REASONING ABOUT COALITIONAL EFFECTIVITY IN MODAL EXTENSION OF ŁUKASIEWICZ LOGIC

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ABSTRACT. We generalize the notion of α-effectivity existing in the context of game forms to deal with situations in which a coalition can enforce a fuzzy set of outcomes. Then we introduce a modal extension of ŁUKASIEWICZ $(n+1)$-valued logic together with a many-valued neighborhood semantic in order to encode the properties of the many-valued effectivity functions arising from strategic game forms. We prove completeness theorem for the studied logic.

1. INTRODUCTION

Modelling collective actions of agents and capturing their effectivity is among the important research topics on the frontiers of game theory, computer science and mathematical logic. The main efforts are concentrated on answering the following question: what is the set of outcome states that can effectively be implemented by a coalition of agents? A game-theoretic framework for studying collective actions and their enforceability is based on the notion of game forms. Loosely speaking, the game form is a pure description of a game and its rules, without regard to the agents' preferences.

The concept of α-effectivity ([1, 8]) is one of the key approaches to characterize the coalitional effectivity within game form models. A set of outcome states X is α-effective for a coalition C if the players in C can choose a joint strategy that enforces the outcome in X no matter what strategies are adopted by the other players. In his seminal paper [11], PAULY introduces a logic $\mathbf{CL}_N$ to reason about α-effectivity in strategic game forms with player set N. The axiomatization of $\mathbf{CL}_N$ is a characterization in a multi-modal language of the class of α-effectivity functions. PAULY also defines a neighborhood semantics with respect to which $\mathbf{CL}_N$ is complete. The logic $\mathbf{CL}_N$ was subsequently considered and extended by many authors (see [2, 3], for instance).

In this paper, we are interested in the generalization of α-effectivity for fuzzy sets of outcomes. In section 2, we introduce the notion of *$Ł_n$-valued effectivity function* whose purpose is to capture the degree with which a coalition can enforce a fuzzy set of outcomes. The meaning of such a generalization is illustrated with examples: in Example 2.5 we show how to model the degree of satisfaction of agents' goals in strategic game forms. In section 3, we develop the tools to capture the properties of many-valued effectivity functions in a many-valued modal language. These developments rely not only on recent advances in modal extensions of ŁUKASIEWICZ logic ([4, 7]), but they also require the introduction of neighborhood semantics, which has never been considered in this modal many-valued setting, to the best of our knowledge. With the notion of *weak $Ł_n$-valued coalitional logic* and *weak $Ł_n$-valued coalitional model*, we provide a general framework for the development of completeness results. In this perspective, our main result is Theorem 3.9.

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We use the following notation throughout the paper. We fix a positive integer n and we denote the set $\left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}$ by $Ł_n$. This set is equipped with Łukasiewicz interpretation of the connectors $\to$ and $\neg$ defined by

$$x \to y = \min\left(1, 1 - x + y\right), \tag{1.1}$$

$$\neg x = 1 - x, \tag{1.2}$$

for every $x, y \in Ł_n$. Moreover, the strong disjunction $\oplus$ and the strong conjunction $\odot$ are defined by $p \oplus q = \neg p \to q$ and $p \odot q = \neg(\neg p \oplus \neg q)$ and hence they are interpreted in $Ł_n$ by the following associative binary operations:

$$x \oplus y = \min\left(x + y, 1\right),$$
$$x \odot y = \max\left(x + y - 1, 0\right).$$

2. Many-valued Effectivity Functions

In what follows, S denotes a nonempty set, $N = \{1, \dots, k\}$ is a finite set and, for any $i \in N$, Σ_i denotes a nonempty set. Recall the following definition [1].

Definition 2.1. A *game form* is a tuple $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$, where N is a set of *players*, Σ_i is a set of *strategies* for each $i \in N$, S is a set of *outcome states*, and $o : \prod_{i \in N} \Sigma_i \to S$ is an *outcome function*.

The game forms are not to be confused with strategic games. While a preference relation over S must be defined for each player $i \in N$ in a strategic game [9], no such requirement exists for a game form. Below we include the basic examples of game forms.

