Numerical Solution of 2D Contact Shape Optimization Problems Involving a Solution-Dependent Coefficient of Friction

Petr Beremlijski, Jaroslav Haslinger, Jiří Outrata, and Róbert Pathó

Abstract This contribution deals with numerical solution of shape optimization problems in frictional contact mechanics. The state problem in our case is given by 2D static Signorini problems with Tresca friction and a solution-dependent coefficient of friction. A suitable Lipschitz continuity assumption on the coefficient of friction is made, ensuring unique solvability of the discretized state problems and Lipschitz continuity of the corresponding control-to-state mapping. The discrete shape optimization problem can be transformed into a nonsmooth minimization problem and handled by the bundle trust method. In each step of the method, the state problem is solved by the method of successive approximations and necessary subgradient information is computed using the generalized differential calculus of B. Mordukhovich.

Keywords Frictional contact • Nonsmooth analysis • Shape optimization

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1 Introduction

Shape optimization is a branch of optimal control theory in which control variables are related to the geometry of optimized structures (size, shape or topology). By an appropriate change of the geometry one tries to get a structure with some desired properties. Usually, its behavior is modeled by partial differential equations. In practice, however, one can meet situations when physical systems are governed by variational inequalities. A common feature of such optimization problems is the fact that the control-to-state mapping might be nonsmooth and, consequently, the whole optimization problem is generally nonsmooth, as well. If it is so, then special tools of nonsmooth analysis have to be used to perform sensitivity analysis which provides necessary gradient-like information for nonsmooth minimization methods. Contact problems represent one of typical applications of variational inequalities in mechanics of solids: one tries to find an equilibrium state of a system of a finite number of loaded deformable bodies which are possibly in mutual contact taking into account effects of friction on common parts. Just the presence of friction complicates the analysis. If the friction obeys the Coulomb law [5], then the respective mathematical model leads to an implicit variational inequality. Shape optimization with contact problems involving Coulomb friction in 2 and 3D has been theoretically studied in [1], and [2], respectively, including numerical experiments. Another type of friction was considered in [8], namely contact problems with given friction and a solution-dependent coefficient of friction. Shape optimization with this type of the state problems has been theoretically analyzed in [7]. The goal of the present chapter is to illustrate applicability of theoretical results concerning sensitivity analysis for numerical realization of model examples.

The paper is organized as follows: after introducing the notation, we recall some basic notions from the theory of generalized differential calculus that will be used later in Sect. 3. In Sect. 2 we present the state problem, the shape optimization problem and also quote some results concerning their solvability. Next, we introduce a suitable discretization and review conditions under which discrete optimal shapes exist and converge to an optimal one as the discretization parameter tends to zero. Assuming unique solvability of the discrete state problems in Sect. 3, we compute shape sensitivities of the cost functional and the discrete state variable employing modern methods of variational analysis [12]. Using these results, numerical examples in Sect. 4 illustrate the feasibility of this approach in solving shape optimization problems involving complicated boundary conditions.

Throughout the paper we use the following notation: the symbol $H^k(\Omega)$ ($k \geq 0$ integer) stands for the Sobolev space of functions which are together with their derivatives up to order k square integrable in Ω, i.e. elements of $L^2(\Omega)$ (we set

$H^0(\Omega) := L^2(\Omega)$). The norm in $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$. Vector-valued functions and the respective spaces of vector-valued functions will be denoted by bold characters. Bold characters will also be used for vectors in $\mathbb{R}^n$, with the Euclidean scalar product $\langle\cdot,\cdot\rangle_n$ and corresponding norm $\|\cdot\|_n$. For a set $A \subset X$, $\overline{A}$ stands for the *closure* of A with respect to the topology of X. For $X = \mathbb{R}^n$ and $\bar{\mathbf{x}} \in A$ we denote by $\hat{N}_A(\vec{x})$ the *Fréchet (regular) normal cone* to A at $\bar{\mathbf{x}}$:

$$\hat{N}_A(\bar{\mathbf{x}}) := \left\{\mathbf{x}^* \in \mathbb{R}^n \;\middle|\; \limsup_{\mathbf{x} \xrightarrow{A} \bar{\mathbf{x}}} \frac{\langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}}\rangle_n}{\|\mathbf{x} - \bar{\mathbf{x}}\|_n} \leq 0\right\},$$

whereas the *limiting (Mordukhovich) normal cone* to A at $\bar{\mathbf{x}}$ will be denoted by $N_A(\bar{\mathbf{x}})$:

$$N_A(\bar{\mathbf{x}}) := \operatorname{limsup}_{\mathbf{x} \xrightarrow{A} \bar{\mathbf{x}}} \hat{N}_A(\mathbf{x}).$$

Here the symbol "Limsup" stands for the Kuratowski-Painlevé outer limit of sets (cf. [16]). Given a multifunction $Q : \mathbb{R}^n \to \mathbb{R}^m$, we denote its graph by $\operatorname{Gr} Q := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathbf{y} \in Q(\mathbf{x})\}$. The *regular coderivative* of Q at a reference point $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \operatorname{Gr} Q$ is given by the multifunction $\hat{D}^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}}) : \mathbb{R}^m \to \mathbb{R}^n$, which is defined as follows:

$$\hat{D}^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}})(\mathbf{y}^*) := \{\mathbf{x}^* \in \mathbb{R}^n \mid (\mathbf{x}^*, -\mathbf{y}^*) \in \hat{N}_{\operatorname{Gr} Q}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\}.$$

Analogously, the multifunction $D^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}}) : \mathbb{R}^m \to \mathbb{R}^n$, defined by

$$D^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}})(\mathbf{y}^*) := \{\mathbf{x}^* \in \mathbb{R}^n \mid (\mathbf{x}^*, -\mathbf{y}^*) \in N_{\operatorname{Gr} Q}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\}$$

is called the *limiting (Mordukhovich) coderivative* of Q at $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Further, we will employ another important notion from the theory of generalized differentiation, namely that of calmness: a multifunction Q is said to be *calm* at $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \operatorname{Gr} Q$ provided $\exists L > 0$ and $\exists$ neighbourhoods U, V of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, respectively, such that:

$$Q(\mathbf{x}) \cap V \subset Q(\bar{\mathbf{x}}) + L\|\mathbf{x} - \bar{\mathbf{x}}\|_n \mathbb{B}_m \quad \forall \mathbf{x} \in U,$$

where $\mathbb{B}_m$ stands for the closed unit ball in $\mathbb{R}^m$, centered at the origin.

2 Problem Formulation and Discretization

Throughout the chapter we assume that the positive real parameters a, b and $0 < C_0 < b$ are fixed.

Let us consider an *elastic body*, represented by the domain $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, a), \alpha(x_1) < x_2 < b\}$, where $\alpha \in C^{0,1}([0, a])$, $0 \leq \alpha \leq C_0$. Suppose that the boundary $\partial\Omega$ is decomposed according to the boundary conditions into three pairwise disjoint, relatively open subsets: along $\Gamma_u \subset \partial\Omega$, $\text{meas}_1 \Gamma_u > 0$ the body is clamped, on $\Gamma_P \subset \partial\Omega$ surface tractions of density $\mathbf{P} = (P_1, P_2) \in \mathbf{L}^2(\Gamma_P)$ act and along $\Gamma_c := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, a),\ x_2 = \alpha(x_1)\} = \text{Gr}\,\alpha$, the body may come in contact with the rigid foundation $M = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$. Due to the special geometry, the *non-penetration conditions* on the contact boundary Γ_c can be expressed exactly and take the following form:

$$\left.\begin{array}{l} u_2(x_1, \alpha(x_1)) \geq -\alpha(x_1), \quad T_2(\mathbf{u})(x_1, \alpha(x_1)) \geq 0, \\ \big(u_2(x_1, \alpha(x_1)) + \alpha(x_1)\big) T_2(\mathbf{u})(x_1, \alpha(x_1)) = 0 \end{array}\right\} \text{ for } x_1 \in (0, a). \tag{2.1}$$

Here $\mathbf{u} = (u_1, u_2) : \Omega \to \mathbb{R}^2$ is a displacement vector and $\mathbf{T}(\mathbf{u}) = (T_1(\mathbf{u}), T_2(\mathbf{u})) : \partial\Omega \to \mathbb{R}^2$ is the stress vector associated with $\mathbf{u}$. In addition to (2.1) we shall consider effects of friction between Ω and M. We use the friction law of Tresca type, i.e. with an a-priori given slip bound $g : \Gamma_c \to \mathbb{R}_+$, but with a coefficient of friction $\mathscr{F} : R_+ \to R_+$ which depends on the solution. Thus the *friction conditions* on Γ_c read as follows:

$$\left.\begin{array}{l} u_1 = 0 \Longrightarrow |T_1(\mathbf{u})| \leq \mathscr{F}(0) g \\ u_1 \neq 0 \Longrightarrow T_1(\mathbf{u}) = -\text{sgn}(u_1) \mathscr{F}(|u_1|) g \end{array}\right\} \text{ on } \Gamma_c. \tag{2.2}$$

Finally, Ω will be subject to body forces of density $\mathbf{F} = (F_1, F_2) \in \mathbf{L}^2(\Omega)$. The equilibrium state of Ω is characterized by a displacement vector $\mathbf{u}$ which satisfies the system of the linear equilibrium equations in Ω, the classical boundary conditions on Γ_P, Γ_u and the unilateral and friction conditions (2.1) and (2.2), resp., on Γ_c.

