A nonlinear domain decomposition technique for scalar elliptic PDEs

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1 Introduction

Nonlinear problems are ubiquitous in a variety of areas, including fluid dynamics, biomechanics, viscoelasticity and finance, to name a few. A number of computational methods exist already for solving such problems, with the general approach being Newton-Krylov type methods coupled with an appropriate preconditioner. However, it is known that the strongest nonlinearity in a domain can directly impact the convergence of Newton-type algorithms. Therefore, local nonlinearities may have a direct impact on the global convergence of Newton's method, as illustrated in both [3] and [5]. Consequently, Newton-Krylov approaches can be expected to struggle when faced with domains containing local nonlinearities.

An attempt to resolve this issue was considered in [4] by Cai and Li. Here, a method based on an overlapping decomposition of the domain was proposed, which involved the development of a nonlinear restrictive additive Schwarz preconditioner for the treatment of high nonlinearities. Effectively, their proposed method ensured that the distribution of nonlinearities was balanced throughout their system, building on earlier work in [9]. While positive results were obtained, it is noted that their numerical experiments display a logarithmic dependence with regard to the mesh size. Additionally, in the situation of the unavailability of sufficient processors, it was found that subdomain problems could become computationally demanding, due in part to the need for a region of overlap. An alternative approach would be to instead consider applying a nonoverlapping decomposition of the domain directly to the nonlinear problem, avoiding the linearisation on a global scale. Methods have been proposed to this effect by both Pebrel et. al. [12] and by Sassi [14]. In [12], the resulting algorithm involved the solution to local nonlinear subproblems, as well as a global interface problem solved by a Newton-type algorithm. As a result, local nonlinearities could be dealt with much more effectively without having a major impact on the solution across the whole domain. While the paper reported speed up in the CPU time when compared directly to a Newton-Krylov approach, the method proposed involves the solution of a global interface problem, which can be both expensive and time consuming to compute. In comparison, [14] considered a preconditioned modified Newton algorithm, which was found to converge independently of the mesh size. However, the diameter of each subdomain was found to have a direct influence on the condition number of the involved operator, and as a result the proposed algorithm struggled with an increasing number of subdomains.

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We propose a splitting of a class of nonlinear problems into a three step procedure wrapped around a fixed point iteration. Section 2 will provide a description of the model problem, before the application of domain decomposition to the nonlinear problem in Section 3. A three step procedure can then be devised by applying an appropriate Picard linearisation (Section 4), which will be wrapped inside a global fixed point iteration. The corresponding weak formulation and finite element discretisation of the problem are given in Section 5, with results from the proposed method illustrated in Section 6.

2 Model Problem

We begin by considering the following problem posed on a two dimensional open and simply connected domain Ω:

$$\begin{cases} \mathscr{N}(u) := -\Delta u + c(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the function $c(u)$ is nonlinear and $\mathscr{N}$ is assumed to be positive. We also assume that (1) has a unique solution. A number of real life situations can be simulated by the nonlinear diffusion equation (1); in particular, notable applications can be found when modelling flow through porous material, in biochemistry, and in the transport of radiation.

An established approach for dealing with problems of type (1) is to employ Newton-Krylov methods and use domain decomposition methods as preconditioners. A number of preconditioning strategies have been considered (e.g. additive-Schwarz [7, 11], approximate-Schur [8, 13]), giving rise to numerous different Newton-Krylov type approaches, which have been applied to a wide range of problems mainly due to the quadratic convergence of Newton's method. However, for domains containing high local nonlinearities, the global convergence of Newton's method becomes entirely dependent on the local phenomena contained within the domain. Therefore, a substantial number of iterations can be expected for certain problems solved using such approaches, even for domains containing predominantly smooth areas, and so it is desirable to consider alternative approaches for determining solutions to systems of the form (1).

3 Nonlinear Domain Decomposition

We consider an approach that applies domain decomposition directly to the nonlinear problem. To do this, we divide our domain Ω into N nonoverlapping subdomains Ω_i with boundary $\partial\Omega_i$ with outer normals $\mathbf{n}_i$. We denote by Γ the resulting skeletal interface $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$, where $\Gamma_i := \partial\Omega_i \backslash \partial\Omega$. The restriction of a function w to a subdomain Ω_i is denoted by w_i. Assuming $u_i|_{\Gamma_i} = \lambda_i$ is given, problem (1) can then be seen to be equivalent to the following subproblems

$$\begin{cases} \mathscr{N}(u_i) := -\Delta u_i + c(u_i) = f_i & \text{in } \Omega_i \\ u_i = 0 & \text{on } \partial\Omega_i \backslash \Gamma_i \\ u_i = \lambda_i & \text{on } \Gamma_i. \end{cases}$$

Let $u = u^1 + u^2$ and assume that the nonlinear function $c(u)$ can be written as $c(u^1+u^2) = c^1(u^1+u^2) + c^2(u^1+u^2)$. The reason for splitting u and c in this way is to attempt to form homogeneous Dirichlet subdomain problems around u^1. The remaining components will then form subdomain problems around u^2.