Example 2.2. (i) Let $N = \{1, 2\}$ and Σ_1, Σ_2 be some strategy sets. Assume that the players choose their strategies simultaneously. Then we may set $S = \Sigma_1 \times \Sigma_2$ and define o as the identity function, which turns $(\{1, 2\}, \{\Sigma_1, \Sigma_2\}, \Sigma_1 \times \Sigma_2, o)$ into a game form.

(ii) Suppose, on the other hand, that player 2 makes his choice only after observing the strategic choice of player 1. This sequential procedure is modeled by a game form such that Σ_2 is the set of all functions $r : \Sigma_1 \to \Sigma'_2$, where Σ'_2 can be viewed as the set of all possible moves that can be played by player 2. Hence, Σ_2 models the replies of player 2 to the selection of a strategy by player 1. The outcome function is given by $o(\sigma_1, r) = (\sigma_1, r(\sigma_1))$, where $(\sigma_1, r) \in \Sigma_1 \times \Sigma_2$ and the set of outcome states is $S = \Sigma_1 \times \Sigma'_2$.

An important example is when the outcome function coincides with some social choice function, as recalled in the next example—see [1, Chapter 1].

Example 2.3. Let S be any set of outcome states and $\Pi(S)$ be a set of admissible preference relations on S. In most applications, $\Pi(S)$ will be either the set of *total preorders* (reflexive, transitive, and complete binary relations) or the set of *linear orders*. A *social choice* function is a map $\pi : \Pi(S)^k \to S$ that implements a collective decision procedure mapping the preferences of the agents into an effective outcome state. If an agent (or a group of agents) wants to enforce some specific outcome, his/her only possible strategy is to declare a preference relation that is likely to bring the collective decision into the desired state. We can describe this scheme as a game form $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ in which $\Sigma_1 = \dots = \Sigma_k = \Pi(S)$ and $o = \pi$.

Elements of the powerset $\mathcal{P}N$ of N are called *coalitions*. For every coalition C we denote by $\overline{C}$ its set-complement in N. If $\sigma_C \in \prod_{i \in C} \Sigma_i$ and $\sigma_{\overline{C}} \in \prod_{i \in \overline{C}} \Sigma_i$, then $\sigma_C\sigma_{\overline{C}}$ is the strategy vector in $\prod_{i\in N}\Sigma_i$ defined by $(\sigma_C\sigma_{\overline{C}})_i = (\sigma_C)_i$ if $i \in C$ and $(\sigma_C\sigma_{\overline{C}})_i = (\sigma_{\overline{C}})_i$ if $i \in \overline{C}$.

The usual definition [1, 8] of the α-effectivity function $E : \mathcal{P}N \to \mathcal{P}\mathcal{P}S$ associated with a game form aims to model the following. For any $C \subseteq N$ and $X \subseteq S$, we set $X \in E(C)$ if coalition C can choose a joint strategy σ_C enforcing the outcome in X no matter what strategies $\sigma_{\overline{C}}$ are adopted by the players in $\overline{C}$. We are going to generalize the concept of effectivity function: our goal is to capture the degree to which a coalition C can enforce a fuzzy set of outcomes $f \in Ł_n^S$.

Definition 2.4. Let $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ be a game form. The $Ł_n$*-valued effectivity function of* G is the map $E_G : \mathcal{P}N \times Ł_n^S \to Ł_n$ defined by

$$E_G(C, f) = \max_{\sigma_C}\min_{\sigma_{\overline{C}}} f(o(\sigma_C\sigma_{\overline{C}})), \qquad C \in \mathcal{P}N, f \in Ł_n^S, \tag{2.1}$$

where σ_C and $\sigma_{\overline{C}}$ range in the set of all joint strategies of coalitions C and $\overline{C}$, respectively.

The meaning of definition (2.1) is the following: coalition C is effective for $f \in Ł_n^S$ to the degree at most $E_G(C, f) \in Ł_n$, disregarding the strategic options of players in opposite coalition $\overline{C}$.

As an example, consider the situation in which we evaluate the capacity of groups of agents to fulfill specified goals.

Example 2.5. Let N be the set of the countries of the European Union (EU) and set $n = k$. Let Σ_i be the set of possible policies that country i can adopt at the local (country) level for each $i \in N$. Assume that for every $i \in N$ we evaluate the following question:

> 'Given a global strategy vector σ_N, does the economy of country $i \in N$ operate at full employment provided that σ_N is applied?'

