conditions in MPCCs and MPECs belong to the area of second-order variational analysis because one differentiates the normal-cone mapping and so the resulting conditions contain a second-order generalized derivative. Consider now an MPCC, where both the objective as well as the complementarity system depend on a joint parameter. Similarly as in stability analysis of parameterized mathematical programs or parameterized variational inequalities one could attempt to analyze the local behavior of the multifunction assigning stationary points of a certain type to this parameter. This problem has been tackled already in [18, Section 5], where the authors proved important results about stability of C(Clarke)-stationary points and the associated multipliers. The proof is based on the famous Kojima's result from [11] combined with the theory of $\mathrm{PC}^1$-functions. In this connection one should mention also the papers [8, 12, 13], where one examines the value function generated by a parameterized MPEC. An upper estimate of its limiting subdifferential is expressed in terms of multipliers and also first- and second-order directional derivatives are computed. Sensitivity analysis of the value functions of MPCCs and bilevel optimization problems has also been conducted in [1]. On the other hand, in [10] the authors discuss local stability (homeomorphy invariance) of the feasible set of MPCCs with respect to small perturbations.

In this paper we will concentrate on the local behavior of M-stationary points and the associated multipliers with respect to small changes of the parameter around a certain reference value. Our main workhorse is the second-order limiting coderivative and the graphical derivative of the limiting coderivative which are amenable for polyhedral multifunctions and enable us to describe the analyzed mappings via standard limiting coderivatives and graphical derivatives. This technique paves the way to some useful stability statements.

The primary motivation for our analysis comes from the following class of problems. Consider a collection of parameterized MPECs, such that the control variable in each particular MPEC plays a role of a parameter in all other MPECs from this collection. Such problems are often referred to as equilibrium problems with equilibrium constraints (EPECs) and may be used, e.g., for modeling of the so called multi-leader-follower games. In this way one can describe, e.g., bidding auctions in deregulated electricity markets, where each energy producer in the market plays a role of the upper-level decision maker while the so called Independent System Operator (ISO), in some sense responsible for the effective use of electricity network and demand satisfaction, plays a role of a single lower-level decision-maker. Each day the producers bid supply functions to the ISO and then, taking into account the demand and the structure of the electricity network, ISO announces production of each producer for each hour of the following day. The procedure is repeated next day with different input data.

Since the producers do not know exactly the true production costs of their concurrents and so they only guess their actual bids, one can hardly expect that some overall noncooperative equilibrium in this hierarchical game can be found. On the other hand, each producer may consider the concurrents' bids as a parameter in his own MPEC, and hence respective qualitative (or even quantitative) stability information is essential in the decision making process. If we now reformulate the ISO problem as a nonlinear complementarity problem, we arrive exactly at the model studied in the present paper.

The structure of the paper is as follows. Section 2 contains the problem formulation and some background from variational analysis needed throughout the whole sequel. In Section 3 we analyze the graph of a polyhedral multifunction in order to efficiently compute the tangent and limiting normal cones to this set. This is the basic tool for the computation of the second-order coderivative of the normal-cone mapping in the case of MPCCs. The results are collected in Section 4, where we obtain, in particular, upper approximations of the graphical derivative and the limiting coderivative of the stationarity map assigning the parameter the corresponding M-stationary points and associated multipliers. Finally, in the last Section 5, we state several stability statements, illustrated by academic examples.

Our notation is basically standard. $\mathbb{B}$ denotes the closed unit ball. We use $\mathbb{R}_+$, $\mathbb{R}_-$, $\mathbb{R}_{++}$ and $\mathbb{R}_{--}$ to denote nonnegative, nonpositive, positive and negative reals, respectively. For a set $\Omega$, $\overline{\Omega}$ denotes its closure, and for a closed cone $D$ with vertex at the origin, $D^\circ$ denotes its negative polar cone. By $x \xrightarrow{\Omega} \bar{x}$ we mean that $x \to \bar{x}$ with $x \in \Omega$. $T_\Omega(x)$ signifies the contingent (Bouligand-Severi, tangent) cone to $\Omega$ at $x$.

For the readers' convenience we now state the definitions of several basic notions from modern variational analysis. For a set $\Omega$ and a point $\bar{x} \in \overline{\Omega}$, the *regular (Fréchet) normal cone* to $\Omega$ at $\bar{x}$ is defined by

$$\widehat{N}_\Omega(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \;\middle|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \le 0 \right\} = (T_\Omega(\bar{x}))^\circ .$$

The *limiting (Mordukhovich) normal cone* to $\Omega$ at $\bar{x}$ is given by

$$N_\Omega(\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}_\Omega(x),$$

where the "Lim sup" stands for the Painlevé-Kuratowski upper (or outer) limit. This limit is defined for a set-valued mapping $M[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ at a point $\bar{x}$ by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} M(x) := \left\{ y \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, \exists y_k \to y \text{ with } y_k \in M(x_k) \right\}.$$

For a convex set $\Omega$, both normal cones $N_\Omega$ and $\widehat{N}_\Omega$ amount to the normal cone of convex analysis, for which we use simply the notation $N_\Omega$.

Given a set-valued mapping $M[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ and a point $(\bar{x}, \bar{y})$ from its graph

$$\operatorname{Gph} M := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in M(x) \right\},$$

the *graphical derivative* $DM(\bar{x}, \bar{y}) \left[\mathbb{R}^n \rightrightarrows \mathbb{R}^m\right]$ of $M$ at $(\bar{x}, \bar{y})$ is defined by

$$DM(\bar{x}, \bar{y})(h) := \left\{ k \in \mathbb{R}^m | (h, k) \in T_{\operatorname{Gph} M}(\bar{x}, \bar{y}) \right\},$$

the *regular (Fréchet) coderivative* $\widehat{D}^* M(\bar{x}, \bar{y}) \left[\mathbb{R}^m \rightrightarrows \mathbb{R}^n\right]$ of $M$ at $(\bar{x}, \bar{y})$ is defined by

$$\widehat{D}^* M(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in \mathbb{R}^n | (x^*, -y^*) \in \widehat{N}_{\operatorname{Gph} M}(\bar{x}, \bar{y}) \right\},$$

and the *limiting (Mordukhovich) coderivative* $D^*M(\bar{x},\bar{y})\left[\mathbb{R}^m \rightrightarrows \mathbb{R}^n\right]$ of $M$ at $(\bar{x},\bar{y})$ is defined by

$$D^*M(\bar{x},\bar{y})(y^*) := \left\{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N_{\mathrm{Gph}M}(\bar{x},\bar{y})\right\}.$$

Finally, throughout the paper we employ the notions of Aubin property, isolated calmness and calmness.

A set-valued mapping $M\left[\mathbb{R}^n \rightrightarrows \mathbb{R}^m\right]$ is said to have the *Aubin (pseudo-Lipschitz, Lipschitz-like) property* around $(\bar{x},\bar{y}) \in \mathrm{Gph}\, M$ with modulus $\ell \geq 0$ if there are neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ such that

$$M(x) \cap \mathcal{V} \subset M(u) + \ell||x-u||\mathbb{B}$$

for all $x, u \in \mathcal{U}$, where $\mathbb{B}$ is closed unit ball. The Mordukhovich criterion [14] provides a characterization of the Aubin property through knowledge of the respective coderivative: a set-valued mapping $M$ has Aubin property around $(\bar{x},\bar{y})$ if and only if

$$D^*M(\bar{x},\bar{y})(0) = \{0\}.$$

A multifunction $M\left[\mathbb{R}^n \rightrightarrows \mathbb{R}^m\right]$ is said to have the *isolated calmness (calmness on selections) property* (to be *isolatedly calm*) at $(\bar{x},\bar{y}) \in \mathrm{Gph}\, M$, provided there exist neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ and a constant $\kappa \geq 0$ such that

$$M(x) \cap \mathcal{V} \subset \{\bar{y}\} + \kappa||x-\bar{x}||\mathbb{B} \quad \text{when } x \in \mathcal{U}.$$

It has been proved (see e.g. [3, Theorem 4C.1]) that $M$ possesses the isolated calmness property at $(\bar{x},\bar{y})$ if and only if

$$DM(\bar{x},\bar{y})(0) = \{0\}.$$

A set-valued mapping $M\left[\mathbb{R}^n \rightrightarrows \mathbb{R}^m\right]$ is said to be *calm (pseudo upper Lipschitz)* at $(\bar{x},\bar{y}) \in \mathrm{Gph}\, M$ with modulus $L \geq 0$ if there are neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ such that

$$M(x) \cap \mathcal{V} \subset M(\bar{x}) + L\|x-\bar{x}\|\mathbb{B} \quad \text{for all } x \in \mathcal{U}.$$

Clearly, both the Aubin and the isolated calmness properties imply calmness, whereas neither converse is true. There does not exist any relationship between these two properties of a multifunction. In the sequel, calmness will be utilized as a suitable qualification condition in the used rules of generalized differential calculus, cf. [6, 9], whereas the Aubin and the isolated calmness properties will be considered as valuable stability concepts for the behavior of M-stationary points with respect to the parameter.

## 2 Problem Statement and Preliminaries

Throughout the whole paper, we shall be concerned with the following parameter-dependent MPCC:

$$\begin{aligned} &\text{minimize } f(p,x,y)\\ &\text{subject to}\\ &\qquad 0 \in F(p,x,y) + N_{\mathbb{R}^m_+}(y), \end{aligned} \tag{1}$$

where $f\left[\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\right]$ and $F\left[\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m\right]$ are at least twice continuously differentiable functions. For simplicity, we do not consider here the so-called non-equilibrial geometric constraints $(x,y) \in \Omega$.