In order to give the weak form of the Signorini problem with given friction and a solution-dependent coefficient of friction, we denote the space of virtual displacements by $\mathbf{V} := \{\mathbf{v} = (v_1, v_2) \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_u\}$ and the closed, convex cone of kinematically admissible displacements by $\mathbf{K} := \{\mathbf{v} \in \mathbf{V} \mid v_2(x_1, \alpha(x_1)) \geq -\alpha(x_1) \text{ a.e. in } (0, a)\}$. Further, let $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $L : \mathbf{V} \to \mathbb{R}$ be defined by:

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \sigma(\mathbf{u}) : \varepsilon(\mathbf{v})\, d\mathbf{x}, \qquad L(\mathbf{v}) := \int_\Omega \mathbf{F} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_P} \mathbf{P} \cdot \mathbf{v}\, ds,$$

where the stress tensor $\sigma(\mathbf{u})$ is linked to the linearized strain tensor $\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ by a linear Hooke's law: $\sigma(\mathbf{u}) = \mathscr{C}\varepsilon(\mathbf{u})$. We assume that the fourth order stiffness tensor $\mathscr{C} \in L^\infty(\Omega)$ satisfies the usual symmetry and ellipticity conditions.

Definition 2.1. By a *weak solution* to the Signorini problem with Tresca friction and a solution-dependent coefficient of friction $\mathscr{F}$ we mean any $\mathbf{u} \in \mathbf{K}$ satisfying:

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_{\Gamma_c} \mathscr{F}(|u_1|) g (|v_1| - |u_1|)\, ds \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}. \tag{$\mathscr{P}$}$$

Note that ($\mathscr{P}$) is an implicit variational inequality of the second kind. Its solvability was addressed in [8] and is summarized in the following theorem.

Theorem 2.2. *(i) For any nonnegative $g \in L^2(\Gamma_c)$ and nonnegative, bounded, continuous $\mathscr{F}$ there exists a solution to ($\mathscr{P}$).*
(ii) There exists a constant $C_{max} > 0$ such that the solution to ($\mathscr{P}$) is unique, provided that $g \in L^\infty(\Gamma_c)$ and $\mathscr{F}$ is bounded, Lipschitz continuous with modulus $C_L > 0$ such that: $C_L \|g\|_{L^\infty(\Gamma_c)} < C_{max}$.

Up to this point we used one fixed domain Ω and solved the corresponding problem ($\mathscr{P}$) on it. When optimizing the contact boundary we consider α to be a parameter, by means of which one can change the shape of Ω. Our aim is to find α^* from an *admissible set* U_{ad} such that the pair $(\alpha^*, \mathbf{u}^*)$, where $\mathbf{u}^*$ solves the corresponding problem ($\mathscr{P}$) on $\Omega(\alpha^*)$, minimizes a given cost functional J on U_{ad}. To emphasize the fact that Ω is parametrized by α, we will write α as the argument. In agreement with this convention, notation $\Omega(\alpha)$, $\Gamma_c(\alpha)$, $\mathbf{V}(\alpha)$, $\mathbf{K}(\alpha)$, $\mathscr{P}(\alpha)$, etc. will be used instead of Ω, Γ_c, $\mathbf{V}$, $\mathbf{K}$, ($\mathscr{P}$), etc.

In what follows we shall restrict ourselves to α belonging to the following admissible set U_{ad}:

$$U_{ad} := \{\alpha \in C^{1,1}([0,a]) \mid 0 \le \alpha \le C_0,\ |\alpha'| \le C_1 \text{ in } [0,a],$$
$$|\alpha''| \le C_2 \text{ a.e. in } (0,a), \text{meas } \Omega(\alpha) = C_3\}, \qquad (2.3)$$

i.e. U_{ad} contains all functions which are together with their first derivatives Lipschitz equi-continuous in $[0,a]$ and preserve the constant area of $\Omega(\alpha)$. We assume that the positive constants C_0, C_1, C_2 and C_3 are chosen in such a way that $U_{ad} \neq \emptyset$. Further, we need to clarify the meaning of all functions appearing in the definition of ($\mathscr{P}$) for various $\alpha \in U_{ad}$. To this end, let $\hat{\Omega} := (0,a) \times (0,b)$ and assume that the functions $\mathscr{C}$, $\mathbf{F}$, $\mathbf{P}$ and g are restrictions of some $\hat{\mathscr{C}} \in L^\infty(\hat{\Omega})$, $\hat{\mathbf{F}} \in \mathbf{L}^2(\hat{\Omega})$, $\hat{\mathbf{P}} \in \mathbf{L}^2(\partial\hat{\Omega})$ and $\hat{g} \in H^1(\hat{\Omega})$, $\hat{g} \ge 0$ onto $\Omega(\alpha)$, $\Gamma_p(\alpha)$ and $\Gamma_c(\alpha)$, respectively.

Let $S : U_{ad} \ni \alpha \mapsto \{\mathbf{u} \in \mathbf{K}(\alpha) \mid \mathbf{u} \text{ solves } \mathscr{P}(\alpha)\}$ denote the *control-to-state* mapping and let $J : \text{Gr } S \to \mathbb{R}$ be a given *cost functional*. Note that S is a *multivalued* mapping, in general.

Definition 2.3. A domain $\Omega(\alpha^*)$ is said to be *optimal* iff there exists $\mathbf{u}^* \in S(\alpha^*)$ satisfying:

$$J(\alpha^*, \mathbf{u}^*) \le J(\alpha, \mathbf{u}) \quad \forall (\alpha, \mathbf{u}) \in \text{Gr } S. \qquad (\mathbb{P})$$

Below we recall the result from [7] stating, that there exists an optimal shape in U_{ad}, defined by (2.3), for a large class of cost functionals.

Theorem 2.4. *Let the assumptions of Theorem 2.2(i) hold and suppose that J is lower semicontinuous in the following sense:*

$$\left.\begin{array}{l}\alpha_n \to \alpha \ \ in\ C^1([0,a]),\ \alpha_n, \alpha \in U_{ad},\\ \mathbf{y}_n \rightharpoonup \mathbf{y} \ \ in\ \mathbf{H}^1(\hat{\Omega}),\ \mathbf{y}_n, \mathbf{y} \in \mathbf{H}^1(\hat{\Omega})\end{array}\right\} \Longrightarrow \liminf_{n\to\infty} J(\alpha_n, \mathbf{y}_n|_{\Omega(\alpha_n)}) \geq J(\alpha, \mathbf{y}|_{\Omega(\alpha)}).$$

Then ($\mathbb{P}$) *has a solution.*

Proof. It is sufficient to prove that Gr S is compact in the above defined topology—see [7, Lemma 1]. □

In the second part of this section we shortly describe a discrete version of ($\mathbb{P}$) and provide sufficient conditions ensuring unique solvability of the discretized state problems and convergence of discrete optimal shapes to an optimal one in the sense of Definition 2.3.

Every discretization of ($\mathbb{P}$) is twofold: (i) one has to approximate the admissible set U_{ad} and (ii) to discretize the state problem. In order to make the forthcoming presentation more straightforward, we shall use continuous, piecewise linear functions α_h as design variables. However, they are not practical from the engineering point of view and therefore will be replaced by Bézier functions in numerical experiments. For the approximation of ($\mathscr{P}$) we shall use standard piecewise linear triangular finite elements.

Let $d \geq 1$ be a given integer and set $h := a/d$. By δ_h we denote the equidistant partition of $[0, a]$:

$$\delta_h : \quad 0 \equiv a_0 < a_1 < \cdots < a_{d(h)} \equiv a, \qquad a_j = a + jh,\ j = 0, \ldots, d. \tag{2.4}$$

With any δ_h we associate the set U_{ad}^h defined by

$$\begin{aligned} U_{ad}^h := \{\alpha_h \in C([0,a]) \mid \alpha_h|_{[a_{i-1},a_i]} \in P_1([a_{i-1}, a_i]) \quad &\forall i = 1, \ldots, d,\\ 0 \leq \alpha_h(a_i) \leq C_0 \quad &\forall i = 0, \ldots, d,\\ |\alpha_h(a_i) - \alpha_h(a_{i-1})| \leq C_1 h \quad &\forall i = 1, \ldots, d,\\ |\alpha_h(a_{i+1}) - 2\alpha_h(a_i) + \alpha_h(a_{i-1})| \leq C_2 h^2, \quad &\forall i = 1, \ldots, d-1,\\ \text{meas}\ \Omega(\alpha_h) = C_3\}, \end{aligned}$$

where $C_0, \ldots, C_3$ are the same as in (2.3). Notice that $U_{ad}^h \not\subset U_{ad}$, i.e. U_{ad}^h is an *external approximation* of U_{ad}.

Since for each $\alpha_h \in U_{ad}^h$ the domain $\Omega(\alpha_h)$ is polygonal, one can construct its triangulation $\mathscr{T}(h, \alpha_h)$ whose nodes lie on the lines $\{a_i\} \times \mathbb{R}$, $i = 0, 1, \ldots, d$.