Problem (1) can be viewed in terms of the following subproblems

$$\begin{cases} -\Delta u_i^1 + c^1(u_i^1 + u_i^2) = f_i & \text{in } \Omega_i \\ u_i^1 = 0 & \text{on } \partial\Omega_i \end{cases} \tag{2a}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \left(\mathbf{n}_i \cdot \nabla(u_i^2)\right) = -\sum_{i=1}^{N} \left(\mathbf{n}_i \cdot \nabla(u_i^1)\right) \text{ on } \Gamma \end{cases} \tag{2b}$$

$$\begin{cases} -\Delta u_i^2 + c^2(u_i^1 + u_i^2) = 0 & \text{in } \Omega_i \\ u_i^2 = 0 & \text{on } \partial\Omega_i \backslash \Gamma_i \\ u_i^2 = \lambda_i & \text{on } \Gamma_i. \end{cases} \tag{2c}$$

The nonlinear subproblems presented in (2a) correspond to obtaining solutions to local copies of (1) with homogeneous Dirichlet conditions enforced on local boundaries $\partial\Omega_i$. In comparison, the nonlinear subdomain problems presented in (2c) use interfacial data found in the intermediate step (2b) to obtain local solutions. The main motivation for considering such a splitting, and indeed for considering a nonoverlapping decomposition of Ω is that each subproblem in both (2a) and (2c) can be solved independently of other subdomains. In the following, we will assume that solution operators exist for problems of the form (2c); these will be denoted by E_i; in particular, we have $u_i^2 = E_i(\lambda_i)$. We will denote by $F_i\mu_i$ any other linear extensions of a given function μ_i defined on Γ_i to Ω_i.

4 Picard Linearisation

We decouple (2a), (2b) and (2c) via the following Picard linearisation

$$\begin{cases} \mathscr{N}_1(u_i^{1,k}) := -\Delta u_i^{1,k} + c^1\left(u_i^{1,k} + u_i^{2,k-1}\right) = f_i & \text{in } \Omega_i \\ u_i^{1,k} = 0 & \text{on } \partial\Omega_i \end{cases} \tag{3a}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \mathbf{n}_i \cdot \nabla(E_i^{k-1}\lambda_i^k) = -\sum_{i=1}^{N} \mathbf{n}_i \cdot \nabla(u_i^{1,k}) \text{ on } \Gamma \end{cases} \tag{3b}$$

$$\begin{cases} \mathscr{N}_2(u_i^{2,k}) := -\Delta u_i^{2,k} + c^2\left(u_i^{1,k} + u_i^{2,k}\right) = 0 & \text{in } \Omega_i \\ u_i^{2,k} = 0 & \text{on } \partial\Omega_i \backslash \Gamma_i \\ u_i^{2,k} = \lambda_i^k & \text{on } \Gamma_i. \end{cases} \tag{3c}$$

Given u^{k-1}, N nonlinear subproblems are first solved independently in (3a). The solution to these subproblems is then used in equation (3b) to obtain the interface update λ_i^k. Finally, the solutions to each nonlinear subproblem in (3c) are obtained independently using the updates from the previous two steps. Note that it is possible to solve each of the two sets of N nonlinear subproblems in (3a) and (3c) in

parallel. Equation (3b) is a linear Steklov-Poincaré equation involving the operator $S^k: H_{00}^{1/2}(\Gamma) \to H_{00}^{-1/2}(\Gamma)$ defined as

$$\langle S^k \lambda^k, \mu \rangle := \sum_{i=1}^{N} \int_{\Gamma_i} (\mathbf{n}_i \cdot \nabla)(E_i^{k-1} \lambda_i) \mu_i \, \mathrm{d}s = \sum_{i=1}^{N} \langle S_i^k \lambda_i^k, \mu_i \rangle,$$

where E_i^{k-1} are linearizations of the nonlinear extension operators E_i corresponding to (3c). We summarize below the proposed iterative scheme for computing the exact solution u^*, given an initial u^0.