The M-stationarity conditions of (1) may be written down as follows [16, Theorem 3.1]:

$$\begin{aligned} 0 &= \nabla_x f(p,x,y) + (\nabla_x F(p,x,y))^\top \nu, \\ 0 &\in \nabla_y f(p,x,y) + (\nabla_y F(p,x,y))^\top \nu + D^* N_{\mathbb{R}^m_+}(y, -F(p,x,y))(\nu), \end{aligned} \tag{2}$$

where $\nu \in \mathbb{R}^m$ is the so-called MPCC-multiplier.

Stationarity conditions in the form (2) originate from [20, Theorem 3.2]. The terminology "M-stationarity" comes from the fact that these conditions are obtained by means of generalized differential calculus of Mordukhovich applied to the limiting normal cone to the graph of the generalized equation in the constraints of (1). Here, however, we do not address any constraint qualification guaranteeing that, at a local minimum of (1), there exists a $\nu$ satisfying these necessary optimality conditions, which has been extensively discussed in the literature, see, e.g., [16, 19]. Rather, we are interested in analysis of local behavior of solutions to (2) around the reference point $(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})$ and so the existence of this reference point is assumed.

To unburden the notation in the formulas which will be derived in the sequel, let us replace (2) by the following generalized equation system:

$$\begin{aligned} 0 &= a(p,x,y) + A^\top(p,x,y)\nu \\ 0 &\in b(p,x,y) + B^\top(p,x,y)\nu + D^* N_{\mathbb{R}^m_+}(y, -F(p,x,y))(\nu), \end{aligned} \tag{3}$$

where the vectors and matrices $a, b, A$ and $B$ correspond to $\nabla_x f$, $\nabla_y f$, $\nabla_x F$ and $\nabla_y F$, respectively. The multifunction in the second line of (3) is the composition of the map $(u, v, w) \rightrightarrows D^* N_{\mathbb{R}^m_+}(u, v)(w)$ with the smooth single-valued mapping

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} y \\ -F(p,x,y) \\ \nu \end{bmatrix}.$$

Denoting $\Lambda :=$ Gph $D^* N_{\mathbb{R}^m_+}$ and

$$\Psi(p,x,y,\nu) = \begin{bmatrix} a(p,x,y) + A^\top(p,x,y)\nu \\ y \\ -F(p,x,y) \\ \nu \\ -b(p,x,y) - B^\top(p,x,y)\nu \end{bmatrix},$$

we can rewrite (3) to

$$\Psi(p,x,y,\nu) \in \{0\} \times \Lambda \tag{4}$$

and associate it with a stationarity map $\mathcal{S}$

$$\mathcal{S}(p) = \{(x,y,\nu) | \Psi(p,x,y,\nu) \in \{0\} \times \Lambda\}. \tag{5}$$

Now assume that for each triple $(p, x, y)$ close to $(\bar{p}, \bar{x}, \bar{y})$ there is a unique multiplier $\nu$ satisfying (4). This holds in particular if mapping $A$ is surjective at $(\bar{p}, \bar{x}, \bar{y})$. Set $C = \left(AA^\top\right)^{-1} A$. Hence, $\nu = -Ca$. Define

$$\Psi^r(p,x,y) = \begin{bmatrix} y \\ -F(p,x,y) \\ -C(p,x,y)a(p,x,y) \\ -b(p,x,y) - d(p,x,y) \end{bmatrix},$$

where $d(p, x, y) = -(B(p, x, y))^\top C(p, x, y)a(p, x, y)$. Then we can reduce the M-stationarity conditions (3) to

$$\Psi^r(p, x, y) \in \Lambda$$

and consider the following "reduced" stationarity map $\mathcal{S}^r$

$$\mathcal{S}^r(p) = \left\{(x, y)|\Psi^r(p, x, y) \in \Lambda\right\}.$$

*Example 1* Let $p, x, y \in \mathbb{R}$ and consider the following perturbed MPCC

$$\begin{aligned} &\text{minimize } (x+1)^2 + y \\ &\text{subject to} \\ &\qquad\qquad 0 \in p + x + N_{\mathbb{R}_+}(y). \end{aligned}$$

Clearly, the M-stationarity conditions (2) can be formulated in the reduced form as

$$\begin{bmatrix} y \\ -x-p \\ -2x-2 \\ -1 \end{bmatrix} \in \text{ Gph } D^* N_{\mathbb{R}_+},$$

where $A = C = (1)_{1\times1}$, $B = (0)_{1\times1}$, $a(p, x, y) = 2x + 2$, $b(p, x, y) = 1$, $d(p, x, y) = 0$. It is easy to check that for $\bar{p} \leq 1$ the conditions (2) have a unique solution $\bar{x} = -\bar{p}$, $\bar{y} = 0$ with the associated MPCC-multiplier $\bar{v} = 2\bar{p} - 2$, while for $\bar{p} \geq 1$ the conditions (2) have a unique solution $\bar{x} = -1$, $\bar{y} = 0$ with the corresponding MPCC-multiplier $\bar{v} = 0$.

As the main tool for investigation of the local behavior of stationarity maps $\mathcal{S}$ and $\mathcal{S}^r$ we introduce the second-order limiting coderivative of mappings.

**Definition 1** (Second-order limiting coderivative)
Let $Q$ be a multifunction with a closed graph. Assume that $(u, v) \in \text{Gph } Q$ and $z \in D^*Q(u, v)(w)$ and put $\text{Gph } D^*Q = \{(u, v, w, z)|z \in D^*Q(u, v)(w)\}$. Then the *second-order limiting coderivative* of $Q$ at $(u, v, w, z)$ is defined by

$$D^{**}Q(u, v, w, z)(z^*) = \left\{(u^*, v^*, w^*)|(u^*, v^*, w^*, -z^*) \in N_{\text{Gph } D^*Q}(u, v, w, z)\right\}.$$

Similarly, one can define other types of second-order coderivatives, e.g. via limiting normal cone to the graph of the regular coderivative. In the sequel we work only with the one introduced in Definition 1 and thus for brevity we omit the word "limiting". The following example illuminates the relationship between the second-order limiting coderivate and the third-order limiting subdifferential analogous to the one between limiting coderivate and the second-order limiting subdifferential, cf. [15, Definitions 1.77, 1.118] for definitions of limiting subdifferential $\partial\varphi$ and second-order limiting subdifferential $\partial^2\varphi$ of a single-valued function $\varphi$.

*Example 2* Consider a function $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and set $Q = \partial\varphi$, $v \in \partial\varphi(u)$ and $z \in \partial^2\varphi(u, v)(w)$. Then the second-order limiting coderivative $D^{**}\partial\varphi(u, v, w, z)$ can be understood as the third-order limiting subdifferential $\partial^3\varphi(u, v, w, z)$ of $\varphi$ at $u$ with respect to $v \in \partial\varphi(u)$ and $(w, z)$ such that $(u, v, w, z) \in \text{Gph } \partial^2\varphi$.

More specifically, in our analysis of the local behavior of stationarity maps $\mathcal{S}$ and $\mathcal{S}^r$ we will need a formula for the graph of the contingent cone $T_{\mathrm{Gph} D^* N_{\mathbb{R}^m_+}}$ and for the graph of $D^{**} N_{\mathbb{R}^m_+}$. In the following section we give a recipe for construction of these sets.

## 3 Generalized Derivatives of $D^* N_{\mathbb{R}^m_+}$

We start this section with calculation of graphs of $T_{\mathrm{Gph} D^* N_{\mathbb{R}_+}}$ and $D^{**} N_{\mathbb{R}_+}$ which will be later generalized to the higher-dimensional case. Nevertheless, we propose a more general procedure dealing with a multifunction $Q\left[\mathbb{R}^m \rightrightarrows \mathbb{R}^n\right]$ satisfying the following condition:

$$\mathrm{Gph}\ Q = \bigcup_{i=1}^{r} \Gamma_i \tag{6}$$

where $r$ is finite and for all components $\Gamma_i \subset \mathbb{R}^m \times \mathbb{R}^n$ one has $\Gamma_i \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}\}^{m+n}$. This is indeed of our interest since Gph $N_{\mathbb{R}_+} = \mathbb{R}_+ \times \{0\} \cup \{0\} \times \mathbb{R}_-$ and also Gph $D^* N_{\mathbb{R}_+}$ have such a structure.

For computation of the normal cone to general polyhedral set we refer an interested reader to [2]. For a more general case of a union of polyhedral sets, we can use the theory developed in [7]. However, it does not exploit the simple structure of the polyhedral multifunction $Q$ we deal with here. On that account, we develop a new procedure to construct the graph of generalized derivatives of such multifunction. Moreover, we will observe that the same procedure may be used to calculate even higher-order generalized derivatives of $Q$.

We work in three steps: first, we obtain a formula for Gph$T_{\mathrm{Gph}\ Q}$, then, based on this result, we compute Gph $\hat{D}^* Q$, and finally we will proceed with a formula for Gph $D^* Q$.