Moreover, we shall assume that for each $h > 0$ the family $\{\mathscr{T}(h, \alpha_h) \mid \alpha_h \in U^h_{ad}\}$ consists of *topologically equivalent* triangulations (cf. [6, p. 32]) and that $\{\mathscr{T}(h, \alpha_h)\}$ are *uniformly regular* with respect to $(h, \alpha_h) \in (0, \infty) \times U^h_{ad}$. The domain $\Omega(\alpha_h)$ with the triangulation $\mathscr{T}(h, \alpha_h)$ will be denoted by $\Omega_h(\alpha_h)$ or just shortly Ω_h.

On $\Omega_h(\alpha_h)$ we construct the following piecewise linear approximations of $\mathbf{V}(\alpha_h)$ and $\mathbf{K}(\alpha_h)$:

$$\mathbf{V}_h(\alpha_h) := \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}_h) \mid \mathbf{v}_h|_T \in (P_1(T))^2 \ \ \forall T \in \mathscr{T}(h, \alpha_h), \ \mathbf{v}_h = \mathbf{0} \text{ on } \overline{\Gamma}_u(\alpha_h)\},$$

and

$$\mathbf{K}_h(\alpha_h) := \{\mathbf{v}_h = (v_{h1}, v_{h2}) \in \mathbf{V}_h(\alpha_h) \mid v_{h2}(a_i, \alpha_h(a_i)) \geq -\alpha_h(a_i) \ \forall a_i \in \mathscr{N}_h\},$$

respectively, where $\mathscr{N}_h$ is the set of all *contact nodes*, i.e. $a_i \in \mathscr{N}_h$ iff $(a_i, \alpha_h(a_i)) \in \overline{\Gamma}_c(\alpha_h) \setminus \overline{\Gamma}_u(\alpha_h)$. Observe, that $\mathbf{K}_h(\alpha_h) \subset \mathbf{K}(\alpha_h) \ \ \forall h > 0 \ \forall \alpha_h \in U^h_{ad}$.

Definition 2.5. By a solution to the discretized Signorini problem with given friction and a solution-dependent coefficient of friction we mean any function $\mathbf{u}_h := \mathbf{u}_h(\alpha_h) \in \mathbf{K}_h(\alpha_h)$ satisfying:

$$\left.\begin{array}{l} a_{\alpha_h}(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \displaystyle\int_0^a \mathscr{F}(r_h|u_{h1} \circ \alpha_h|) g \circ \alpha_h (|v_{h1} \circ \alpha_h| - \\ |u_{h1} \circ \alpha_h|) \sqrt{1 + (\alpha_h')^2}\, dx_1 \geq L_{\alpha_h}(\mathbf{v}_h - \mathbf{u}_h) \quad \forall \mathbf{v}_h \in \mathbf{K}_h(\alpha_h), \end{array}\right\} \qquad (\mathscr{P}_h(\alpha_h))$$

where $r_h : C([0, a]) \to C([0, a])$ stands for the piecewise linear Lagrange interpolation operator on δ_h and for any $w \in H^1(\Omega(\alpha_h))$ the symbol $w \circ \alpha_h$ denotes the function $x \mapsto w(x, \alpha_h(x))$, $x \in (0, a)$.

Theorem 2.6. *(i) Let the assumptions of Theorem 2.2(i) be satisfied. Then* $(\mathscr{P}_h(\alpha_h))$ *has a solution for any* $h > 0$ *and* $\alpha_h \in U^h_{ad}$.

(ii) There exists a constant $C^h_{max} > 0$ *such that the solution to* $(\mathscr{P}_h(\alpha_h))$ *is unique, provided that the following conditions hold:* $g \in C(\overline{\hat{\Omega}})$ *and* $\mathscr{F}$ *is nonnegative, bounded, Lipschitz continuous with modulus* $C_L > 0$ *such that* $C_L\|g\|_{C(\overline{\hat{\Omega}})} < C^h_{max}$.

Proof. It can be found in [8] and [15]. □

Note, that by looking at the explicit form of the constants $C_{\max}$ and $C^h_{\max}$ appearing in Theorems 2.2 and 2.6 we find that: (1) $C^h_{\max}$ can be chosen *independently* of h and (2) $C^h_{\max} < C_{\max}$. Hence, Theorem 2.6(*ii*) implies Theorem 2.2(*ii*).

Analogously to S and ($\mathbb{P}$) we define the discrete control-to-state mapping S_h and the discrete shape optimization problem ($\mathbb{P}_h$):

$$\left.\begin{array}{ll}\text{minimize} & J(\alpha_h, \mathbf{u}_h) \\ \text{subj. to} & (\alpha_h, \mathbf{u}_h) \in \text{Gr } S_h\end{array}\right\} \tag{$\mathbb{P}_h$}$$

In the next theorem the symbol "$\tilde{}$" above a function $\mathbf{v} \in \mathbf{H}^1(\Omega(\alpha_h))$ denotes its extension to $\hat{\Omega}$ satisfying: $\|\tilde{\mathbf{v}}\|_{1,\hat{\Omega}} \le \tilde{C}\|\mathbf{v}\|_{1,\Omega(\alpha_h)}$ $\forall \mathbf{v} \in \mathbf{H}^1(\Omega(\alpha_h))$, such that the constant $\tilde{C}$ is independent of α_h and $h > 0$. Since $\{\Omega(\alpha_h) \mid \alpha_h \in U_{ad}^h, h > 0\}$ is a system satisfying the *uniform cone property*, such an extension exists (see [3]). For details we refer to [7].

Theorem 2.7. *Suppose that J is continuous in the following sense:*

$$\left.\begin{array}{l}\alpha_h \to \alpha \ \ \text{in } C([0,a]),\ \alpha_h \in U_{ad}^h \\ \mathbf{u}_h \rightharpoonup \mathbf{u} \ \ \text{in } \mathbf{H}^1(\hat{\Omega}),\ \mathbf{u}_h, \mathbf{u} \in \mathbf{H}^1(\hat{\Omega})\end{array}\right\} \Longrightarrow \lim_{h \to 0_+} I(\alpha_h, \mathbf{u}_h|_{\Omega(\alpha_h)}) = I(\alpha, \mathbf{u}|_{\Omega(\alpha)})$$

and let the assumptions of Theorem 2.6(ii) hold. Then:

(j) there exists at least one solution $(\alpha_h^*, \mathbf{u}_h^*)$ *to* ($\mathbb{P}_h$) $\forall h > 0$;

(jj) for every sequence of discrete optimal pairs $\{(\alpha_h^*, \mathbf{u}_h^*)\}$, $h \to 0_+$ *there exists a subsequence* $\{(\alpha_{h_j}^*, \mathbf{u}_{h_j}^*)\}$, $j \to \infty$ *and functions* $\alpha^* \in U_{ad}$, $\mathbf{u}^* \in \mathbf{H}^1(\hat{\Omega})$ *such that:*

$$\alpha_{h_j}^* \to \alpha^* \ \ \text{in} \ \ C([0,a]) \quad \text{and} \quad \tilde{\mathbf{u}}_{h_j}^* \rightharpoonup \mathbf{u}^* \ \ \text{in} \ \ \mathbf{H}^1(\hat{\Omega}),\ j \to \infty, \tag{2.5}$$

where $(\alpha^*, \mathbf{u}^*|_{\Omega(\alpha^*)})$ *solves* ($\mathbb{P}$). *In addition, every accumulation point of* $\{(\alpha_h^*, \mathbf{u}_h^*)\}$ *in the sense of* (2.5) *has this property.*

Proof. It follows from Theorems 6, 7 and Lemma 7 in [7]. □

We conclude this section with the algebraic form of the discrete state problem ($\mathscr{P}_h(\alpha_h)$), in particular with its reduced version involving only state variables defined on the contact boundary Γ_c. In the rest of the paper $h > 0$ shall be *fixed*.