This situation can be modeled by a set of states $S = \mathcal{P}N$. The outcome state $A \in \mathcal{P}N$ means that only the countries in A realize full employment. Experts define the outcome function $o : \prod_{i\in N}\Sigma_i \to \mathcal{P}N$ by setting $i \in o(\sigma_N)$ if and only if the global strategy σ_N leads to a state in which country $i \in N$ has reached full employment.

In this setting, an example of a relevant fuzzy set $f \in Ł_k^S$ can be defined by $f(A) = \frac{|A|}{k}$, where $|A|$ is the cardinality of $A \in S = \mathcal{P}N$. For any coalition C, the value $E_G(C, f)$ measures the level with which the countries in C can collaborate to enforce full employment in Europe, regardless of the policies adopted by the countries in $\overline{C}$. As another example, consider now that $n = \mathrm{lcm}(1, \dots, k)$ and define $g_C : \mathcal{P}N \to Ł_n$ by setting $g_C(A) = \frac{|A\cap C|}{|C|}$ for every $A, C \in \mathcal{P}N$ with $C \neq \varnothing$. Then the value of $E_G(C, g_C)$ can be seen as measuring how efficient is coalition C in enforcing full employment in the countries of C (regardless of the result of their strategies in countries in $\overline{C}$).

The previous example is just an instance of a more general scheme. Assume that each player in N is trying to achieve his/her own specific goal and that each goal can be partially achieved with a degree of achievement quantified in the scale $Ł_n$. As usual, let Σ_i be the set of possible strategies that player $i \in N$ can adopt for that purpose. This situation is modeled by the game form $G = (N, \{\Sigma_i \mid i \in N\}, Ł_n^N, o)$ where $o\colon \prod_{i\in N}\Sigma_i \to Ł_n^N$ is an output function that maps any global strategy vector σ to the *achievement vector* $o(\sigma)$ where for every $i \in N$, the value of $o(\sigma)_i$ measures the degree of achievement of the ith player's goal if strategy σ is adopted. A fuzzy set $f : Ł_n^N \to Ł_n$ can be considered as an aggregation function mapping any achievement vector into a single value, summarizing achievement on the global level of the grand coalition. For instance, for every nonempty coalition $C \subseteq N$ one can consider the aggregation function $f_C : Ł_n^N \to Ł_n$ such that

$$f_C(\mathbf{a}) = \min\{a_i \mid i \in C\}, \text{ where } \mathbf{a} = (a_1, \ldots, a_k) \in Ł_n^N.$$

The value $E_G(C, f_C)$ is thus the minimal degree of achievement of the members of coalition C, regardless of the strategies of players in $\overline{C}$.

It is shown in the next example that $Ł_n$-valued effectivity functions arise naturally from particular social choice functions, when the agents' preferences are represented in a finite numerical scale.

Example 2.6. Assume that $\Sigma_1 = \cdots = \Sigma_k = Ł_n^S$. Every $f \in Ł_n^S$ can be viewed as a function assigning utility—measured in the scale $Ł_n$—to each outcome state in S. Each such function f is thus called a *utility function.* This situation can be viewed as a special case of the social choice framework of Example 2.3. Indeed, for any $f \in Ł_n^S$ let us denote by $\preceq_f$ the linear order defined on S by setting $s_1 \preceq_f s_2$ whenever $f(s_1) \leq f(s_2)$. Conversely, every linear order on S arises in this way. Analogously to Example 2.3, the player's strategic choice boils down to declaring his/her own preferences over S, that is, his/her own utility function over S. Assume that an outcome function $o\colon (Ł_n^S)^k \to S$ is fixed. What is the meaning of $Ł_n$-valued effectivity function $E_G(C, g)$ in this setting? The number $E_G(C, g) \in Ł_n$ is the degree to which coalition C enforces the utility distribution described by $g \in Ł_n^S$.

Analogously to the classical literature on effectivity functions, we can study the notion of effectivity in a more general setting independent on game forms. In this sense a $Ł_n$*-valued effectivity function* is a map $E : \mathcal{P}N \times Ł_n^S \to Ł_n$.