To find Gph $T_{\mathrm{Gph}\ Q}$ we need to evaluate $T_{\mathrm{Gph}\ Q}(u, v)$ for any $u, v$ such that $v \in Q(u)$. To this end, we partition Gph $Q$ following [7] as follows

$$\mathrm{Gph}\ Q = \bigcup_{\emptyset \neq I \subset \{1,\ldots,r\}} \Gamma_I, \tag{7}$$

where for any $\emptyset \neq I \subset \{1, \ldots, r\}$ we define

$$\Gamma_I = \left( \bigcap_{i \in I} \Gamma_i \right) \setminus \left( \bigcup_{i \in \{1,\ldots,r\} \setminus I} \Gamma_i \right). \tag{8}$$

To ease the evaluation of $\Gamma_I$ in higher dimensions, we propose the following lemma.

**Lemma 1** *Consider $D = \times_{i=1}^k D_i$, $E = \times_{i=1}^k E_i$ with $k$ finite and $D_i, E_i \subset \mathbb{R}^{d_i}$. Then, it holds*

$$D \setminus E = \left( \mathop{\times}_{i=1}^{k} D_i \right) \setminus \left( \mathop{\times}_{i=1}^{k} E_i \right) = \bigcup_{i:(D_i \setminus E_i) \neq \emptyset} D_1 \times \ldots \times D_{i-1} \times (D_i \setminus E_i) \times D_{i+1} \times \ldots \times D_k.$$

*In particular, we have $D \setminus E = D$ whenever there exists $j \in \{1, \ldots, k\}$ such that $D_j \cap E_j = \emptyset$.*

*Proof* For $k = 2$ we observe $(D_1 \times D_2) \setminus (E_1 \times E_2) = ((D_1 \setminus E_1) \times D_2) \cup (D_1 \times (D_2 \setminus E_2))$, for a general $k > 2$ the statement may be shown by induction. $\square$

With regards to this lemma we see that each $\Gamma_I$ defined by (8) has to be a finite union of sets from $\{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}, \mathbb{R}_{--}, \mathbb{R}_{++}, \mathbb{R} \setminus \{0\}\}^{m+n}$, since this is a closure of set system $\{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}\}^{m+n}$ with respect to set complements. Further, we will observe that $\Gamma_I$ may be represented also by a finite union of sets from $\{\mathbb{R}, \mathbb{R}_{--}, \mathbb{R}_{++}, \{0\}\}^{m+n}$ with a straight impact on the calculation of the tangent cone. Recall, that $\mathbb{R}_{++}$ and $\mathbb{R}_{--}$ stands for the set of positive and negative reals, respectively.

**Lemma 2** *Consider $\emptyset \neq I \subset \{1, \ldots, r\}$ and $\Gamma_I$ defined as above. Then, there exists a finite number $q$ and a finite partition $\{\Gamma_{I,k}\}_{k=1}^{q}$ of $\Gamma_I$ such that $\Gamma_{I,k} \in \{\mathbb{R}, \mathbb{R}_{++}, \mathbb{R}_{--}, \{0\}\}^{m+n}$. Moreover, for any $k \in \{1, \ldots, q\}$, $i \in I$ and $x, y \in \Gamma_{I,k} \subset \Gamma_I$ it holds $T_{\Gamma_i}(x) = T_{\Gamma_i}(y)$. We denote this constant value by $T_{\Gamma_i}(\Gamma_{I,k})$. For $q$ we further use notation $q(I)$ since it generally depends on $I$.*

*Proof* We know that $\Gamma_I$ is a finite union of sets from the system $\{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}, \mathbb{R}_{--}, \mathbb{R}_{++}, \mathbb{R} \setminus \{0\}\}^{m+n}$. Expressing each involved $\mathbb{R}_+$, $\mathbb{R}_-$, and $\mathbb{R} \setminus \{0\}$ in terms of $\mathbb{R}_{++} \cup \{0\}$, $\mathbb{R}_{--} \cup \{0\}$ and $\mathbb{R}_{--} \cup \mathbb{R}_{++}$, respectively, we have shown the first part of the statement. Next, we prove the second part only in one dimension for simplicity. For any $x, y \in \Gamma_{I,k} \subset \Gamma_I$ we may verify $T_{\Gamma_i}(x) = T_{\Gamma_i}(y)$ by direct evaluation of all the possible combinations of $\Gamma_i$ and $\Gamma_{I,k}$. For example, for the case of $\Gamma_i = \mathbb{R}_+$ we may have either $\Gamma_{I,k} = \mathbb{R}_{++}$ or $\Gamma_{I,k} = \{0\}$. Then, we observe that $T_{\mathbb{R}_+}(\mathbb{R}_{++})$ and $T_{\mathbb{R}_+}(\{0\})$ are indeed well-defined. Finally we note that the proof goes in a similar way even in a higher dimension. □

Now, for any $\emptyset \neq I \subset \{1, \ldots, r\}$ and $k \in \{1, \ldots, q(I)\}$ we show that even $T_{\mathrm{Gph}\ Q}(x)$ is constant with respect to $x \in \Gamma_{I,k}$. Indeed, we see that there exists a neighborhood $\mathcal{V}$ of $x$ such that $\mathrm{Gph}\ Q \cap \mathcal{V} = \bigcup_{i \in I} \Gamma_i \cap \mathcal{V}$ owing to the definition of $\Gamma_I$. Next, with regards to Lemma 2, we may write

$$T_{\mathrm{Gph}\ Q}(\Gamma_{I,k}) = T_{\bigcup_{i\in I} \Gamma_i}(\Gamma_{I,k}) = \bigcup_{i \in I} T_{\Gamma_i}(\Gamma_{I,k}), \tag{9}$$

where $T_{\mathrm{Gph}\ Q}(\Gamma_{I,k})$ is used to denote the constant value of $T_{\mathrm{Gph}\ Q}(x)$ with respect to $x \in \Gamma_{I,k}$. Now, we may directly construct the whole Gph $T_{\mathrm{Gph}\ Q}$, but some more effort is needed to construct also Gph $D^* Q$. On that account, we introduce a set operator to ease the further notation.

**Definition 2** For $m, n$ finite we define set-valued operator $\mathcal{T}_{mn} : \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ in such a way that for set $D \subset \mathbb{R}^m \times \mathbb{R}^n$ we have

$$\mathcal{T}_{mn}[D] = \left\{(-y, x) \in \mathbb{R}^n \times \mathbb{R}^m | (x, y) \in D\right\}.$$

We introduce this operator in order to establish equivalence of $(u, v, w, z) \in \mathrm{Gph}\ D^* Q$ with $(w, z) \in \mathcal{T}_{mn}\left[N_{\mathrm{Gph}\ Q}(u, v)\right]$.

**Theorem 1** *Let $Q\left[\mathbb{R}^m \rightrightarrows \mathbb{R}^n\right]$ be a multifunction such that Gph $Q = \bigcup_{i=1}^{r} \Gamma_i$ and $\Gamma_i \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}\}^{m+n}$. Then*

$$Gph\ T_{GphQ} = \bigcup_{\emptyset \neq I \subset \{1,\ldots,r\}} \bigcup_{k=1}^{q(I)} \left(\Gamma_{I,k} \times \bigcup_{i \in I} T_{\Gamma_i}(\Gamma_{I,k})\right). \tag{10}$$

*and*

$$Gph\ D^*Q = \bigcup_{\emptyset \neq I \subset \{1,\ldots,r\}} \bigcup_{k=1}^{q(I)} \left( \overline{\Gamma_{I,k}} \times \mathcal{T}_{mn} \left[ \bigcap_{i \in I} \hat{N}_{\Gamma_i}(\Gamma_{I,k}) \right] \right). \tag{11}$$

*where* $\{\Gamma_{I,k}\}_{k=1}^{q(I)}$ *is the finite partition of* $\Gamma_I$ *from Lemma 2.*

*Proof* First, observe that formula (10) stems directly from (9) and the fact that Gph $Q = \bigcup_{\emptyset \neq I \subset \{1,\ldots,r\}} \bigcup_{k=1}^{q(I)} \Gamma_{I,k}$. Then, one can easily find a formula for Gph $\hat{N}_{Gph\,Q}$; it reads

$$\text{Gph } \hat{N}_{\text{Gph } Q} = \bigcup_{\emptyset \neq I \subset \{1,\ldots,r\}} \bigcup_{k=1}^{q(I)} \left( \Gamma_{I,k} \times \bigcap_{i \in I} \hat{N}_{\Gamma_i}(\Gamma_{I,k}) \right), \tag{12}$$

where the fact that also $\hat{N}_{\Gamma_i}(x)$ is constant with respect to $x \in \Gamma_{I,k}$ is used. Now, according to the definition of the regular coderivative, we have

$$(u, v, w, z) \in \text{Gph } \hat{D}^*Q \Leftrightarrow (z, -w) \in \hat{N}_{\text{Gph } Q}(u, v)$$

with $(u, v) \in$ Gph $Q$. Thus, applying operator $\mathcal{T}_{mn}$ in (12), we obtain

$$\text{Gph } \hat{D}^*Q = \bigcup_{I \subset \{1,\ldots,r\}} \bigcup_{k=1}^{q(I)} \left( \Gamma_{I,k} \times \mathcal{T}_{mn} \left[ \bigcap_{i \in I} \hat{N}_{\Gamma_i}(\Gamma_{I,k}) \right] \right).$$

Now, since Gph $D^*Q$ is simply the closure of Gph $\hat{D}^*Q$, and since all $\hat{N}_{\Gamma_i}(\Gamma_{I,k})$ are closed, we have proved (11). □

Using Theorem 1, we shall now compute the graphs of $T_{\text{Gph } D^*N_{\mathbb{R}_+}}$ and $D^{**}N_{\mathbb{R}_+}$.