Let us set $n := \dim \mathbf{V}_h(\alpha_h)$ and $p := \operatorname{card} \mathscr{N}_h$, i.e. p is the number of the contact nodes. For the sake of simplicity let us further assume that $p = d(h) + 1$ (cf. (2.4)). Then U_{ad}^h is isomorphic to a convex, compact set $\mathscr{U}_{ad} \subset \mathbb{R}_+^p$ by means of the mapping $\alpha_h \mapsto \boldsymbol{\alpha} = (\alpha_h(a_0), \ldots, \alpha_h(a_{d(h)}))$. Further, the set $\mathbf{K}_h(\alpha_h)$ may be identified with the closed, convex set:

$$\mathscr{K}(\boldsymbol{\alpha}) := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}_\nu \ge -\boldsymbol{\alpha}\}, \quad \boldsymbol{\alpha} \in \mathscr{U}_{ad},$$

where $\mathbf{v}_\nu \in \mathbb{R}^p$ stands for the subvector of $\mathbf{v} \in \mathbb{R}^n$ consisting of the second components of the displacement vector $\mathbf{v}$ at all contact nodes, i.e. $(\mathbf{v}_\nu)_i = v_{h2}(a_{i-1}, \alpha_h(a_{i-1}))$ for each $i = 1, \ldots, p$. Analogously, $\mathbf{v}_\tau \in \mathbb{R}^p$ consists of the first components of $\mathbf{v}$ at the contact nodes. The frictional term in ($\mathscr{P}_h(\alpha_h)$) should be *approximated* by a quadrature formula whose integration nodes coincide with the contact nodes. Hence, the algebraic formulation of the discrete Signorini problem with a solution-dependent coefficient of friction reads as:

$$\left.\begin{aligned} &\text{Find } \mathbf{u} \in \mathscr{K}(\boldsymbol{\alpha}) \text{ such that for every } \mathbf{v} \in \mathscr{K}(\alpha) : \\ &\langle \mathbb{A}(\boldsymbol{\alpha})\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_n + \sum_{i=1}^{p} \omega_i(\boldsymbol{\alpha}) \mathscr{F}(|(\mathbf{u}_\tau)_i|)\big(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\big) \geq \langle \mathbf{L}(\boldsymbol{\alpha}), \mathbf{v} - \mathbf{u}\rangle_n \end{aligned}\right\} \qquad (\mathscr{P}'(\boldsymbol{\alpha}))$$

where $\mathbb{A} \in C^1(\mathscr{U}_{ad}; \mathbb{R}^{n\times n})$ and $\mathbf{L} \in C^1(\mathscr{U}_{ad}; \mathbb{R}^n)$ denote the matrix- and vector-valued function, resp., associating with any $\boldsymbol{\alpha} \in \mathscr{U}_{ad}$ the stiffness matrix $\mathbb{A}(\boldsymbol{\alpha})$ and the load vector $\mathbf{L}(\boldsymbol{\alpha})$. Finally, let us assume that $\omega_i \in C^1(\mathscr{U}_{ad}; (0, \infty))$ $\forall i = 1, \ldots, p$.

Instead of dealing with ($\mathscr{P}'(\boldsymbol{\alpha})$) directly, we shall introduce Lagrange multipliers $\lambda \in \mathbb{R}^p_+$ to release the constraint $\mathbf{v} \in \mathscr{K}(\boldsymbol{\alpha})$, and employ the Schur complement technique to eliminate all internal variables and reduce the state problem to the contact boundary. Since it will be more convenient for sensitivity analysis, the resulting variational inequality is formulated as a *generalized equation* (GE) (for details the reader is kindly referred to [7] and also [1, 2]):

$$\left.\begin{aligned} &\mathbf{0} \in \mathbb{A}_{\tau\tau}(\boldsymbol{\alpha})\mathbf{u}_\tau + \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha})\mathbf{u}_\nu - \mathbf{L}_\tau(\boldsymbol{\alpha}) + Q_1(\boldsymbol{\alpha}, \mathbf{u}_\tau) \\ &\mathbf{0} = \mathbb{A}_{\nu\tau}(\boldsymbol{\alpha})\mathbf{u}_\tau + \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha})\mathbf{u}_\nu - \mathbf{L}_\nu(\boldsymbol{\alpha}) - \boldsymbol{\lambda} \\ &\mathbf{0} \in \mathbf{u}_\nu + \boldsymbol{\alpha} + N_{\mathbb{R}^p_+}(\boldsymbol{\lambda}). \end{aligned}\right\} \qquad (2.6)$$

In our case the multifunction $Q_1 : \mathscr{U}_{ad} \times \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is defined as:

$$\big(Q_1(\boldsymbol{\alpha}, \mathbf{u}_\tau)\big)_i := \omega_i(\boldsymbol{\alpha})\mathscr{F}(|(\mathbf{u}_\tau)_i|)\partial|(\mathbf{u}_\tau)_i| \quad \forall i = 1, \ldots, p,$$

where "∂" denotes the subdifferential of convex functions, $N_{\mathbb{R}^p_+}(\cdot)$ is the normal cone in the sense of convex analysis and submatrices $\mathbb{A}_{\tau\tau}$, $\mathbb{A}_{\tau\nu}$, $\mathbb{A}_{\nu\nu} \in C^1(\mathscr{U}_{ad}; \mathbb{R}^{p\times p})$ are parts of the Schur complement to the stiffness matrix with $\mathbb{A}_{\nu\tau} = \mathbb{A}^T_{\tau\nu}$. In addition, note that $\mathbb{A}_{\tau\tau}$ and $\mathbb{A}_{\nu\nu}$ are positive definite uniformly with respect to $\boldsymbol{\alpha} \in \mathscr{U}_{ad}$.

The next theorem states that GE (2.6) is uniquely solvable and its solution depends Lipschitz continuously on the shape variable α.

Theorem 2.8. *There exists a constant $C_L > 0$, independent of h and $\alpha \in \mathscr{U}_{ad}$ such that if $\mathscr{F}$ is Lipschitz continuous with modulus C_L, then the corresponding control-to-state mapping $S : \mathscr{U}_{ad} \to \mathbb{R}^{3p}$, $\alpha \mapsto \{(\mathbf{u}_\tau, \mathbf{u}_\nu, \lambda) \mid (\mathbf{u}_\tau, \mathbf{u}_\nu, \lambda)$ solves (2.6)$\}$ is single-valued and Lipschitz continuous.*

3 Discrete Sensitivity Analysis

Introducing the state variable $\mathbf{y} := (\mathbf{u}_\tau, \mathbf{u}_\nu, \boldsymbol{\lambda}) \in \mathbb{R}^{3p}$, the GE (2.6) may be written in the following compact form:

$$\mathbf{0} \in F(\boldsymbol{\alpha}, \mathbf{y}) + Q(\boldsymbol{\alpha}, \mathbf{y}), \tag{3.1}$$

with $\boldsymbol{\alpha} \in \mathscr{U}_{ad}$ being the control variable and

$$F(\boldsymbol{\alpha}, \mathbf{y}) := \begin{pmatrix} \mathbb{A}_{\tau\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha}) & 0 \\ \mathbb{A}_{\nu\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha}) & -\mathbb{I} \\ 0 & \mathbb{I} & 0 \end{pmatrix} \mathbf{y} - \begin{pmatrix} \mathbf{L}_\tau(\boldsymbol{\alpha}) \\ \mathbf{L}_\nu(\boldsymbol{\alpha}) \\ -\boldsymbol{\alpha} \end{pmatrix}, \quad Q(\boldsymbol{\alpha}, \mathbf{y}) := \begin{pmatrix} Q_1(\boldsymbol{\alpha}, \mathbf{y}_1) \\ 0 \\ N_{\mathbb{R}^p_+}(\mathbf{y}_3) \end{pmatrix}.$$

Note that F is single-valued, continuously differentiable in its domain of definition and Q is a closed-graph multifunction. The algebraic *shape optimization problem* reads as the following Mathematical Program with Equilibrium Constraints (MPEC):

$$\left.\begin{array}{ll} \text{minimize} & J(\boldsymbol{\alpha}, \mathbf{y}) \\ \text{subj. to} & \mathbf{0} \in F(\boldsymbol{\alpha}, \mathbf{y}) + Q(\boldsymbol{\alpha}, \mathbf{y}), \\ & \boldsymbol{\alpha} \in \mathscr{U}_{ad}, \end{array}\right\} \tag{$\mathbb{P}$}$$

where J is a given continuously differentiable cost functional. In what follows we shall assume that the assumptions of Theorem 2.8 are satisfied. Then ($\mathbb{P}$) may be equivalently reformulated as a nonlinear optimization problem:

$$\left.\begin{array}{ll} \text{minimize} & \mathscr{J}(\boldsymbol{\alpha}) := J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha})) \\ \text{subj. to} & \boldsymbol{\alpha} \in \mathscr{U}_{ad}. \end{array}\right\} \tag{$\tilde{\mathbb{P}}$}$$

Since $\mathscr{J}$ is locally Lipschitz continuous, ($\tilde{\mathbb{P}}$) can be solved by standard methods of nonsmooth optimization. Such algorithms, however, require typically knowledge of some subgradient information, usually in the form of one (arbitrary) subgradient from the Clarke subdifferential $\overline{\partial}\mathscr{J}$ (cf. [4, Theorem 2.5.1]) in each iteration step. This can be conducted by using the chain rule from [4, Theorem 2.6.6]:

$$\overline{\partial}\mathscr{J}(\overline{\boldsymbol{\alpha}}) = \nabla_\alpha J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}) + \left(\overline{\partial} S(\overline{\boldsymbol{\alpha}})\right)^T \nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}), \tag{3.2}$$

valid at any reference point $\overline{\boldsymbol{\alpha}} \in \mathscr{U}_{ad}$, $\bar{\mathbf{y}} := S(\overline{\boldsymbol{\alpha}})$. Thus, for the required subgradient information it is sufficient to compute an element from $\left(\overline{\partial} S(\overline{\boldsymbol{\alpha}})\right)^T \nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$, where $\overline{\partial} S(\overline{\boldsymbol{\alpha}})$ stands for the *generalized Jacobian* of Clarke, defined in [4, Definition 2.6.1]. The rest of the section is devoted to this task.