1. Run through the following three steps to compute the solution $u^k = u^{1,k} + u^{2,k}$:

$$\begin{cases} \mathscr{N}_1(u_i^{1,k}) = f & \text{in } \Omega_i \\ u_i^{1,k} = 0 & \text{on } \partial\Omega_i \end{cases} \qquad i = 1, \ldots, N. \tag{4a}$$

$$\begin{cases} S^k \lambda^k = -\sum_{i=1}^{N} \mathbf{n}_i \cdot \nabla(u_i^{1,k}) \text{ on } \Gamma \end{cases} \tag{4b}$$

$$\begin{cases} \mathscr{N}_2(u_i^{2,k}) = 0 & \text{in } \Omega_i \\ u_i^{2,k} = 0 & \text{on } \partial\Omega_i \backslash \Gamma_i \\ u_i^{2,k} = \lambda_i^k & \text{on } \Gamma_i. \end{cases} \qquad i = 1, \ldots, N. \tag{4c}$$

2. Compute the residual $\mathscr{R}^k = \mathscr{N}(u^k) - f$. If $\|\mathscr{R}^k\| < \tau$, set $u^* = u^k$ and terminate. Else, set $k = k+1$ and return to step 1.

5 Finite Element Discretisation

Define now local bilinear forms

$$a_i^l(v, w; z) := \int_{\Omega_i} \nabla v \nabla w \, \mathrm{d}x + \int_{\Omega_i} c^l(v+z) \, w \, \mathrm{d}x,$$

for $l = 1, 2$. Using the above notation, the weak formulation of (4) is

$$\begin{cases} \text{Find } u_i^{1,k} \in H_0^1(\Omega_i) \text{ such that } \forall v_i \in H_0^1(\Omega_i) \\ a_i^1(u_i^{1,k}, v_i; u_i^{2,k-1}) = (f_i, v_i) \end{cases} \tag{5a}$$

$$\begin{cases} \text{Find } \lambda^k \in H_{00}^{1/2}(\Gamma) \text{ such that } \forall \mu \in H_{00}^{1/2}(\Gamma) \\ s(\lambda^k, \mu) = \sum_{i=1}^{N} (f_i, F_i \mu_i) - a_i^1(u_i^{1,k}, F_i \mu_i; u_i^{2,k-1}) \end{cases} \tag{5b}$$

$$\begin{cases} \text{Find } u_i^{2,k} \in E(\lambda_i^k) + H_0^1(\Omega_i) \text{ such that } \forall v_i \in H_0^1(\Omega_i) \\ a_i^2(u_i^{2,k}, v_i; u_i^{1,k}) = 0. \end{cases} \tag{5c}$$

Let now $V_h \subset H_0^1(\Omega) \cap C^0(\Omega)$ be a space of continuous piecewise polynomials of degree m defined on an isotropic subdivision of Ω into simplices of maximum diameter h. In our tests we choose $m = 1$, though other values are equally possible.

Let the corresponding basis be denoted by $\{\psi_r\}$. Let B denote the index set corresponding to basis elements ψ_r with support on Γ. Let $S_h := \text{span}\{\gamma_0(\Gamma)\psi_r : r \in B\}$ where γ_0 denotes the trace operator. The finite element discretisation of the systems in (5) can then be written for $i = 1, \ldots, N$ as

$$\begin{cases} \text{Find } u_{i,h}^{1,k} \in V_{i,h} \text{ such that } \forall v_{i,h} \in V_{i,h} \\ a_i^1(u_{i,h}^{1,k}, v_{i,h}; u_{i,h}^{2,k-1}) = (f_i, v_{i,h}) \end{cases} \tag{6a}$$

$$\begin{cases} \text{Find } \lambda_h^k \in S_h \text{ such that } \forall \mu_h \in S_h \\ s(\lambda_h^k, \mu_h) = \sum_{i=1}^{N} (f_i, F_i\mu_i) - a_i^1(u_{i,h}^{1,k}, F_i\mu_{i,h}; u_{i,h}^{2,k-1}) \end{cases} \tag{6b}$$

$$\begin{cases} \text{Find } u_{i,h}^{2,k} \in (E\lambda)_{i,h}^k + V_{i,h} \text{ such that } \forall v_{i,h} \in V_{i,h} \\ a_i^2(u_{i,h}^{2,k}, v_{i,h}; u_{i,h}^{1,k}) = 0. \end{cases} \tag{6c}$$

The system (6) can be represented systematically by matrices and vectors in the usual way. In particular, the Schur complement of the system matrix corresponds to the matrix representation of $s(\cdot,\cdot)$ in the basis of S_h. We can therefore describe our proposed method as follows:

1. Run through the following three step procedure to determine $\mathbf{u}$.
   a. Solve the N decoupled nonlinear subdomain problems (6a) written in matrix form as
   $$A_{II}^{i,1}(\mathbf{u}_{I,i}^{1,k})\mathbf{u}_{I,i}^{1,k} = \mathbf{f}_{I,i}^1, \tag{7a}$$
   using a Newton-Krylov method with line search and adaptive tolerances $\tau_{1,i}$.
   b. Calculate interface values $\boldsymbol{\lambda}^k$ using
   $$S^k\boldsymbol{\lambda}^k = \mathbf{f}_\Gamma - \sum_{i=1}^{N} A_{\Gamma I}^{i,1}(\mathbf{u}_{I,i}^{1,k})\mathbf{u}_{I,i}^{1,k}. \tag{7b}$$
   c. Solve the N decoupled nonlinear subdomain problems (6c) written in matrix form as
   $$A_{II}^{i,2}(\mathbf{u}_{I,i}^{2,k})\mathbf{u}_{I,i}^{2,k} = -A_{I\Gamma}^{i,2}(\mathbf{u}_{I,i}^{2,k})\boldsymbol{\lambda}_i^k, \tag{7c}$$
   using a Newton-Krylov method with line search and adaptive tolerances $\tau_{2,i}$.
2. Set $\mathbf{u}^k = \mathbf{u}^{1,k} + \mathbf{u}^{2,k}$, where $\mathbf{u}^{1,k} = [\mathbf{u}_I^{1,k}, 0]^T$ and $\mathbf{u}^{2,k} = [\mathbf{u}_I^{2,k}, \boldsymbol{\lambda}^k]^T$. Assemble the global stiffness matrix $A^k(\mathbf{u})$ and compute the residual $\mathscr{R}^k(\mathbf{u}^k) = A(\mathbf{u}^k)\mathbf{u}^k - \mathbf{f}$. If $\|\mathscr{R}^k\| < \tau$ set $\mathbf{u}^* = \mathbf{u}^k$ and exit; else, return to Step 1.

The subindices I and Γ indicate permutations involving the index sets corresponding to the interior and boundary nodes in the subdivision of Ω. The adaptive tolerances $\tau_{1,i}, \tau_{2,i}$ are chosen in relation to the norm of the global nonlinear residual $\|\mathscr{R}^k\|$, following the strategy in [6].

We solve the system (7b) using iterative methods of Krylov type with preconditioning. The matrix S^k is the interface Schur complement corresponding to the

reaction-diffusion problem $-\Delta u^{2,k-1} + c^2(u^{1,k-1} + u^{2,k-1})$; as such, it can be preconditioned by any domain decomposition preconditioner designed for elliptic problems. The preconditioner employed in this work is based on [2], where discrete norms corresponding to finite element discretisations of fractional Sobolev spaces are presented. In particular, it was shown that a discrete norm on $S_h \subset H_{00}^{1/2}(\Gamma)$ which is spectrally equivalent to S^k is given by

$$H_{1/2} = M_\Gamma \left(M_\Gamma^{-1} L_\Gamma\right)^{1/2}.$$

In [2], M_Γ and L_Γ correspond to the mass and Laplacian matrices, respectively, assembled on Γ. We adapt the definition of $H_{1/2}$ to include the contribution from the reaction term as suggested in [1]; this involves replacing L_Γ with

$$L_\Gamma^k = L_\Gamma + M_\Gamma^k,$$

where M_Γ^k is the mass matrix assembled on Γ and weighted by the trace on the interface Γ of $c^2\left(u^{1,k-1} + u^{2,k-1}\right)$. For more details, see [15].

Note that M_Γ, L_Γ^k are assembled globally on Γ and hence $H_{1/2}$ is a dense matrix. However, in our computations we use sparse techniques to circumvent this issue. In particular, the application of both Lanczos and inverse Lanczos factorisations has been considered in [2], and will be applied in this work in a similar manner.