*Example 3* Set $Q = N_{\mathbb{R}_+}$. It is easy to verify that Gph $N_{\mathbb{R}_+} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \mathbb{R}_+ \times \{0\}$ and $\Gamma_2 = \{0\} \times \mathbb{R}_-$. We observe that $\Gamma_{\{1\}}$, $\Gamma_{\{2\}}$, and $\Gamma_{\{12\}}$ are already subsets of $\{\mathbb{R}, \mathbb{R}_{++}, \mathbb{R}_{--}, \{0\}\}^2$, and so there is no need to use Lemma 2. Thus, to find $D^*N_{\mathbb{R}_+}$ we may compute regular normal cones directly as follows.

$$\begin{aligned}
\hat{N}_{\text{Gph } N_{\mathbb{R}_+}}(\Gamma_{\{12\}}) = \hat{N}_{\Gamma_1 \cup \Gamma_2}(\Gamma_1 \cap \Gamma_2) &= \hat{N}_{\Gamma_1}(\{0\} \times \{0\}) \cap \hat{N}_{\Gamma_2}(\{0\} \times \{0\}) = \mathbb{R}_- \times \mathbb{R}_+, \\
\hat{N}_{\text{Gph } N_{\mathbb{R}_+}}(\Gamma_{\{1\}}) = \hat{N}_{\Gamma_1}(\Gamma_1 \setminus \Gamma_2) &= \hat{N}_{\Gamma_1}(\mathbb{R}_{++} \times \{0\}) = \{0\} \times \mathbb{R}, \\
\hat{N}_{\text{Gph } N_{\mathbb{R}_+}}(\Gamma_{\{2\}}) = \hat{N}_{\Gamma_2}(\Gamma_2 \setminus \Gamma_1) &= \hat{N}_{\Gamma_2}(\{0\} \times \mathbb{R}_{--}) = \mathbb{R} \times \{0\}.
\end{aligned}$$

Next, involving the $\mathcal{T}_{mn}$-operator according to (11) we obtain

$$\begin{aligned}
\text{Gph } \hat{D}^*N_{\mathbb{R}_+} = (\{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}_-) &\bigcup (\mathbb{R}_{++} \times \{0\} \times \mathbb{R} \times \{0\}) \\
&\bigcup (\{0\} \times \mathbb{R}_{--} \times \{0\} \times \mathbb{R}),
\end{aligned}$$

and finally we conclude that Gph $D^*N_{\mathbb{R}_+} = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ with

$$\begin{aligned}
\Lambda_1 &= \{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}_-, \\
\Lambda_2 &= \mathbb{R}_+ \times \{0\} \times \mathbb{R} \times \{0\}, \\
\Lambda_3 &= \{0\} \times \mathbb{R}_- \times \{0\} \times \mathbb{R}.
\end{aligned}$$

We note that Gph $D^*N_{\mathbb{R}_+}$ may be constructed even without an explicit knowledge of Gph $T_{\text{Gph } N_{\mathbb{R}_+}}$.

Now, we see that also Gph $D^*N_{\mathbb{R}_+}$ satisfies the assumptions of Theorem 1 and so we may find Gph $D^{**}N_{\mathbb{R}_+}$ analogously. First we find the partition

$$\text{Gph } D^*N_{\mathbb{R}_+} = \bigcup_{\emptyset \neq I \subset \{1,2,3\}} \Lambda_I$$

using Lemma 1 and the partitions of the relevant $\Lambda_I$'s according to Lemma 2. One obtains

$$\begin{aligned}
\Lambda_{\{123\}} &= \{0\} \times \{0\} \times \{0\} \times \{0\},\\
\Lambda_{\{12\}} &= \{0\} \times \{0\} \times \mathbb{R}_{--} \times \{0\},\\
\Lambda_{\{13\}} &= \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_{--},\\
\Lambda_{\{23\}} &= \emptyset,\\
\Lambda_{\{1\}} &= \{0\} \times \{0\} \times \mathbb{R}_{--} \times \mathbb{R}_{--},\\
\Lambda_{\{2\},1} &= \mathbb{R}_{++} \times \{0\} \times \mathbb{R} \times \{0\},\\
\Lambda_{\{2\},2} &= \{0\} \times \{0\} \times \mathbb{R}_{++} \times \{0\},\\
\Lambda_{\{3\},1} &= \{0\} \times \mathbb{R}_{--} \times \{0\} \times \mathbb{R},\\
\Lambda_{\{3\},2} &= \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_{++},
\end{aligned}$$

and by virtue of (10) one has

$$\text{Gph } T_{\text{Gph } D^*N_{\mathbb{R}_+}} = \begin{cases}
\{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}_-\\
\bigcup \{0\} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R} \times \{0\}\\
\bigcup \{0\} \times \{0\} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}_- \times \{0\} \times \mathbb{R}\\
\bigcup \{0\} \times \{0\} \times \mathbb{R}_{--} \times \{0\} \times \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}_-\\
\bigcup \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_{--} \times \{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}\\
\bigcup \{0\} \times \{0\} \times \mathbb{R}_{--} \times \mathbb{R}_{--} \times \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}\\
\bigcup \mathbb{R}_{++} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}\\
\bigcup \{0\} \times \mathbb{R}_{--} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}.
\end{cases} \tag{13}$$

Finally, we employ (11) and conclude that

$$\text{Gph } D^{**}N_{\mathbb{R}_+} = \begin{cases}
\{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}_+ \times \{0\}\\
\bigcup \{0\} \times \{0\} \times \mathbb{R}_- \times \{0\} \times \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R} \times \{0\}\\
\bigcup \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_- \times \{0\} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+\\
\bigcup \{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}_- \times \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\}\\
\bigcup \mathbb{R}_+ \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}\\
\bigcup \{0\} \times \{0\} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R} \times \mathbb{R}_- \times \mathbb{R} \times \{0\}\\
\bigcup \{0\} \times \mathbb{R}_- \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}\\
\bigcup \{0\} \times \{0\} \times \{0\} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.
\end{cases} \tag{14}$$

In order to handle the general case where $Q = N_{\mathbb{R}^m_+}$ we will associate with each pair $(u, v) = (y, -F(p, x, y)) \in \operatorname{Gph} N_{\mathbb{R}^m_+}$ the index sets

$$\begin{aligned} L(p,x,y) &:= \{i \in \{1,\ldots,m\} | (y_i, -F_i(p,x,y)) \in \mathbb{R}_{++} \times \{0\}\}, \\ I_+(p,x,y) &:= \{i \in \{1,\ldots,m\} | (y_i, -F_i(p,x,y)) \in \{0\} \times \mathbb{R}_{--}\}, \\ I_0(p,x,y) &:= \{i \in \{1,\ldots,m\} | (y_i, -F_i(p,x,y)) \in \{0\} \times \{0\}\}. \end{aligned}$$

These index sets are related to the complementarity constraints in (1) and they are frequently called index sets of inactive, strongly active and weakly active inequalities, respectively. Similarly, we shall introduce the following disjunctive decomposition of the set of indices of weakly active inequalities associated with quadruple $(u, v, w, z) = (y, -F(p, x, y), \nu, \mu) \in \operatorname{Gph} D^* N_{\mathbb{R}^m_+}$:

$$\begin{aligned} I_0^{--}(p,x,y) &:= \{i \in I_0(p,x,y) | (\nu_i, \mu_i) \in \mathbb{R}_{--} \times \mathbb{R}_{--}\}, \\ I_0^{-0}(p,x,y) &:= \{i \in I_0(p,x,y) | (\nu_i, \mu_i) \in \mathbb{R}_{--} \times \{0\}\}, \\ I_0^{0-}(p,x,y) &:= \{i \in I_0(p,x,y) | (\nu_i, \mu_i) \in \{0\} \times \mathbb{R}_{--}\}, \\ I_0^{+0}(p,x,y) &:= \{i \in I_0(p,x,y) | (\nu_i, \mu_i) \in \mathbb{R}_{++} \times \{0\}\}, \\ I_0^{0+}(p,x,y) &:= \{i \in I_0(p,x,y) | (\nu_i, \mu_i) \in \{0\} \times \mathbb{R}_{++}\}, \\ I_0^{00}(p,x,y) &:= \{i \in I_0(p,x,y) | (\nu_i, \mu_i) \in \{0\} \times \{0\}\}. \end{aligned}$$

Note that these sets cover all possibilities for values of $(y, -F(p, x, y), \nu, \mu)$ with respect to $\operatorname{Gph} D^* N_{\mathbb{R}^m_+}$. We shall omit the arguments of these index sets whenever it cannot cause any confusion.