First, observe that Lipschitz continuity of S and formula (2.23) in [11] yield:

$$\left(\overline{\partial} S(\overline{\boldsymbol{\alpha}})\right)^T \mathbf{y}^* = \operatorname{conv} D^* S(\overline{\boldsymbol{\alpha}}(\mathbf{y}^*)) \quad \forall \mathbf{y}^* \in \mathbb{R}^{3p}.$$

Comparing with (3.2), we see that it is sufficient to determine one element from the set $D^* S(\overline{\boldsymbol{\alpha}})(\nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))$ and we are done. The latter task will be accomplished using the following theorem.

Theorem 3.1. *Let $(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}) \in Gr\ S$ be fixed and introduce the mapping: $\Phi : \mathbb{R}^p \times \mathbb{R}^{3p} \to \mathbb{R}^p \times \mathbb{R}^{3p} \times \mathbb{R}^{3p}$, $(\alpha, \mathbf{y}) \mapsto \left(\boldsymbol{\alpha}, \mathbf{y}, -F(\boldsymbol{\alpha}, \mathbf{y})\right)^T$. Then the following hold:*

(i) The multifunction $M : \mathbb{R}^p \times \mathbb{R}^{3p} \times \mathbb{R}^{3p} \to \mathbb{R}^p \times \mathbb{R}^{3p}$, $\mathbf{p} \mapsto \{ (\boldsymbol{\alpha}, \mathbf{y}) \mid \mathbf{p} + \Phi(\boldsymbol{\alpha}, \mathbf{y}) \in Gr\ Q \}$ is calm at $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}})^T$.

(ii) For every $\mathbf{p}^* \in D^* S(\overline{\boldsymbol{\alpha}})(\nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))$ there exists a vector $\mathbf{v}^* \in \mathbb{R}^{3p}$ such that $(\mathbf{p}^*, \mathbf{v}^*)$ is a solution of the (limiting) adjoint GE:

$$\begin{pmatrix} \mathbf{p}^* \\ -\nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}) \end{pmatrix} \in \nabla F(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}})^T \mathbf{v}^* + D^* Q(\Phi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))(\mathbf{v}^*). \tag{AGE}$$

Proof. Part (i) was proved in [7, Lemma 8], whereas part (ii) follows from (i) and [9, Theorem 4.1]. For details see [7]. □

Note that due to Lipschitz continuity of S, (AGE) attains at least one solution $\mathbf{p}^*$ and at points $(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$, where Q is normally regular, i.e. $\hat{N}_{\mathrm{Gr}\ Q}(\Phi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}})) = N_{\mathrm{Gr}\ Q}(\Phi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))$, every solution $\mathbf{p}^*$ of (AGE) belongs to $D^* S(\overline{\boldsymbol{\alpha}})(\nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))$. In the nonregular case, however, the set of solutions of (AGE) is in general larger than $D^* S(\overline{\boldsymbol{\alpha}})(\nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))$ and so this procedure may lead to a subgradient, which lies outside of $(\overline{\partial} S(\overline{\boldsymbol{\alpha}}))^T \nabla_y J(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$. Nevertheless, numerical experience shows that this phenomenon occurs very rarely and typically does not negatively influence the behavior of bundle methods, which we use for the solution of ($\tilde{\mathbb{P}}$) (cf. [17]).

In the rest of this section we will devote our attention to the solution of (AGE), in particular to expression of the coderivative $D^* Q$ in terms of the problem data. To begin with, note that the components of Q are *decoupled* (this is a consequence of the assumed model of given friction), hence its coderivative can be computed componentwise:

$$D^* Q(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}, \bar{\mathbf{q}})(\mathbf{q}^*) = \begin{pmatrix} D^* Q_1(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}_1, \bar{\mathbf{q}}_1)(\mathbf{q}_1^*) \\ 0 \\ D^* N_{\mathbb{R}^p_+}(\bar{\mathbf{y}}_3, \bar{\mathbf{q}}_3)(\mathbf{q}_3^*) \end{pmatrix} \quad \forall \mathbf{q}^* \in \mathbb{R}^{3p}, \tag{3.3}$$

at any given point $(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{y}}, \bar{\mathbf{q}}) \in \text{Gr } Q$. The third component is standard and the exact formula for it may be found e.g. in [13, Lemma 2.2]. In order to deal with the first component, let us write the multifunction $Q_1 : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ as a composition of an outer multifunction Z_1 and an inner single-valued, smooth mapping Ψ:

$$Q_1(\boldsymbol{\alpha}, \mathbf{u}) = \begin{pmatrix} \omega_1(\boldsymbol{\alpha})\mathscr{F}(|u_1|)\partial|u_1| \\ \omega_2(\boldsymbol{\alpha})\mathscr{F}(|u_2|)\partial|u_2| \\ \vdots \\ \omega_p(\boldsymbol{\alpha})\mathscr{F}(|u_p|)\partial|u_p| \end{pmatrix} = (Z_1 \circ \Psi)(\boldsymbol{\alpha}, \mathbf{u}), \tag{3.4}$$

where

$$\Psi = (\Psi_1, \ldots, \Psi_p) : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^{3p}, \qquad \Psi_j(\boldsymbol{\alpha}, \mathbf{u}) := (\omega_j(\boldsymbol{\alpha}), u_j)^T,$$

and

$$Z_1 : \mathbb{R}^{3p} \to \mathbb{R}^p, \quad \mathbf{y} \mapsto (Z(\mathbf{y}_1), \ldots, Z(\mathbf{y}_p))^T,$$

with

$$Z : (0, \infty) \times \mathbb{R} \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_1 \mathscr{F}(|x_2|)\partial|x_2|.$$

Now the chain rule from [16, Theorem 10.40] allows us to compute the coderivative of the composite multifunction (3.4) as follows:

Theorem 3.2. *Let* $(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in Gr\ Q_1$ *be such that the following condition holds:*

$$Ker\, \nabla\Psi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}})^T \cap D^* Z_1(\Psi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(\mathbf{0}) = \{\mathbf{0}\}. \tag{3.5}$$

Then:

$$\begin{aligned} \forall \mathbf{q}^* \in \mathbb{R}^p : \quad D^* Q_1(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}, \bar{\mathbf{q}})(\mathbf{q}^*) &\subset \nabla\Psi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}})^T D^* Z_1(\Psi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(\mathbf{q}^*) \\ &= \nabla\Psi(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}})^T \begin{pmatrix} D^* Z(\Psi_1(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{q}_1)(\bar{q}_1^*) \\ D^* Z(\Psi_2(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{q}_2)(\bar{q}_2^*) \\ \vdots \\ D^* Z(\Psi_p(\overline{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{q}_p)(\bar{q}_p^*) \end{pmatrix}. \end{aligned} \tag{3.6}$$

By means of (3.3) and (3.6) we have reduced the computation of D^*Q to that of D^*Z. Due to the particularly simple structure of Z, this can be done relatively easily and has been investigated in detail in [7, Section 6.2]. We summarize these results in the next theorem.

Theorem 3.3. *Let $(\bar{x}_1, \bar{x}_2, \bar{z}) \in Gr\ Z$ be a given point and $z^* \in \mathbb{R}$ arbitrary. Then exactly one from the following cases holds true:*

(1) $\bar{x}_2 > 0$, *then:* $D^*Z(\bar{x}_1, \bar{x}_2, \bar{z})(z^*) = \{z^*\mathscr{F}(\bar{x}_2)\} \times D^*\mathscr{F}(\bar{x}_2)(\bar{x}_1 z^*)$;
(2) $\bar{x}_2 < 0$, *then:* $D^*Z(\bar{x}_1, \bar{x}_2, \bar{z})(z^*) = \{-z^*\mathscr{F}(-\bar{x}_2)\} \times \big(-D^*\mathscr{F}(-\bar{x}_2)(-\bar{x}_1 z^*)\big)$;
(3) $\bar{x}_2 = 0$, $|\bar{z}| < \bar{x}_1\mathscr{F}(0)$, *then:*

$$D^*Z(\bar{x}_1, 0, \bar{z})(z^*) = \begin{cases} \{0\} \times \mathbb{R}\ , & \text{if } z^* = 0, \\ \emptyset\ , & \text{otherwise}; \end{cases}$$

(4) $\bar{x}_2 = 0$, $\bar{z} = \bar{x}_1\mathscr{F}(0)$, *then:*

$$D^*Z(\bar{x}_1, 0, \bar{x}_1\mathscr{F}(0))(z^*) \begin{cases} \subset \{z^*\mathscr{F}(0)\} \times D^*\mathscr{F}(0)(\bar{x}_1 z^*), & \text{if } z^* > 0, \\ = \{z^*\mathscr{F}(0)\} \times (-\infty,\ \bar{x}_1 z^* D^+\mathscr{F}(0)], & \text{if } z^* < 0, \\ = \{0\} \times \mathbb{R}, & \text{if } z^* = 0, \end{cases}$$

where the symbol $D^+\mathscr{F}(0) := \limsup_{\eta \to 0_+} \frac{\mathscr{F}(\eta) - \mathscr{F}(0)}{\eta}$ *stands for the upper Dini derivative of* $\mathscr{F}$ *at* 0;
(5) $\bar{x}_2 = 0$, $\bar{z} = -\bar{x}_1\mathscr{F}(0)$, *then:*

$$D^*Z(\bar{x}_1, 0, -\bar{x}_1\mathscr{F}(0))(z^*) \begin{cases} = \{-z^*\mathscr{F}(0)\} \times [\bar{x}_1 z^* D^+\mathscr{F}(0),\ +\infty), & \text{if } z^* > 0, \\ \subset \{-z^*\mathscr{F}(0)\} \times \big(-D^*\mathscr{F}(0)(-\bar{x}_1 z^*)\big), & \text{if } z^* < 0, \\ = \{0\} \times \mathbb{R}, & \text{if } z^* = 0. \end{cases}$$

Using this result one may construct and solve the (AGE). Moreover, one has:

Corollary 3.4. *The condition (3.5) holds at each* $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in Gr\ Q_1$.