Definition 2.7. Let E be a $Ł_n$-valued effectivity function. We say that E is

(1) *output monotonic* whenever $E(C, f) \geq E(C, g)$, for every $C \in \mathcal{P}N$ and every $f, g \in Ł_n^S$ with $f \geq g$;
(2) *weakly-playable* if it is output monotonic and satisfies the two conditions $E(C, f \oplus f) = E(C, f) \oplus E(C, f)$ and $E(C, f \odot f) = E(C, f) \odot E(C, f)$, for every $C \in \mathcal{P}N$ and every $f \in Ł_n^S$.

The readers familiar with the Boolean approach to effectivity functions may be surprised not to find a $Ł_n$-valued version of superadditivity. As explained in Section 4, this is the topic of current research. Moreover, it may be difficult to interpret the two conditions in (2) of Definition 2.7 in the game form framework. We refer to Remark 1 for an equivalent formulation of these two conditions.

The following result is straigthforward.

Lemma 2.8. *If G is a game form, then E_G is a weakly-playable $Ł_n$-valued effectivity function.*

Hence weak-playability can be seen as a minimal framewok for studying $Ł_n$-valued generalizations of effectivity functions.

3. Weak $Ł_n$-valued coalitional logic

In this section, we build a modal logic framework in the spirit of [11] to capture the properties of the $Ł_n$-valued effectivity functions.

Let $\mathcal{L}$ be the language $\{\to, \neg, 1\} \cup \{[C] \mid C \in \mathcal{P}N\}$ where $\to$ is binary, $\neg$ and $[C]$ are unary for every $C \in \mathcal{P}N$ and 1 is constant. The set $\mathsf{Form}_\mathcal{L}$ of formulas is defined inductively from the infinite set Prop of propositional variables by the following rules:

$$\phi ::= 1 \mid p \mid \phi \to \phi \mid \neg\phi \mid [C]\phi$$

where $p \in \mathsf{Prop}$ and $C \in \mathcal{P}N$.

Definition 3.1. A *weak* $Ł_n$*-valued coalitional logic* is a subset $\mathbf{L}$ of $\mathsf{Form}_{\mathcal{L}}$ that is closed under Modus Ponens, Equivalence, Uniform Substitution, Monotonicity (if $\phi \to \psi \in \mathbf{L}$ then $[C]\phi \to [C]\psi \in \mathbf{L}$ for any $C \in \mathcal{P}N$) and that contains an axiomatic base of Łukasiewicz $(n+1)$-valued logic together with the axioms

$$[C](p \odot p) \quad \leftrightarrow \quad [C]p \odot [C]p, \tag{3.1}$$

$$[C](p \oplus p) \quad \leftrightarrow \quad [C]p \oplus [C]p, \tag{3.2}$$

for any $C \in \mathcal{P}N$.

We denote by $\mathbf{WC}_n$ the smallest weak $Ł_n$-valued coalitional logic, that is, the intersection of all the weak $Ł_n$-valued coalitional logics. We conform with common usage and we often write $\mathbf{WC}_n \vdash \phi$ for $\phi \in \mathbf{WC}_n$.

The formula $[C]\phi$ reads as 'coalition C can enforce a state in which ϕ holds'. The axioms (3.1) and (3.2) and the rule of Monotonicity reflect the properties of weak-playability. In Remark 1 at the end of section 3.1, we give an equivalent and more intuitive axiomatization.

Now, we introduce a $Ł_n$-valued generalization of neighborhood semantics in order to interpret $\mathcal{L}$-formulas.

Definition 3.2. A *weak* $Ł_n$*-valued coalitional frame*, or simply a *frame*, is a couple $\mathfrak{F} = (S, E)$ where S is a nonempty set and $E : S \to (\mathcal{P}N \times Ł_n^S \to Ł_n)$ assigns a weakly-playable function $E(s) : \mathcal{P}N \times Ł_n^S \to Ł_n$ to every $s \in S$. Elements of S are called *states*.

A *weak* $Ł_n$*-valued coalitional model*, or simply *a model*, is a couple $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ where $\mathfrak{F} = (S, E)$ is a weak $Ł_n$-valued coalitional frame and $\mathrm{Val} : S \times \mathsf{Prop} \to Ł_n$.