Consider $(\alpha, \beta, \gamma, \delta) \in T_{\operatorname{Gph} D^* N_{\mathbb{R}^m_+}}(y, -F(p, x, y), \nu, \mu)$ and $(u^*, v^*, w^*, -z^*) \in N_{\operatorname{Gph} D^* N_{\mathbb{R}^m_+}}(y, -F(p, x, y), \nu, \mu)$; then for each $i \in \{1. \ldots, m\}$

| | | |
|---|---|---|
| for $i \in L$ | $\alpha_i \in \mathbb{R}, \beta_i = 0, \gamma_i \in \mathbb{R}, \delta_i = 0$ | $u_i^* = 0, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R}$ |
| for $i \in I_+$ | $\alpha_i = 0, \beta_i \in \mathbb{R}, \gamma_i = 0, \delta_i \in \mathbb{R}$ | $u_i^* \in \mathbb{R}, v_i^* = 0, w_i^* \in \mathbb{R}, z_i^* = 0$ |
| for $i \in I_0^{--}$ | $\alpha_i = 0, \beta_i = 0, \gamma_i \in \mathbb{R}, \delta_i \in \mathbb{R}$ | $u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* = 0$ |
| for $i \in I_0^{-0}$ | $\begin{cases} \alpha_i = 0, \beta_i = 0, \gamma_i \in \mathbb{R}, \delta_i \in \mathbb{R}_- \text{ or} \\ \alpha_i \in \mathbb{R}_+, \beta_i = 0, \gamma_i \in \mathbb{R}, \delta_i = 0 \end{cases}$ | $\begin{cases} u_i^* = 0, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R} \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* = 0 \text{ or} \\ u_i^* \in \mathbb{R}_-, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R}_- \end{cases}$ |
| for $i \in I_0^{0-}$ | $\begin{cases} \alpha_i = 0, \beta_i = 0, \gamma_i \in \mathbb{R}_-, \delta_i \in \mathbb{R} \text{ or} \\ \alpha_i = 0, \beta_i \in \mathbb{R}_-, \gamma_i = 0, \delta_i \in \mathbb{R} \end{cases}$ | $\begin{cases} u_i^* \in \mathbb{R}, v_i^* = 0, w_i^* \in \mathbb{R}, z_i^* = 0 \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* = 0 \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}_+, w_i^* \in \mathbb{R}_+, z_i^* = 0 \end{cases}$ |
| for $i \in I_0^{+0}$ | $\alpha_i \in \mathbb{R}_+, \beta_i = 0, \gamma_i \in \mathbb{R}, \delta_i = 0$ | $\begin{cases} u_i^* = 0, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R} \text{ or} \\ u_i^* \in \mathbb{R}_-, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R} \end{cases}$ |
| for $i \in I_0^{0+}$ | $\alpha_i = 0, \beta_i \in \mathbb{R}_-, \gamma_i = 0, \delta_i \in \mathbb{R}$ | $\begin{cases} u_i^* \in \mathbb{R}, v_i^* = 0, w_i^* \in \mathbb{R}, z_i^* = 0 \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}_+, w_i^* \in \mathbb{R}, z_i^* = 0 \end{cases}$ |

$$\text{for } i \in I_0^{00} \quad \begin{cases} \alpha_i = 0, \beta_i = 0, \gamma_i \in \mathbb{R}_-, \delta_i \in \mathbb{R}_- \text{ or} \\ \alpha_i \in \mathbb{R}_+, \beta_i = 0, \gamma_i \in \mathbb{R}, \delta_i = 0 \text{ or} \\ \alpha_i = 0, \beta_i \in \mathbb{R}_-, \gamma_i = 0, \delta_i \in \mathbb{R} \end{cases} \quad \begin{cases} u_i^* = 0, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R}, \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* = 0, w_i^* \in \mathbb{R}, z_i^* = 0 \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* = 0 \text{ or} \\ u_i^* \in \mathbb{R}_-, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R}_- \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}_+, w_i^* \in \mathbb{R}_+, z_i^* = 0, \text{ or} \\ u_i^* \in \mathbb{R}_-, v_i^* \in \mathbb{R}, w_i^* = 0, z_i^* \in \mathbb{R}, \text{ or} \\ u_i^* \in \mathbb{R}, v_i^* \in \mathbb{R}_+, w_i^* \in \mathbb{R}, z_i^* = 0, \text{ or} \\ u_i^* \in \mathbb{R}_-, v_i^* \in \mathbb{R}_+, w_i^* = 0, z_i^* = 0 \end{cases} \tag{15}$$

To illuminate the computation of generalized derivatives in the case of $m > 1$, we conclude this section with the following example.

*Example 4* Let $m = 2, y = (0,1), F(p,x,y) = (0,0), \nu = (0,-1), \mu = (-1,0)$ and $z^* = (0,1)$. Then, clearly, $I_0^{0-} = \{1\}, L = \{2\}$. Thus $(u^*, v^*, w^*) \in D^{**}N_{\mathbb{R}_+^m}(y, -F(p,x,y), \nu, \mu)(z^*)$ whenever

$$\begin{array}{lclcl} u^* \in \mathbb{R} \times \{0\}, & & u^* \in \mathbb{R} \times \{0\}, & & u^* \in \mathbb{R} \times \{0\}, \\ v^* \in \{0\} \times \mathbb{R}, & \text{or} & v^* \in \mathbb{R} \times \mathbb{R}, & \text{or} & v^* \in \mathbb{R}_+ \times \mathbb{R}, \\ w^* \in \mathbb{R} \times \{0\}, & & w^* \in \{0\} \times \{0\}, & & w^* \in \mathbb{R}_+ \times \{0\}. \end{array}$$

## 4 Generalized Derivatives of Stationarity Maps

This section is devoted to computation of the graphical derivative and the coderivative of both stationarity and reduced stationarity maps $\mathcal{S}$ and $\mathcal{S}^r$, respectively. In what follows, $(\bar{x}, \bar{y})$ is called the M-stationary pair with a corresponding MPCC-multiplier $\bar{\nu}$.

To justify the need for the corresponding formulas for *both types* of the stationarity maps, we give an example of an MPCC for which the reduction of the stationarity map $\mathcal{S}$ into $\mathcal{S}^r$ is not possible.

*Example 5* Consider $p, x, y \in \mathbb{R}$ and the following parameter-dependent MPCC

$$\text{minimize } \frac{1}{2}x^2 + y(p+1)$$

subject to

$$0 \in p + y(p+1) - \frac{1}{2}x^2 + N_{\mathbb{R}_+}(y).$$

The stationarity map $\mathcal{S}$ is thus given by map $\Psi$

$$\Psi(p,x,y,\nu) = \begin{pmatrix} x(1-\nu) \\ y \\ -p - y(p+1) + \frac{1}{2}x^2 \\ \nu \\ -(p+1)(\nu+1) \end{pmatrix}.$$

For $\bar{p} \le -1$ the corresponding M-stationarity conditions do not have any solution while

| | |
|---|---|
| for $\bar{p} \in (-1, 0)$ | there is a unique M-stationary pair $\bar{x} = 0$, $\bar{y} = -\dfrac{p}{p+1}$ with $\bar{\nu} = -1$; |
| for $\bar{p} = 0$ | there is a unique M-stationary pair $\bar{x} = 0$, $\bar{y} = 0$ with $\bar{\nu} \in [-1, 0]$; |
| for $\bar{p} > 0$ | there is a unique M-stationary pair $\bar{x} = 0$, $\bar{y} = 0$ with $\bar{\nu} = 0$. |

Thus, due to multiplicity of MPCC-multipliers for $\bar{p} = 0$ the reduction to $\mathcal{S}^r$ is not possible. Note that our assumption for reducibility of the stationarity map is violated for all feasible values of $p$.

By using the standard rules of the generalized differential calculus we can now compute upper approximations of the graphical derivative and the limiting coderivative of the stationarity map $\mathcal{S}$. Let us first introduce the *perturbation mapping* associated with $\mathcal{S}$

$$M(z) := \{(p, x, y, \nu) | \Psi(x, y, p, \nu) + z \in \{0\} \times \Lambda\}.$$

**Lemma 3** *Let $(\bar{x}, \bar{y})$ be the M-stationary pair of MPCC($\bar{p}$) with a corresponding MPCC-multiplier $\bar{\nu}$. Then*

$$T_{Gph\ \mathcal{S}}(\bar{p}, \bar{x}, \bar{y}, \bar{\nu}) \subset \{h | (\nabla\Psi(\bar{x}, \bar{y}, \bar{p}, \bar{\nu}))h \in T_{\{0\}\times\Lambda}(\Psi(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})). \tag{16}$$

*Proof* The statement follows from [17, Theorem 6.31]. □

**Lemma 4** *Let $(\bar{x}, \bar{y})$ be the M-stationary pair of MPCC($\bar{p}$) with a corresponding MPCC-multiplier $\bar{\nu}$. Further, let $M$ be calm at $(0, \bar{p}, \bar{x}, \bar{y}, \bar{\nu})$. Then*

$$N_{Gph\mathcal{S}}(\bar{p}, \bar{x}, \bar{y}, \bar{\nu}) \subset (\nabla\Psi(\bar{x}, \bar{y}, \bar{p}, \bar{\nu}))^\top N_{\{0\}\times\Lambda}(\Psi(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})), \tag{17}$$

*where*

$$N_{\{0\}\times\Lambda}(\Psi(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})) = \mathbb{R}^n \times N_{Gph\ D^* N_{\mathbb{R}^m_+}}(\bar{y}, -F(\bar{p}, \bar{x}, \bar{y}), \bar{\nu}, -b(\bar{p}, \bar{x}, \bar{y}) - B^\top(\bar{p}, \bar{x}, \bar{y})\bar{\nu}).$$

*Proof* The statement follows from [6, Theorem 4.1]. □

In our case we have (for simplicity, we do not state the arguments of maps in the formulas)

$$\nabla\Psi = \begin{pmatrix} \nabla_p a + \nabla_p(A^\top\bar{\nu}) & \nabla_x a + \nabla_x(A^\top\bar{\nu}) & \nabla_y a + \nabla_y(A^\top\bar{\nu}) & A \\ 0 & 0 & I & 0 \\ -\nabla_p F & -\nabla_x F & -\nabla_y F & 0 \\ 0 & 0 & 0 & I \\ -\nabla_p b - \nabla_p(B^\top\bar{\nu}) & -\nabla_x b - \nabla_x(B^\top\bar{\nu}) & -\nabla_y b - \nabla_y(B^\top\bar{\nu}) & -B \end{pmatrix}.$$