Proof. See [7, Corollary 2]. □

4 Numerical Results

The theoretical results of the previous sections will now be used for computation of model examples. We assume that the friction coefficient $\mathscr{F}$ is defined by

$$\mathscr{F}(t) = 0.25 \cdot \frac{1}{t^2 + 1} \quad \forall t \in \mathbb{R}_+, \tag{4.1}$$

and the a-priori given slip bound is $g = 150$. Further we assume that the cost functional J is continuously differentiable so that the composite map $\mathscr{J}$ is locally Lipschitzian. Therefore one can use the implicit programming approach [14] to solve the shape optimization problem ($\mathbb{P}$). For the minimization of $\mathscr{J}$ we used the Matlab implementation of the bundle trust method BT (see [17]). This method is very robust and was designed just for the minimization of nonsmooth functions. It requires in every step the value of the objective function and its arbitrary Clarke subgradient (for more details see [4]), i.e., for each admissible $\boldsymbol{\alpha}$ we have to be able to find a solution of the state problem $(\mathbf{u}, \lambda) = S(\boldsymbol{\alpha})$ and to compute one arbitrary Clarke subgradient of $\mathscr{J}$ at $\boldsymbol{\alpha}$. This issue was discussed in the preceding section.

Since the Signorini problem with given friction and a solution-dependent coefficient of friction can be equivalently formulated as a fixed-point problem (see [8]), the method of successive approximations will be used for its numerical solution. Each iterative step is represented by the Signorini problem with given friction and the given coefficient friction computed from the previous iteration.

These techniques were implemented and the following experiments were solved by MatSol library [10] developed in the Matlab environment.

For the solution of model examples, we slightly modify the set U^h_{ad}. The purpose of this modification is to decrease the number of control variables and, at the same time, to get a smooth shape of the contact boundary. Therefore, the boundary Γ_c will be modelled by Bézier functions of order d. The system of points $\{A_i\}_{i=0}^d$, where $A_i = (ih, \alpha_i), \alpha_i \in \mathbb{R}, i = 0, 1, \ldots, d, h = a/d$ defines the so-called control points of the Bézier function F_α of order d on $[0, a]$:

$$F_\alpha(x) = \sum_{i=0}^{d} \alpha^i \beta_d^i(x), \quad \beta_d^i(x) = \frac{1}{a^d} \binom{d}{i} x^i (a - x)^{d-i}, \; x \in [0, a].$$

Discretized shapes are determined by the vector $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_d)$, where α_i is the second component of A_i, $i = 0, \ldots, d$. The end points of F_α coincide with the first and last control point. The graph of F_α itself lies in the convex envelope of the control points. This means that any upper and lower bounds imposed on the components of $\boldsymbol{\alpha}$ are automatically satisfied for F_α, too.

The new shape optimization problem using this type of the design variables is defined as follows:

$$\left.\begin{array}{ll} \text{minimize} & J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha})) \\ \text{subj. to} & \boldsymbol{\alpha} \in \mathscr{U}, \end{array}\right\} \qquad (\mathbb{P}_B)$$

where

$$\begin{aligned} \mathscr{U} = \{\boldsymbol{\alpha} \in \mathbb{R}^{d+1} \mid 0 \le \alpha^i \le C_0,\ i = 0, 1, \dots, d; \\ |\alpha^{i+1} - \alpha^i| \le C_1 h,\ i = 0, 1, \dots, d-1; \\ |\alpha^{i+1} - 2\alpha^i + \alpha^{i-1}| \le C_2 h^2,\ i = 1, 2, \dots, d-1; \\ C_{31} \le \text{meas}\, \Omega(\boldsymbol{\alpha}) \le C_{32}\} \end{aligned}$$

and $C_0, C_1, C_2, C_{31}, C_{32}$ are given positive constants. The first $d+1$ box constraints guarantee that $|F_\alpha(x)| \le C_0\ \forall x \in [0, a]$. The second and the third set of the constraints take care of the smoothness of the optimal shape. It is well known that if the control points satisfy these two conditions, then $|F'_\alpha(x)| \le C_1$ and $|F''_\alpha(x)| \le C_2\ \forall x \in [0, a]$. The last constraint is added to control the volume of the domain. Unlike the constant volume constraint considered in the theoretical part of this paper, this time we use the inequality constraints for the volume of $\Omega(\boldsymbol{\alpha})$. The last constraint has a physical meaning of preserving the weight of the structure in prescribed limits.

We will present results of two examples solved by the mentioned implicit programming technique combined with the BT code. In both examples we use the same data and change only the cost function J. The shape of the elastic body $\Omega(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathscr{U}$, is defined through a Bézier function F_α as follows (cf. Fig. 1):

$$\Omega(\boldsymbol{\alpha}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, a),\ F_\alpha(x_1) < x_2 < b\}.$$

From Fig. 1 one also sees the distribution of external pressures on the boundary Γ_P, given as $\mathbf{P}^1 = (0; -60\text{ MPa})$ on $(0, 1.8) \times \{1\}$ and zero on $(1.8, 2) \times \{1\}$, while $\mathbf{P}^2 = (50\text{ MPa}; 30\text{ MPa})$ on $\{2\} \times (0, 1)$. Further, Γ_u is the part of the boundary where the zero displacements are prescribed.

The set of the admissible designs $\mathscr{U}$ is specified as follows: $a = 2, b = 1$ and $C_0 = 0.75, C_1 = 0.85, C_2 = 10, C_{31} = 1.88, C_{32} = 1.95$. In both examples

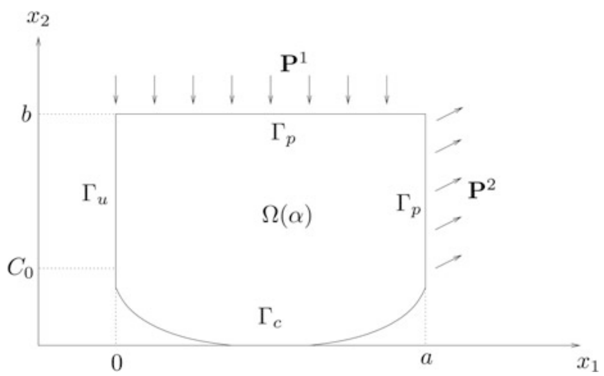

Fig. 1 The elastic body and applied loads

the Young modulus $E = 1$ GPa and the Poisson constant $\sigma = 0.3$ are used. The state problems on $\Omega(\boldsymbol{\alpha})$ are discretized by isoparametric quadrilateral elements of Lagrange type. The total number of nodes (vertices of quadrilaterals) is 1,800 for any $\boldsymbol{\alpha} \in \mathscr{U}$. The dimension of the control vector $\boldsymbol{\alpha}$, generating the Bézier function and defining $\Omega(\boldsymbol{\alpha})$, is 20.

Example 1. In the first example we try to smooth down peaks of the normal contact stress distribution. To this aim, one should minimize the max norm of the discrete normal contact stress $\boldsymbol{\lambda}$. The objective function $\mathscr{J}$, however, must be continuously differentiable, so we will use (p power of) p-norm of vectors with $p = 4$. The shape optimization problem then reads as follows:

$$\begin{array}{ll} \text{minimize} & \|\lambda\|_4^4 \\ \text{subj. to} & \boldsymbol{\alpha} \in \mathscr{U}. \end{array}$$

In Fig. 2 we depict the initial shape and the distribution of the von Mises stress in the loaded body. Figure 3 shows the optimal shape and the von Mises stress in