In a model $\mathcal{M}$, the valuation map Val is extended inductively to $S \times \mathsf{Form}_{\mathcal{L}}$ by using Łukasiewicz interpretation of the connectors $\neg, \to, 1$ in $[0,1]$ (see (1.1) and (1.2)) and by setting

$$\mathrm{Val}(u, [C]\phi) = E(u)\big(C, \mathrm{Val}(-, \phi)\big)$$

for every $C \in \mathcal{P}N$ and every $\phi \in \mathsf{Form}_{\mathcal{L}}$. We use the standard notation and terminology: we say that a formula ϕ is *true in* $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ and we write $\mathcal{M} \models \phi$ if $\mathrm{Val}(u, \phi) = 1$ for every state u of $\mathcal{M}$.

The following result can be proved by a standard induction argument.

Lemma 3.3. *If* $\mathcal{M}$ *is a weak* $Ł_n$*-valued coalitional model and if* $\mathbf{WC}_n \vdash \phi$ *then* $\mathcal{M} \models \phi$.

3.1. **Construction of the canonical model.** To prove completeness of $\mathbf{WC}_n$ with respect to the class of the weak $Ł_n$-valued coalitional models, we use the technique of the canonical model.

Let us denote by $\mathfrak{F}_{\mathbf{WC}_n}$ the Lindenbaum - Tarski algebra of $\mathbf{WC}_n$, *i.e.*, the quotient of $\mathsf{Form}_{\mathcal{L}}$ under the syntactic equivalence relation $\equiv$ defined by

$$\phi \equiv \psi \quad \text{if} \quad \mathbf{WC}_n \vdash \phi \leftrightarrow \psi,$$

equipped with the operations 1, $\neg$, $\to$ and $[C]$ defined by $1 = 1/\equiv$, $\neg(\phi/\equiv) = \neg\phi/\equiv$, $\phi/\equiv \to \psi/\equiv\ = (\phi \to \psi)/\equiv$ and $[C](\phi/\equiv)\ = [C]\phi/\equiv$ for any $C \in \mathcal{P}N$ and any $\phi, \psi \in \mathsf{Form}_{\mathcal{L}}$ (these definitions are allowed because $\mathbf{WC}_n$ is closed under Equivalence). By abuse of notation, we denote by ϕ the class $\phi/\equiv$.

Since $\mathbf{WC}_n$ contains every tautology of Łukasiewicz $n+1$-valued logic, the $\{\to, \neg, 1\}$-reduct of $\mathfrak{F}_{\mathbf{WC}_n}$ is an MV-algebra that belongs to the variety $\mathcal{MV}_n$ generated by $Ł_n$. Recall that MV-algebras are the algebraic counterpart of Łukasiewicz infinite-valued logic exactly as Boolean algebras are the algebraic counterpart of propositional logic. We refer to [6] for a survey of the theory of MV-algebras and to [5] for a monograph on the subject.

In the Boolean setting, one of the key ingredient of the construction of the canonical model is the ultrafilter theorem that allows to separate by an ultrafilter any two different non-top elements of any Boolean algebra $\mathbf{B}$. We can rephrase this separation result using the bijective correspondence between ultrafilters of $\mathbf{B}$ and homomorphisms from $\mathbf{B}$ to the two-element Boolean algebra $\mathbf{2}$: for any $a \neq b \in \mathbf{B} \setminus \{1\}$, there is a homomorphism $u : \mathbf{B} \to \mathbf{2}$ such that $u(a) = 1$ and $u(b) = 0$. It turns out (see [5]) that the variety $\mathcal{MV}_n$ enjoys a similar property: if $\mathbf{A} \in \mathcal{MV}_n$ then for any $a \neq b$ of $\mathbf{A} \setminus \{1\}$, there is a $\{\neg, \to, 1\}$-homomorphism $u : \mathbf{A} \to Ł_n$ such that $u(a) = 1$ and $u(b) \neq 1$.

This separation property explains why we choose the set $\mathcal{MV}(\mathfrak{F}_{\mathbf{WC}_n}, Ł_n)$ of $\{\neg, \to, 1\}$-homomorphisms from $\mathfrak{F}_{\mathbf{WC}_n}$ to $Ł_n$ as universe for the canonical model of $\mathbf{WC}_n$.

Before going into the details of the construction, we need some technical preliminaries.