6 Results

In this section, we will consider a number of examples to highlight the benefits of our proposed method. In particular, we will consider models for which

$$\text{(a) } c(u) = u^{q+1}, \qquad \text{and} \qquad \text{(b) } c(u) = u^{q+1}\sin(10u),$$

where q is a positive integer. For both choices, we note that by substituting $u = u^1 + u^2$ into the function, we can write

$$c(u^1+u^2) = (u^1+u^2)^{q+1} = (u^1+u^2)^q u^1 + (u^1+u^2)^q u^2.$$

Table 1 displays performance comparisons of our proposed method to the standard Newton-Krylov approach for two test problems. We used piecewise linear discretizations for a range of mesh parameters h. Each nonlinear problem was solved with a zero initial guess. We consider four different representations for the preconditioner $\tilde{S}$, namely the exact Schur complement, the exact discrete fractional Sobolev norm $H_{1/2}$, and both the Lanczos (L) and inverse Lanczos (I) approximations to $H_{1/2}$. The performance recorded in the table indicates that our method delivers promising results when directly compared to the corresponding Newton-Krylov method. In particular, it can be seen that the results indicate independence with respect to both the mesh size and the number of subdomains used. By comparing the columns in Table 1, an indication is given on how well both methods adapt to the increase in nonlinearity. Notably, it is clear that the Newton-Krylov method struggled when faced with the increased nonlinearity, confirming results noted earlier. How-

		(a), $q=2$						(b), $q=9$					
		Newton-Krylov			3-Step Procedure			Newton-Krylov			3-Step Procedure		
$\tilde{S}$	h	4	16	64	4	16	64	4	16	64	4	16	64
S	1/16	4 (8)	4 (8)	3 (6)	4 (8)	4 (8)	3 (5)	11 (22)	10 (20)	10 (20)	6 (12)	5 (10)	5 (10)
	1/32	4 (8)	3 (6)	3 (6)	4 (8)	3 (6)	3 (5)	10 (20)	10 (20)	9 (18)	5 (10)	5 (10)	5 (10)
	1/64	3 (6)	3 (6)	3 (6)	3 (6)	3 (6)	3 (5)	10 (20)	9 (18)	8 (16)	5 (10)	5 (10)	4 (8)
$H_{1/2}$	1/16	9 (24)	11 (35)	6 (48)	6 (29)	6 (26)	5 (35)	15 (66)	16 (89)	12 (110)	7 (48)	8 (60)	6 (47)
	1/32	12 (38)	7 (42)	5 (49)	6 (36)	5 (25)	4 (30)	17 (79)	12 (103)	10 (110)	7 (53)	6 (49)	5 (52)
	1/64	6 (33)	4 (29)	4 (42)	5 (28)	4 (25)	4 (41)	11 (75)	9 (96)	8 (103)	6 (47)	4 (28)	4 (43)
$H_{1/2}^{(L)}$	1/16	9 (24)	11 (35)	6 (48)	6 (29)	6 (26)	5 (35)	15 (66)	16 (89)	12 (110)	7 (48)	8 (60)	6 (47)
	1/32	12 (38)	7 (42)	5 (49)	6 (36)	5 (25)	4 (30)	17 (79)	12 (103)	10 (110)	7 (53)	6 (49)	5 (52)
	1/64	6 (33)	4 (29)	4 (42)	5 (28)	4 (25)	4 (41)	11 (75)	9 (96)	8 (103)	6 (47)	4 (28)	4 (43)
$H_{1/2}^{(I)}$	1/16	17 (39)	106 (148)	7 (51)	6 (31)	6 (34)	5 (39)	21 (78)	132 (193)	12 (102)	8 (64)	7 (56)	6 (58)
	1/32	11 (40)	6 (35)	4 (36)	6 (37)	5 (35)	4 (33)	14 (77)	11 (92)	5 (53)	7 (54)	6 (49)	5 (55)
	1/64	6 (37)	4 (34)	4 (40)	5 (36)	4 (27)	3 (27)	11 (76)	9 (90)	8 (92)	5 (40)	5 (44)	4 (45)

Table 1 Nonlinear iterations (total GMRES iterations) for a global tolerance $\tau = 10^{-7}$.

ever, in comparison our method was found to deal with the increase in nonlinearity in a much more efficient manner. This would suggest that our method would adapt quite well to domains containing high local nonlinearities confined to a particular region of the domain. It is also noted that by directly inverting the Schur complement, an adaptation of the result presented in [10] is also shown for our method, namely that the interface problem (7b) solved with GMRES can be expected to converge in a number of iterations no more than the dimension of Ω per fixed point iteration.

7 Conclusion

In this paper, we introduced a three step procedure for solving a class of nonlinear PDEs. We have demonstrated that our method is able to deliver results independent of both the mesh size and the number of subdomains used. Furthermore, we have shown that our procedure is competitive when directly compared to the corresponding Newton-Krylov method. Future work will involve further testing to include problems that contain a high nonlinearity confined to a particular region of the domain together with an appropriate analysis of the method. We will also adapt our method to problems in topology optimization [15].

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