Thus the graphical derivative of $\mathcal{S}$ can be approximated as follows

$$D\mathcal{S}(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})(h_p) \subset \left\{ (h_x, h_y, h_\nu) \;\middle|\; \begin{array}{c} (\nabla a + \nabla(A^\top\bar{\nu}))h = 0 \\ \begin{pmatrix} h_y \\ -(\nabla F)h \\ h_\nu \\ -(\nabla b - \nabla(B^\top\bar{\nu}))h \end{pmatrix} \in T_{\mathrm{Gph}\ D^* N_{\mathbb{R}^m_+}}(\bar{y}, -F, \bar{\nu}, -b - B^\top\bar{\nu}) \end{array} \right\}. \tag{18}$$

If $M$ is calm at $(0, \bar{p}, \bar{x}, \bar{y}, \bar{\nu})$, then we get the following upper approximation of the coderivative of the stationarity map $\mathcal{S}$.

$$
\begin{aligned}
&D^*\mathcal{S}(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})(x^*, y^*, \nu^*) \\
&\subset \left\{ p^* \,\middle|\, \begin{array}{l}
p^* = (\nabla_p a + \nabla_p(A^\top \bar{\nu}))^\top q^* - (\nabla_p F)^\top v^* - (\nabla_p b + \nabla_p(B^\top \bar{\nu}))^\top z^* \\
-x^* = (\nabla_x a + \nabla_x(A^\top \bar{\nu}))^\top q^* - (\nabla_x F)^\top v^* - (\nabla_x b + \nabla_x(B^\top \bar{\nu}))^\top z^* \\
-y^* = (\nabla_y a + \nabla_y(A^\top \bar{\nu}))^\top q^* + u^* - (\nabla_y F)^\top v^* - (\nabla_x b + \nabla_x(B^\top \bar{\nu}))^\top z^* \\
-\nu^* = A^\top q^* + w^* - B^\top z^* \\
q^* \in \mathbb{R}^m, (u^*, v^*, w^*) \in D^{**} N_{\mathbb{R}^m_+}(\bar{y}, -F, \bar{\nu}, -b - B^\top \bar{\nu})(-z^*)
\end{array} \right\}.
\end{aligned} \tag{19}
$$

Moreover, (16), (17) and thus also (18) and (19) become equalities, whenever $\nabla\Psi$ is surjective at $(\bar{p}, \bar{x}, \bar{y}, \bar{\nu})$, cf. [17, Exercise 6.7].

Whenever mapping $M$ has the Aubin property around the reference point, the calmness condition stated in the Lemma 4 is satisfied. The standard constraint qualification (CQ) that implies the Aubin property of $M$ can be formulated as follows

$$
\left.\begin{array}{l}
0 = (\nabla\Psi(\bar{p}, \bar{x}, \bar{y}, \bar{\nu}))^\top u \\
u \in \mathbb{R}^m \times N_{\mathrm{Gph}\, D^* N_{\mathbb{R}^m_+}}(\bar{y}, -F(\bar{p}, \bar{x}, \bar{y}), \bar{\nu}, -b(\bar{p}, \bar{x}, \bar{y}) - (B(\bar{p}, \bar{x}, \bar{y}))^\top \bar{\nu})
\end{array}\right\} \Rightarrow u = 0. \tag{20}
$$

Analogously to Lemmas 3 and 4, using [17, Theorem 6.31] and [6, Theorem 4.1] we can obtain upper approximations to the contingent and the limiting normal cone to the graph of $\mathcal{S}^r$. This time, the corresponding chain rules contain

$$
\nabla\Psi^r = \begin{pmatrix}
0 & 0 & I \\
-\nabla_p F & -\nabla_x F & -\nabla_y F \\
-\nabla_p(Ca) & -\nabla_x(Ca) & -\nabla_y(Ca) \\
-\nabla_p b - \nabla_p d & -\nabla_x b - \nabla_x d & -\nabla_y b - \nabla_y d
\end{pmatrix}.
$$

We can thus provide upper approximations of the graphical derivative and the coderivative of $\mathcal{S}^r$. One obtains that

$$
D\mathcal{S}^r(\bar{p}, \bar{x}, \bar{y})(h'_p) \subset \left\{ (h'_x, h'_y) \,\middle|\, \begin{pmatrix} h'_y \\ -(\nabla F)h' \\ -(\nabla(Ca))h' \\ -(\nabla b - \nabla d)h' \end{pmatrix} \in T_{\mathrm{Gph}\, D^* N_{\mathbb{R}^m_+}}(\Psi^r(\bar{p}, \bar{x}, \bar{y})) \right\} \tag{21}
$$

and, under calmness of the multifunction $M^r$ given by

$$
M^r(z') = \{(p, x, y) | \Psi^r(x, y, p) + z' \in \Lambda\}
$$

at $(0, \bar{p}, \bar{x}, \bar{y})$,

$$
\begin{aligned}
&D^*\mathcal{S}^r(\bar{p}, \bar{x}, \bar{y})(x^*, y^*) \subset \\
&\left\{ p^* \,\middle|\, \begin{array}{ll}
p^* & = -(\nabla_p F)^\top v^* - (\nabla_p(Ca))^\top w^* - (\nabla_p b + \nabla_p d)^\top z^* \\
-x^* & = -(\nabla_x F)^\top v^* - (\nabla_x(Ca))^\top w^* - (\nabla_x b + \nabla_x d)^\top z^* \\
-y^* & = u^* - (\nabla_y F)^\top v^* - (\nabla_y(Ca))^\top w^* - (\nabla_y b + \nabla_y d)^\top z^* \\
\multicolumn{2}{l}{(u^*,\ v^*, w^*) \in D^{**} N_{\mathbb{R}^m_+}(\Psi^r(\bar{p}, \bar{x}, \bar{y}))(-z^*)}
\end{array} \right\}.
\end{aligned} \tag{22}
$$

Again, whenever $\nabla\Psi^r$ is surjective at $(\bar{p}, \bar{x}, \bar{y})$, inclusions (21) and (22) become equalities, cf. [17, Exercise 6.7].

Analogously to the non-reduced case, the following standard CQ implies the Aubin property (and thus also the calmness) of mapping $M^r$ at $(\bar{p}, \bar{x}, \bar{y})$:

$$\left.\begin{array}{l} 0 = (\nabla\Psi^r(\bar{p}, \bar{x}, \bar{y}))^\top u' \\ u' \in N_{\mathrm{Gph}\, D^* N_{\mathbb{R}^m_+}}(\bar{y}, -F(\bar{p}, \bar{x}, \bar{y}), -C(\bar{p}, \bar{x}, \bar{y})a(\bar{p}, \bar{x}, \bar{y}), -b(\bar{p}, \bar{x}, \bar{y}) - d(\bar{p}, \bar{x}, \bar{y})) \end{array}\right\} \Rightarrow u' = 0. \tag{23}$$

We close this section with a continuation of Example 1 and provide upper approximations for both graphical derivative and coderivative of the respective reduced stationarity map.

*Example 1 (continued)*
In our case for any $(p, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$\nabla\Psi^r(p, x, y) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence

$$D\mathcal{S}^r(p, x, y)(h'_p) \subset \left\{ (h'_x, h'_y) \;\middle|\; \begin{pmatrix} h'_y \\ -h'_p - h'_x \\ -2h'_x \\ 0 \end{pmatrix} \in T_{\mathrm{Gph}\, D^* N_{\mathbb{R}^m_+}}(\Psi^r(p, x, y)) \right\}.$$

Thus, by virtue of (15)

1. for $\bar{p} < 1$, $I_0^{--} = \{1\}$ and for $h'_p \in \mathbb{R}$, we have $D\mathcal{S}^r(\bar{p}, -\bar{p}, 0)\left(h'_p\right) \subset \left\{(-h'_p, 0)\right\}$.
2. for $\bar{p} = 1$, $I_0^{0-} = \{1\}$ and for $h'_p \leq 0$, we have $D\mathcal{S}^r(1, -1, 0)\left(h'_p\right) \subset \left\{(-h'_p, 0)\right\}$, while for $h'_p \geq 0$ we have $D\mathcal{S}^r(1, -1, 0)\left(h'_p\right) \subset \{(0, 0)\}$.
3. for $\bar{p} > 1$, $I_+ = \{1\}$ and for $h'_p \in \mathbb{R}$, we have $D\mathcal{S}^r(\bar{p}, -1, 0)\left(h'_p\right) \subset \{(0, 0)\}$.

The CQ (23) has the form:

$$\left.\begin{array}{r} -u'_2 = 0 \\ -u'_2 - 2u'_3 = 0 \\ u'_1 = 0 \\ \left(u'_1, u'_2, u'_3\right) \in D^{**}N_{\mathbb{R}_+}(y, -x, -2x + 2p, -2y + 2p)\left(-u'_4\right) \end{array}\right\} \Rightarrow u' = 0,$$

Clearly, the equality conditions imply $u'_1 = u'_2 = u'_3 = 0$, while $u'_4 = 0$ follows for all values of $p$ from the last condition as we are interested only in index sets $I_0^{--}$, $I_0^{0-}$ and $I_+$, recall the formula (15). Thus, the CQ (23) is satisfied at each point from the graph of $\mathcal{S}^r$.