Definition 3.4. Let i be an element of $\{1, \dots, n\}$. We denote by $\tau_{i/n}$ a composition (fixed throughout the paper) of the formulas $p \oplus p$ and $p \odot p$ whose Łukasiewicz interpretation on $Ł_n$ satisfies $\tau_{i/n}(x) = 0$ if $x < \frac{i}{n}$ and $\tau_{i/n}(x) = 1$ if $x \geq \frac{i}{n}$ (see [10] for the existence and the construction of such formulas). For the sake of readability, we denote by $\tau_{\frac{i}{2n}}$ the formula $\tau_{\lceil \frac{i}{2n} \rceil}$.

Definition 3.5. The *canonical model* of $\mathbf{WC}_n$ is the model $\mathcal{M} = (\mathfrak{F}, \mathrm{Val}^c)$ with $\mathfrak{F} = (W^c, E^c)$ where $W^c = \mathcal{MV}(\mathfrak{F}_{\mathbf{WC}_n}, Ł_n)$, where $E^c(u)(C, -)$ is defined for any $u \in W^c$ and any $C \in \mathcal{P}N$ by

$$E^c(u)\big(C, f\big) \geq \frac{i}{n} \quad \text{if} \quad \exists \phi \big(u([C]\phi) \geq \frac{i}{n} \;\&\; \forall v \big(v(\phi) \geq \frac{i}{n} \implies f(v) \geq \frac{i}{n}\big)\big), \tag{3.3}$$

for any $f \in Ł_n^S$, and where Val^c is defined by

$$\mathrm{Val}^c(u, p) = u(p), \tag{3.4}$$

for any $p \in \mathsf{Prop}$ and $u \in W^c$.

The following technical result states that condition (3.3) can legitimately be used to define a function $E^c(C, -) : Ł_n^S \to Ł_n$.

Lemma 3.6. *If $E^c(u)\big(C, f\big) \geq \frac{i}{n}$ for some $i \in \{1, \dots, n\}$, then $E^c(u)\big(C, f\big) \geq \frac{i-1}{n}$.*

Proof. Assume that ϕ satisfies condition (3.3) for C, f and $i > 0$ and set $\rho = \tau_{i/n}(\phi)$. Then $u([C]\rho) = \tau_{i/n}\big(u([C]\phi)\big) = 1 \geq \frac{i-1}{n}$. Moreover if $v(\rho) \geq \frac{i-1}{n}$ then $v(\rho) = 1$ since ρ is an idempotent element of $\mathfrak{F}_{\mathbf{WC}_n}$. It follows that $v(\phi) \geq \frac{i}{n}$ which implies that $f(v) \geq \frac{i}{n} \geq \frac{i-1}{n}$. □

The next proposition proves that in the canonical model the identity (3.4) remains true if we replace p by any formula $\phi \in \mathsf{Form}_{\mathcal{L}}$.

Proposition 3.7 (Truth Lemma)**.** *For any $\phi \in \mathsf{Form}_{\mathcal{L}}$ and any $u \in W^c$, the canonical model satisfies* $\mathrm{Val}^c(u, \phi) = u(\phi)$.

Proof. We proceed by induction on the number of connectors in ϕ. For propositional variables, the result is obtained by (3.4). If $\phi = \neg\psi$ or $\phi = \psi \to \rho$, the result is easily obtained. Assume that $\phi = [C]\psi$ for some $\psi \in \mathsf{Form}_{\mathcal{L}}$ and $C \in \mathcal{P}N$. We prove that for any $u \in W^c$ and any $i \leq n$,

$$E^c(u)\big(C, \mathrm{Val}^c(-, \psi)\big) \geq \frac{i}{n} \iff u([C]\psi) \geq \frac{i}{n}. \tag{3.5}$$

First, assume that $E^c(u)\big(C, \mathrm{Val}^c(-, \psi)\big) \geq \frac{i}{n}$. Then, by definition of E^c, there is a $\rho \in \mathsf{Form}_{\mathcal{L}}$ such that $u([C]\rho) \geq \frac{i}{n}$ and such that $\mathrm{Val}^c(v, \psi) \geq \frac{i}{n}$ for any $v \in W^c$

such that $v(\rho) \geq \frac{i}{n}$. The latter condition is equivalent to

$$v(\rho) \geq \frac{i}{n} \implies v(\psi) \geq \frac{i}{n}, \qquad v \in W^c,$$

by induction hypothesis.