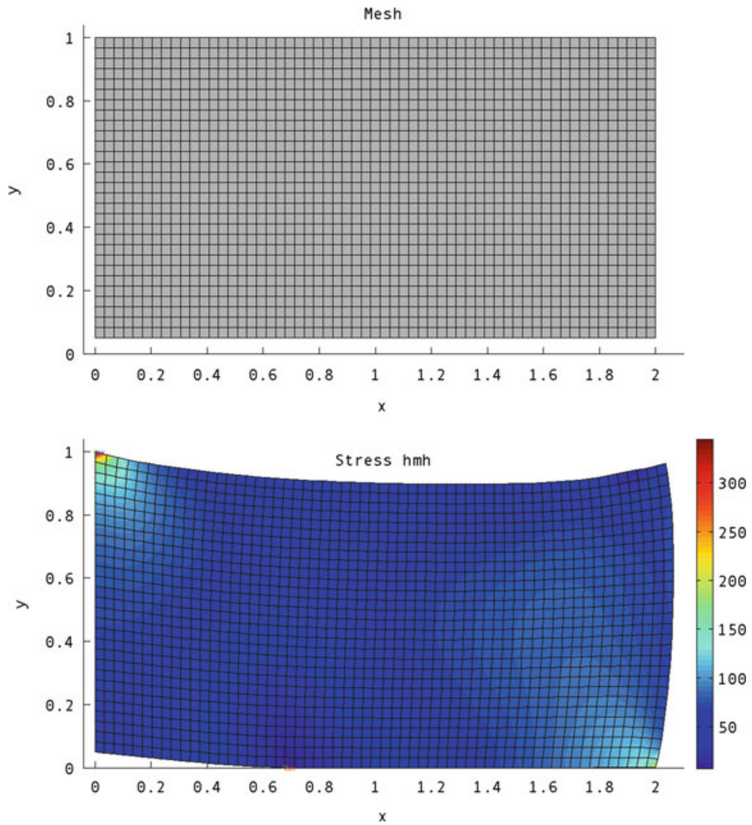

Fig. 2 Example 1, initial design

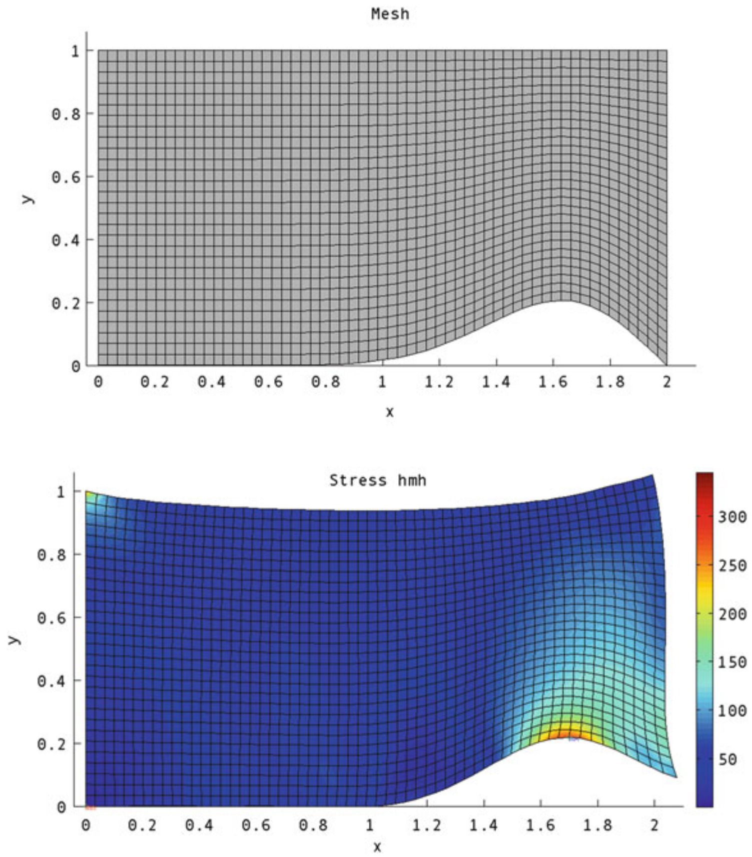

Fig. 3 Example 1, optimal design

the deformed optimal body. Finally, Fig. 4 compares the contact normal stresses for the initial $\Omega(\boldsymbol{\alpha}_0)$ (left) and optimal $\Omega(\boldsymbol{\alpha}_{\text{opt}})$ (right) shape, respectively. The obtained optimal value of the cost functional $\mathscr{J}(\boldsymbol{\alpha}_{\text{opt}}) = 1.9623 \cdot 10^8$ compared to $\mathscr{J}(\boldsymbol{\alpha}_0) = 6.0151 \cdot 10^8$ represents a decrease by 67 %. The decrease of the peak stress is also quite significant.

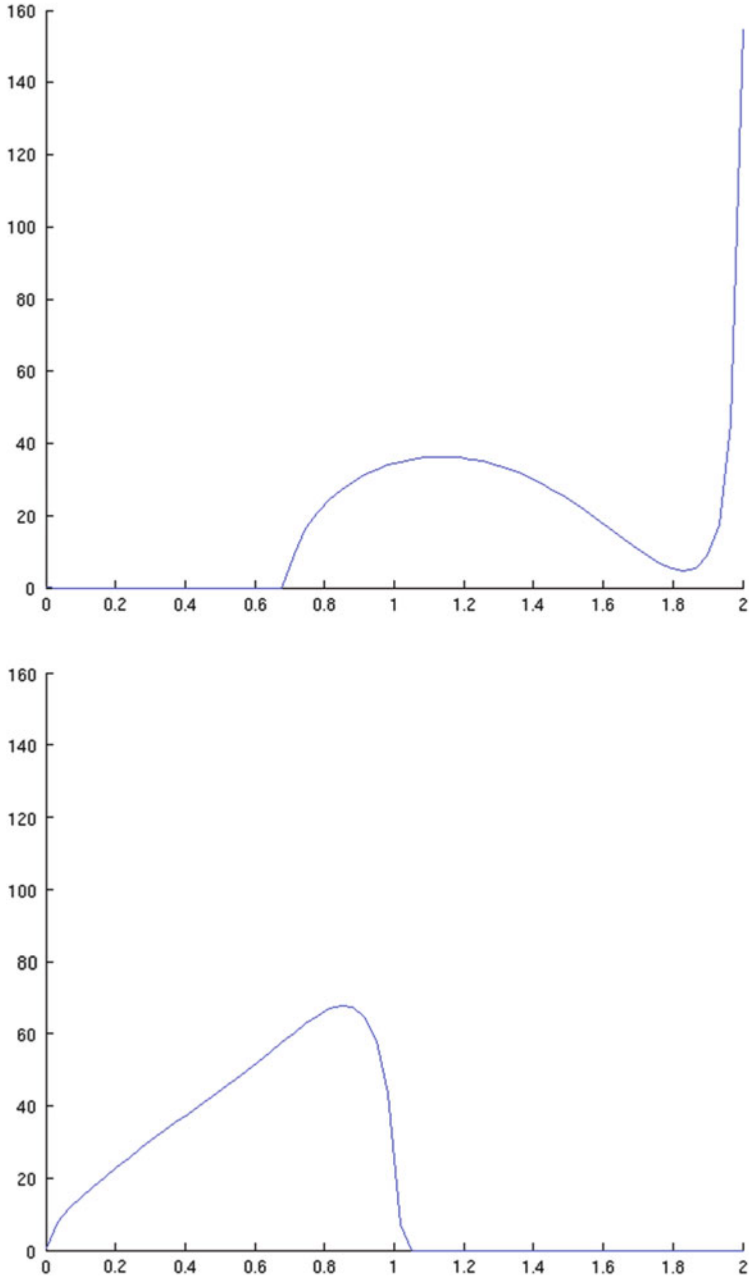

Fig. 4 Example 1, normal stress for initial (*left*) and optimal (*right*) design

Example 2. Here we try to identify the contact normal stress $\boldsymbol{\lambda}$ with a prescribed value $\bar{\boldsymbol{\lambda}}$. The shape optimization problem can be written as

$$\begin{aligned} &\text{minimize} \quad \|\bar{\lambda} - \lambda\|_2^2 \\ &\text{subj. to} \quad \boldsymbol{\alpha} \in \mathscr{U}. \end{aligned}$$

This vector was chosen to model a function, depicted in Fig. 7 by the dotted line.

The initial design and its deformation with the distribution of the von Mises stress is presented in Fig. 5, while Fig. 6 shows the optimal design before and