So, we obtain

$$D^*\mathcal{S}^r(p, x, y)(x^*, y^*) \subset \left\{ p^* \;\middle|\; \begin{array}{l} p^* = -v^* \\ -x^* = -v^* - 2w^* \\ -y^* = u^* \\ (u^*, v^*, w^*) \in D^{**}N_{\mathbb{R}_+}(y, -x - p, -2x - 2, -1)(-z^*) \end{array} \right\}.$$

Using formula (15), one can easily obtain the following results.

1. For $\bar{p} < 1$, $D^*\mathcal{S}^r(\bar{p}, -\bar{p}, 0)(x^*, y^*) \subset \{-x^*\}$ for $x^*, y^* \in \mathbb{R}$.
2. For $\bar{p} = 1$, taking into account all three possibilities for values of $(u^*, v^*, w^*, z^*)$, the inclusion (22) gives $D^*\mathcal{S}^r(1, -1, 0)(x^*, y^*) \subset [-x^*, 0]$ with $x^*, y^* \in \mathbb{R}$.
3. For $\bar{p} > 1$, $D^*\mathcal{S}^r(\bar{p}, -1, 0)(x^*, y^*) \subset \{0\}$ for $x^*, y^* \in \mathbb{R}$.

## 5 Qualitative Stability of Stationarity Maps

In the last section we provide sufficient (and sometimes also necessary) conditions for isolated calmness and Aubin property for both types of stationarity maps. As in the previous section, we start first with the stationarity map $\mathcal{S}$.

**Theorem 2** *The stationarity map $\mathcal{S}$ has the isolated calmness property at $(\bar{p}, \bar{x}, \bar{y}, \bar{v})$ if the following condition holds true:*

$$\left.\begin{array}{c} (\nabla a + \nabla(A^\top \bar{v}))h = 0 \\ \begin{pmatrix} h_y \\ -(\nabla F)h \\ h_v \\ -(\nabla b - \nabla(B^\top \bar{v}))h \end{pmatrix} \in T_{Gph\ D^*N_{\mathbb{R}^m_+}}(\bar{y}, -F, \bar{v}, -b - B^\top \bar{v}) \\ h_p = 0 \end{array}\right\} \Rightarrow \begin{cases} h_x = 0, \\ h_y = 0, \\ h_v = 0. \end{cases} \tag{24}$$

*Proof* Using the upper approximation (18), the condition (24) implies that

$$D\mathcal{S}(\bar{p}, \bar{x}, \bar{y}, \bar{v})(0) \subset \{0\}.$$

The statement then follows from [3, Theorem 4C.1]). □

**Theorem 3** *Let $M$ be calm at $(0, \bar{p}, \bar{x}, \bar{y}, \bar{v})$. The stationarity map $\mathcal{S}$ has the Aubin property around $(\bar{p}, \bar{x}, \bar{y}, \bar{v})$ if the following condition holds true:*

$$\left.\begin{array}{l} p^* = (\nabla_p a + \nabla_p(A^\top \bar{v}))^\top q^* - (\nabla_p F)^\top v^* - (\nabla_p b + \nabla_p(B^\top \bar{v}))^\top z^* \\ \ \ 0 = (\nabla_x a + \nabla_x(A^\top \bar{v}))^\top q^* - (\nabla_x F)^\top v^* - (\nabla_x b + \nabla_x(B^\top \bar{v}))^\top z^* \\ \ \ 0 = (\nabla_y a + \nabla_y(A^\top \bar{v}))^\top q^* + u^* - (\nabla_y F)^\top v^* - (\nabla_x b + \nabla_x(B^\top \bar{v}))^\top z^* \\ \ \ 0 = A^\top q^* + w^* - B^\top z^* \\ q^* \in \mathbb{R}^m, (u^*, v^*, w^*) \in D^{**}N_{\mathbb{R}^m_+}(\bar{y}, -F, \bar{v}, -b - B^\top \bar{v})(-z^*) \end{array}\right\} \Rightarrow p^* = 0. \tag{25}$$

*Proof* Using the upper approximation (19), the condition (25) implies that

$$D^*\mathcal{S}(\bar{p}, \bar{x}, \bar{y}, \bar{v})(0) \subset \{0\}.$$

The statement then follows from the Mordukhovich criterion. □

Note that both Theorems 2 and 3 provide just the sufficient conditions for the respective properties of $\mathcal{S}$. In the case when $\nabla\Psi$ is surjective at the reference point, these conditions become also necessary.

Replacing calmness condition on $M$ by stronger CQ (20) we can obtain alternative condition on Aubin property of $\mathcal{S}$ which combines conditions for the Aubin property of $M$ and $\mathcal{S}$ into one.

**Corollary 1** *The stationarity map $\mathcal{S}$ has the Aubin property around $(\bar{p}, \bar{x}, \bar{y}, \bar{v})$ if*

$$\left.\begin{array}{l} 0 = (\nabla_x a + \nabla_x (A^\top \bar{v}))^\top q^* - (\nabla_x F)^\top v^* - (\nabla_x b + \nabla_x (B^\top \bar{v}))^\top z^* \\ 0 = (\nabla_y a + \nabla_y (A^\top \bar{v}))^\top q^* + u^* - (\nabla_y F)^\top v^* - (\nabla_x b + \nabla_x (B^\top \bar{v}))^\top z^* \\ 0 = A^\top q^* + w^* - B^\top z^* \\ q^* \in \mathbb{R}^m, (u^*, v^*, w^*) \in D^{**} N_{\mathbb{R}^m_+}(\bar{y}, -F, \bar{v}, -b - B^\top \bar{v})(-z^*) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} q^* = 0, \\ u^* = 0, \\ v^* = 0, \\ w^* = 0, \\ z^* = 0. \end{array}\right.$$

Analogously, we can derive conditions for isolated calmness and Aubin property of the reduced stationarity map $\mathcal{S}^r$.

**Theorem 4** *The reduced stationarity map $\mathcal{S}^r$ is isolatedly calm at $(\bar{p}, \bar{x}, \bar{y})$ if the following condition holds true:*

$$(\nabla \Psi^r(\bar{p}, \bar{x}, \bar{y})) \begin{pmatrix} 0 \\ h'_x \\ h'_y \end{pmatrix} \in T_{Gph\ D^* N_{\mathbb{R}^m_+}}(\Psi^r(\bar{p}, \bar{x}, \bar{y})) \Rightarrow \left\{\begin{array}{l} h'_x = 0, \\ h'_y = 0. \end{array}\right.$$

Based on the inclusion (22) we can formulate the condition for the Aubin property of the reduced stationarity map.

**Theorem 5** *Let $M^r$ be calm at $(0, \bar{p}, \bar{x}, \bar{y})$. The reduced stationarity map $\mathcal{S}^r$ has the Aubin property around $(\bar{p}, \bar{x}, \bar{y})$ if the following condition holds true:*

$$\left.\begin{array}{l} p^* = -(\nabla_p F)^\top v^* - (\nabla_p (Ca))^\top w^* - (\nabla_p b + \nabla_p d)^\top z^* \\ 0 = -(\nabla_x F)^\top v^* - (\nabla_x (Ca))^\top w^* - (\nabla_x b + \nabla_x d)^\top z^* \\ 0 = u^* - (\nabla_y F)^\top v^* - (\nabla_y (Ca))^\top w^* - (\nabla_y b + \nabla_y d)^\top z^* \\ (u^*, v^*, w^*) \in D^{**} N_{\mathbb{R}^m_+}(\Psi^r(p, x, y))(-z^*) \end{array}\right\} \Rightarrow p^* = 0.$$

Both conditions become necessary and sufficient whenever $\nabla \Psi^r$ is surjective at $(\bar{p}, \bar{x}, \bar{y})$. Replacing calmness condition on $M^r$ by stronger CQ (23) we can derive an alternative condition ensuring the Aubin property of $\mathcal{S}^r$.

**Corollary 2** *The stationarity map $\mathcal{S}^r$ has the Aubin property around $(\bar{p}, \bar{x}, \bar{y})$ if*

$$\left.\begin{array}{l} 0 = -(\nabla_x F)^\top v^* - (\nabla_x (Ca))^\top w^* - (\nabla_x b + \nabla_x d)^\top z^* \\ 0 = u^* - (\nabla_y F)^\top v^* - (\nabla_y (Ca))^\top w^* - (\nabla_y b + \nabla_y d)^\top z^* \\ (u^*, v^*, w^*) \in D^{**} N_{\mathbb{R}^m_+}(\Psi^r(\bar{p}, \bar{x}, \bar{y}))(-z^*) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u^* = 0, \\ v^* = 0, \\ w^* = 0, \\ z^* = 0. \end{array}\right.$$

We conclude this section with illustrative examples. First we proceed with Example 1 and show that we can verify the isolated calmness and the Aubin properties of the respective reduced stationarity map $\mathcal{S}^r$ using the conditions derived above.