It follows that $v\big(\tau_{i/n}(\rho) \to \tau_{i/n}(\psi)\big) = 1$ for any $v \in W^c$. This means that $\mathbf{WC}_n \vdash \tau_{i/n}(\rho) \to \tau_{i/n}(\psi)$ since the $\{\to, \neg, 1\}$-reduct of $\mathfrak{F}_{\mathbf{WC}_n}$ enjoys the separation property discussed before Definition 3.4. As $\mathbf{WC}_n$ is closed under Monotonicity, we obtain that $\mathbf{WC}_n \vdash [C]\tau_{i/n}(\rho) \to [C]\tau_{i/n}(\psi)$. Thanks to axioms (3.1) and (3.2) and Uniform Substitution, this is equivalent to $\mathbf{WC}_n \vdash \tau_{i/n}([C]\rho) \to \tau_{i/n}([C]\psi)$. Hence, for any $v \in W^c$, if $v([C]\rho) \geq \frac{i}{n}$ then $v([C]\psi) \geq \frac{i}{n}$. We can thus conclude that $u([C]\psi) \geq \frac{i}{n}$, which is the desired result.

Now, assume that $u([C]\psi) \geq \frac{i}{n}$. We want to prove that $E^c(u)\big(C, \mathrm{Val}^c(-,\psi)\big) \geq \frac{i}{n}$. This inequality is simply obtained by induction hypothesis by considering $\phi = \psi$ in the definition (3.3) of E^c. □

In order to use the canonical model to prove completeness of $\mathbf{WC}_n$ with respect to the class of the $Ł_n$-valued coalitional models, we need the following result.

Lemma 3.8. *The canonical model of $\mathbf{WC}_n$ is a weak $Ł_n$-valued coalitional model.*

Proof. Let $u \in W^c$. It is easily checked that $E^c(u)$ is output monotonic. Then we have to prove that $E^c(u)$ is weakly playable. Let $C \in \mathcal{P}N$ and $f \in Ł_n^S$. We first prove that for any $i \in \{0, \ldots, n\}$,

$$E^c(u)(C, f \oplus f) \geq \frac{i}{n} \iff E^c(u)(C, f) \oplus E^c(u)(C, f) \geq \frac{i}{n}. \tag{3.6}$$

First, assume that $E^c(u)(C,f) \oplus E^c(u)(C,f) \geq \frac{i}{n}$, or equivalently that $E^c(u)(C,f) \geq \frac{i}{2n}$. By definition of E^c, there is a formula ρ such that $u([C]\rho) \geq \frac{i}{2n}$ and such that $f(v) \geq \frac{i}{2n}$ for any v that satisfies $v(\rho) \geq \frac{i}{2n}$. On the one hand, by considering $\phi = \tau_{\frac{i}{2n}}(\rho)$ we obtain thanks to axioms (3.1) and (3.2) that

$$u([C]\phi) = \tau_{\frac{i}{2n}}(u([C]\rho)) = 1 \geq \frac{i}{n}.$$

On the other hand, if v is any world of W^c such that $v(\phi) \geq \frac{i}{n}$, then $v(\phi) = 1$ since ϕ is an idempotent element of $\mathfrak{F}_{\mathbf{WC}_n}$ and so $v(\rho) \geq \frac{i}{2n}$. This implies $f(v) \geq \frac{i}{2n}$, or equivalently $(f \oplus f)(v) \geq \frac{i}{n}$. We conclude that $E^c(u)(C, f \oplus f) \geq \frac{i}{n}$.