*Example 1 (continued)*
According to results derived in the previous sections,

$$(\nabla\Psi^r(p,x,y))\begin{pmatrix}0\\ h'_x\\ h'_y\end{pmatrix}=\begin{pmatrix}h'_y\\ -h'_x\\ -2h'_x\\ 0\end{pmatrix}.$$

Thus, invoking formula (15),

1. for $\bar{p}<1$, we clearly have $h'_x=h'_y=0$, hence the reduced stationarity map $\mathcal{S}^r$ is isolatedly calm at $(\bar{p},-\bar{p},0)$. Also, $D^*\mathcal{S}^r(\bar{p},-\bar{p},0)(0,0)\subset\{0\}$ and thus the reduced stationarity map $\mathcal{S}^r$ also has the Aubin property at $(\bar{p},-\bar{p},0)$.
2. for $\bar{p}=1$, $h'_x=h'_y=0$, and the reduced stationarity map $\mathcal{S}^r$ is isolatedly calm also at $(1,-1,0)$. Also in this case, $D^*\mathcal{S}^r(1,-1,0)(0,0)\subset\{0\}$ and thus the reduced stationarity map $\mathcal{S}^r$ has the Aubin property at $(1,-1,0)$.
3. for $\bar{p}>1$, analogously to previous cases, the reduced stationarity map $\mathcal{S}^r$ has both the isolated calmness and the Aubin properties at $(\bar{p},-1,0)$.

As can be seen in Example 6 below, our sufficient conditions may be far from necessity when $\nabla\Psi^r$ is not surjective. Our sufficient conditions for the Aubin property may very well be violated at some point despite the fact that the stationarity map has the Aubin property there.

*Example 6* Let $p,x,y\in\mathbb{R}$ and consider the following perturbed MPCC

$$\begin{aligned}&\text{minimize } (x-p)^2+(y-p)^2\\ &\text{subject to}\\ &\qquad\qquad 0\in x+N_{\mathbb{R}_+}(y).\end{aligned}$$

The solution to this problem is the projection of the point $(p,p)$ onto the boundary of the positive orthant in $\mathbb{R}^2$.

Clearly, the M-stationarity conditions can be formulated in the reduced form as

$$\begin{bmatrix}y\\ -x\\ -2x+2p\\ -2y+2p\end{bmatrix}\in\operatorname{Gph} D^*N_{\mathbb{R}_+},$$

where $A=C=(1)_{1\times1}, B=(0)_{1\times1}, a(p,x,y)=2x-2p, b(p,x,y)=2y-2p, d(p,x,y)=0$.

The CQ (23) has the form:

$$\left.\begin{aligned}(\nabla\Psi^r(p,x,y))^\top u'=0\\ (u'_1,u'_2,u'_3)\in D^{**}N_{\mathbb{R}_+}(y,-x,-2x+2p,-2y+2p)(-u'_4)\end{aligned}\right\}\Rightarrow u'=0,$$

where for any $(p,x,y)$

$$\nabla\Psi^r(p,x,y)=\begin{pmatrix}0&0&1\\ 0&-1&0\\ 2&-2&0\\ 2&0&-2\end{pmatrix}.$$

The solution of $(\nabla\Psi^r(p,x,y))^\top u' = 0$ is $\bar{u}' = (2t, 2t, -t, t)$, $t \in \mathbb{R}$. It remains to check that for each of the M-stationary points of the MPEC (which coincide with solution), the second condition of the CQ implies that at least one of the components of $u'$ vanishes. Recall the formula (15).

1. For $\bar{p} < 0$, the unique solution is the origin $(\bar{x}, \bar{y}) = (0, 0)$. $I_0^{--} = \{1\}$ and the second condition of CQ implies that $u_3'$ and $u_4'$ vanish. Thus $t = 0$ and ultimately $u' = 0$.
2. For $\bar{p} = 0$, the unique solution is the origin $(\bar{x}, \bar{y}) = (0, 0)$. In this case $I_0^{00} = \{1\}$ thus there are eight possible combinations of values of components of $u'$. Note that in each of them at least one component of $u'$ vanishes. Hence $t = 0$ and ultimately $u' = 0$.
3. For $\bar{p} > 0$, the MPCC has two solutions
   (a) $(\bar{x}, \bar{y}) = (\bar{p}, 0)$. In this case $I_+ = \{1\}$ and the second condition of CQ implies that $u_2'$ and $u_4'$ vanish. Thus $t = 0$ and ultimately $u' = 0$.
   (b) $(\bar{x}, \bar{y}) = (0, \bar{p})$. In this case $L = \{1\}$ and the second condition of CQ implies that $u_1'$ and $u_3'$ vanish. Thus $t = 0$ and ultimately $u' = 0$.

Thus the CQ (23) holds for all values of $p$.

So, we obtain

$$D^*\mathcal{S}^r(\bar{p}, \bar{x}, \bar{y})(x^*, y^*) \subset \left\{ p^* \;\middle|\; \begin{array}{l} p^* = 2w^* + 2z^* \\ -x^* = -v^* - 2w^* \\ -y^* = u^* - 2z^* \\ (u^*, v^*, w^*) \in D^{**}N_{\mathbb{R}_+}(\bar{y}, -\bar{x}, -2\bar{x} + 2\bar{p}, -2\bar{y} + 2\bar{p})(-z^*) \end{array} \right\}.$$

Using formula (15), one can easily obtain the following results.

1. For $\bar{p} < 0$, $D^*\mathcal{S}^r(\bar{p}, 0, 0)(x^*, y^*) \subset \{0\}$ for $x^*, y^* \in \mathbb{R}$. Thus the reduced stationarity map $\mathcal{S}^r$ has the Aubin property at $(\bar{p}, 0, 0)$.
2. For $\bar{p} = 0$, taking into account all eight possibilities for values of $(u^*, v^*, w^*, z^*)$, the inclusion (22) gives $D^*\mathcal{S}^r(0, 0, 0)(x^*, y^*) \subset (-\infty, \max\{x^*, y^*\}]$ with $x^*, y^* \in \mathbb{R}$. Hence, we are unable to check the Aubin property of $\mathcal{S}^r$ at $(0, 0, 0)$ since the condition of Theorem 5 is violated.
3. For $\bar{p} > 0$, the MPCC has two solutions
   (a) $(\bar{x}, \bar{y}) = (\bar{p}, 0)$. In this case $D^*\mathcal{S}^r(\bar{p}, \bar{p}, 0)(x^*, y^*) \subset \{x^*\}$ for $x^*, y^* \in \mathbb{R}$.
   (b) $(\bar{x}, \bar{y}) = (0, \bar{p})$. In this case $D^*\mathcal{S}^r(\bar{p}, 0, \bar{p})(x^*, y^*) \subset \{y^*\}$ for $x^*, y^* \in \mathbb{R}$.

   The reduced stationarity map $\mathcal{S}^r$ has the Aubin property at points $(\bar{p}, \bar{p}, 0)$ as well as at points $(\bar{p}, 0, \bar{p})$.

In this example, taking into account the exact analytic form of $\mathcal{S}^r$, one can compute precisely also its coderivative $D^*\mathcal{S}^r(0, 0, 0)(x^*, y^*) = [0, \min\{x^*, y^*\}]$. The Mordukhovich criterion is thus satisfied at this point too and the Aubin property of $\mathcal{S}^r$ follows.

We conclude this section with an example of a parameterized MPCC, where the stationarity map is single-valued for nonpositive values of the parameter $p \leq 0$ while it is empty for $p > 0$. By means of this example we show effectiveness of our sufficient condition for the isolated calmness property of the stationarity map at $p = 0$.

*Example 7* Let $p, x, y \in \mathbb{R}$ and consider the following parameterized MPCC

$$\begin{aligned} &\text{minimize} \;\; -x-(y+p)^2 \\ &\text{subject to} \\ &\qquad\qquad 0 \in x + N_{\mathbb{R}_+}(y). \end{aligned}$$

It is easy to check that the corresponding reduced M-stationarity conditions

$$\begin{bmatrix} y \\ -x \\ 1 \\ 2(y+p) \end{bmatrix} \in \text{ Gph } D^* N_{\mathbb{R}_+},$$

have a unique solution $(\bar{x}, \bar{y}) = (0, -\bar{p})$ with multiplier $\bar{\nu} = 1$ for $\bar{p} \leq 0$ while it does not have a solution for $\bar{p} > 0$. From (21) we get the following upper approximation of the graphical coderivate of the reduced stationarity map

$$D\mathcal{S}^r(0,0,0)(h'_p) \subset \left\{ (h'_x, h'_y) \;\middle|\; \begin{array}{c} h'_y \geq 0 \\ -h'_x = 0 \\ 0 \in \mathbb{R} \\ 2h'_p + 2h'_y = 0 \end{array} \right\} = \{(h'_x, h'_y) | h'_x = 0, h'_y = -h'_p \geq 0\}.$$

Clearly, by means of Theorem 4, the reduced stationarity map is isolatedly calm at (0, 0, 0).

## 6 Conclusion

Various stability properties of multifunctions can be ensured or even characterized by appropriate generalized derivatives. In the case of stationarity maps of standard optimization problems (containing subdifferentials/normal-cone operators) one can exploit the methods and tools of second-order analysis, but in the case of MPECs/MPCCs or bilevel programs one has to differentiate also second-order terms (typically coderivatives of normal-cone mappings). To this end we have introduced the second-order limiting coderivative and shown how this notion and the graphical derivative of the limiting coderivative can be used in analysis of M-stationarity points of parameterized MPCCs.

**Acknowledgments** The authors would like to thank to anonymous referee for careful reading and valuable suggestions and comments. This research was partially supported by the Grant Agency of the Czech Republic, projects P102/11/0437 and P402/12/1309 and by the Australian Research Council, project DP110102011.

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