Conversely, assume that $E^c(u)(C, f \oplus f) \geq \frac{i}{n}$ for some $i > 0$. By definition of E^c, there is a formula ρ such that $u([C]\rho) \geq \frac{i}{n}$ and $f(v) \geq \frac{i}{2n}$ for any $v \in W^c$ such that $v(\rho) \geq \frac{i}{n}$. By considering $\phi = \tau_{i/n}(\rho)$, we obtain on the one hand that

$$u([C]\phi) = \tau_{i/n}\big(u([C]\rho)\big) = 1 \geq \frac{i}{2n}.$$

On the other hand, if v is any world of W^c such that $v(\phi) \geq \frac{i}{2n}$ then $v(\phi) = 1$. It follows that $v(\rho) \geq \frac{i}{n}$ so that $f(v) \geq \frac{i}{2n}$. We have proved that $E^c(u)(C,f) \geq \frac{i}{2n}$ or equivalently that $E^c(u)(C,f) \oplus E^c(u)(C,f) \geq \frac{i}{n}$. This concludes the proof of equivalence (3.6). One proceeds in a similar way to prove that

$$E^c(u)(C, f \odot f) \geq \frac{i}{n} \iff E^c(u)(C, f) \odot E^c(u)(C, f) \geq \frac{i}{n}, \tag{3.7}$$

which concludes the proof. □

We have gathered the necessary ingredients to obtain the following completeness result.

Theorem 3.9 (Completeness of $\mathbf{WC}_n$)**.** *For any $\phi \in \mathsf{Form}_{\mathcal{L}}$ we have $\mathbf{WC}_n \vdash \phi$ if and only if $\mathcal{M} \models \phi$ for any weak $Ł_n$-valued coalitional model.*

Proof. The necessity is Lemma 3.3. For the sufficiency, note that according to Proposition 3.7, $\mathcal{M}^c \models \phi$ means that the class of ϕ is equal to 1 in $\mathfrak{F}_{\mathbf{WC}_n}$, or equivalently that $\mathbf{WC}_n \vdash \phi$. □

Remark 1. We can use the formulas $\tau_{i/n}$ to replace the axioms (3.1) and (3.2) by a family of axioms, which are more easy to understand. Indeed, let us set

$$A = \{[C](p \star p) \leftrightarrow ([C]p \star [C]p) \mid \star \in \{\odot, \oplus\}, C \in \mathcal{P}N\},$$
$$B = \{[C]\tau_{i/n}(p) \leftrightarrow \tau_{i/n}([C]p) \mid i \in \{1, \ldots, n\}, C \in \mathcal{P}N\}.$$

It follows from the definition of a $Ł_n$-valued coalitional logic that $B \subseteq \mathbf{WC}_n$. Now, a careful analysis of the proofs of Lemma 3.6, Proposition 3.7, Lemma 3.8 and Theorem 3.9 shows that we have only used the axioms in A in the form of substitutions in formulas of B. Now denote by $\mathbf{WC}'_n$ the smallest set of formulas that contains an axiomatic base of Łukasiewicz logic and the set B, and that is closed under Modus Ponens, Uniform Substitution, Equivalence and Monotonicity. It follows from the previous observation that for any $\phi \in \mathsf{Form}_{\mathcal{L}}$ we have $\mathbf{WC}'_n \vdash \phi$ if and only if $\mathcal{M} \models \phi$ for every weak $Ł_n$-valued coalitional model. It results that $\mathbf{WC}'_n = \mathbf{WC}_n$.

In other words, the set of axioms A can be equivalently replaced by B. Hence, informally speaking, the content of axioms (3.1) and (3.2) is essentially the following:

> For any $i \leq n$, the truth value of the statement 'coalition C can enforce a state in which ϕ holds' is at least $\frac{i}{n}$ if and only if it holds that 'coalition C can enforce a state in which the truth value of ϕ is at least $\frac{i}{n}$'.

4. Conclusions and Future Work

In this paper, we have generalized the notion of α-effectivity in the context of game forms to fuzzy set of outcomes. We also have introduced the class of weakly-playable $Ł_n$-valued effectivity functions together with a characterization of this class in a multi-modal many-valued language through the logic $\mathbf{WC}_n$. Any $Ł_n$-valued effectivity function arising from a game form is weakly-playable. Hence, $\mathbf{WC}_n$ and its semantic counterpart (the class of the $Ł_n$-valued coalitional models) can be seen as a *minimal framework* in which studying the properties of α-effectivity for fuzzy sets of outcomes.

In this paper we do not consider the generalization of *superadditivity*, which is a fundamental property of Boolean effectivity functions. Finding the appropriate $Ł_n$-valued version of superadditivity and obtaining a characterization of the $Ł_n$-valued effectivity functions arising from a game form in the spirit of Pauly's result [11, Theorem 3.2] constitute the main thread of ongoing research.

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