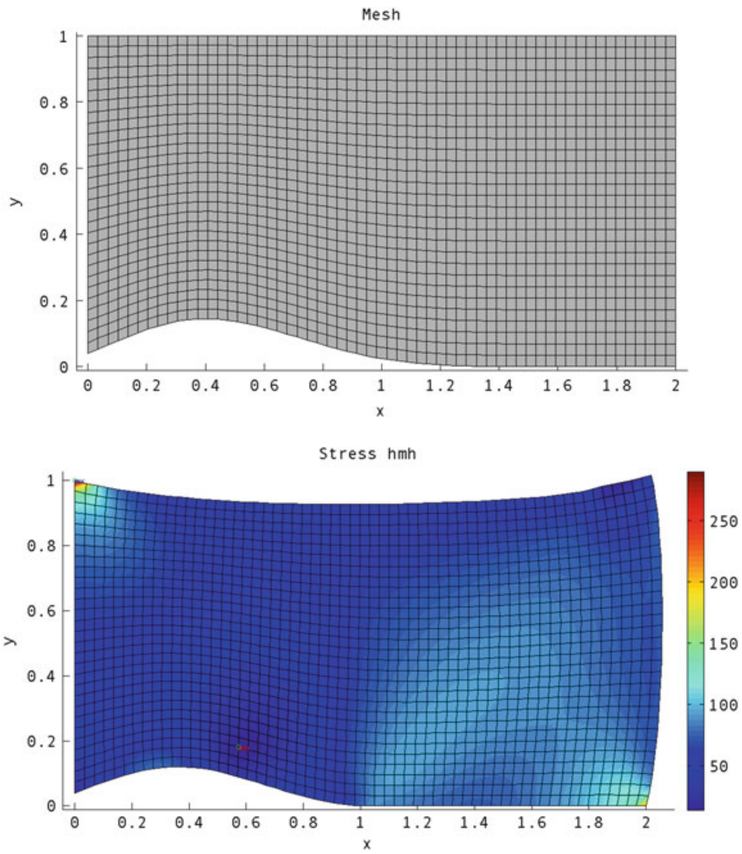

Fig. 5 Example 2, initial design

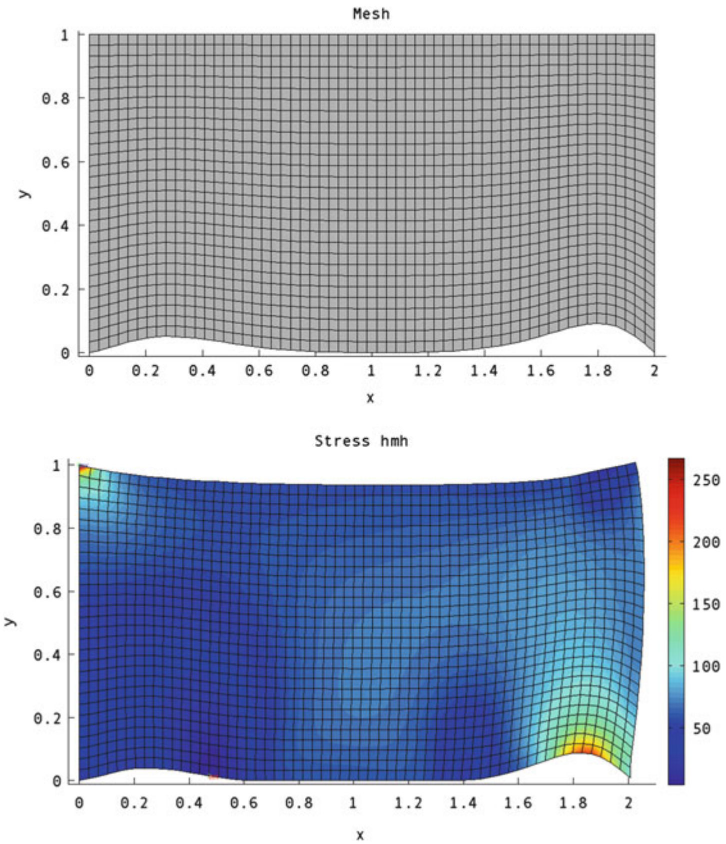

Fig. 6 Example 2, optimal design

after deformation. Finally, Fig. 7 compares the contact normal stresses with the prescribed values. While the initial contact stresses are far from the prescribed values, the stresses for the optimal shape follow very closely $\overline{\boldsymbol{\lambda}}$. Note that during the optimization process the initial value $\mathscr{J}(\boldsymbol{\alpha}_0) = 5.910 \cdot 10^4$ of the cost functional dropped by two orders of magnitude to $\mathscr{J}(\boldsymbol{\alpha}_{\text{opt}}) = 9.1457 \cdot 10^2$.

In order to emphasize the importance of proper modelling of contact problems, let us compute the same example, now with a coefficient of friction which *does not depend* on the solution. In particular, we set $\mathscr{F}(t) = 0.25$ for every $t \geq 0$, but keep all other parameters of Example 2 unchanged. Starting from the same

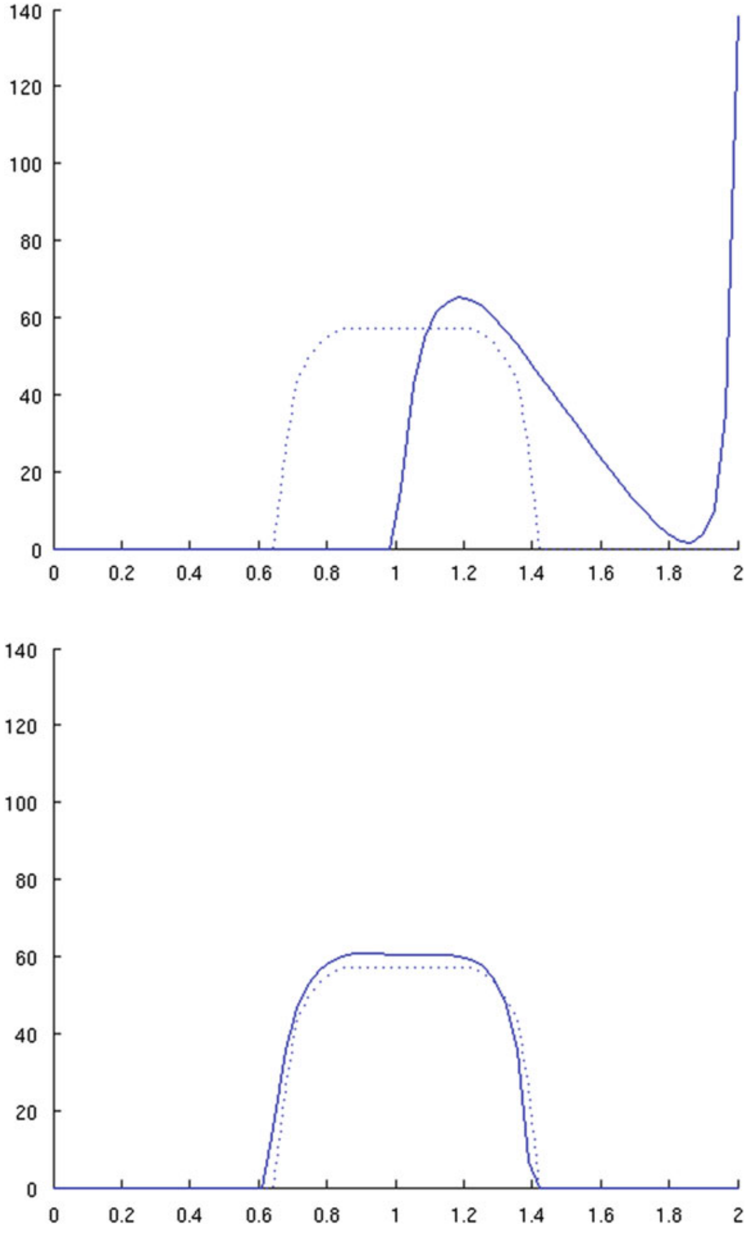

Fig. 7 Example 2, normal stress for initial (*left*) and optimal (*right*) design

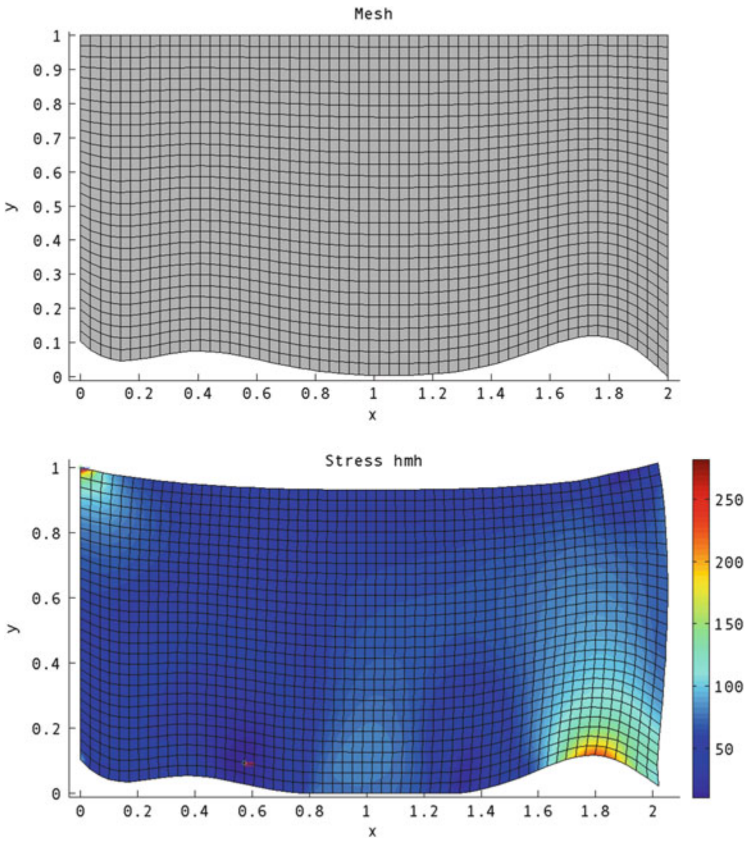

Fig. 8 Example 2 with $\mathscr{F} = \text{const}$; optimal design $\Omega(\overline{\boldsymbol{\alpha}}_{\text{opt}})$

initial configuration $\Omega(\boldsymbol{\alpha}_0)$ as in Example 2, the algorithm converges to a solution $\Omega(\overline{\boldsymbol{\alpha}}_{\text{opt}})$—see Fig. 8.

Then we solve the original contact problem with the coefficient of friction given by (4.1) on $\Omega(\overline{\boldsymbol{\alpha}}_{\text{opt}})$ and show the distribution of the normal stress along $\Gamma_c(\overline{\boldsymbol{\alpha}}_{\text{opt}})$ in Fig. 9. Comparing Fig. 9 with Fig. 7, one can see that the "approximate optimal design" $\Omega(\overline{\boldsymbol{\alpha}}_{\text{opt}})$, obtained by replacing the original state problem with a simpler one, is not optimal at all.

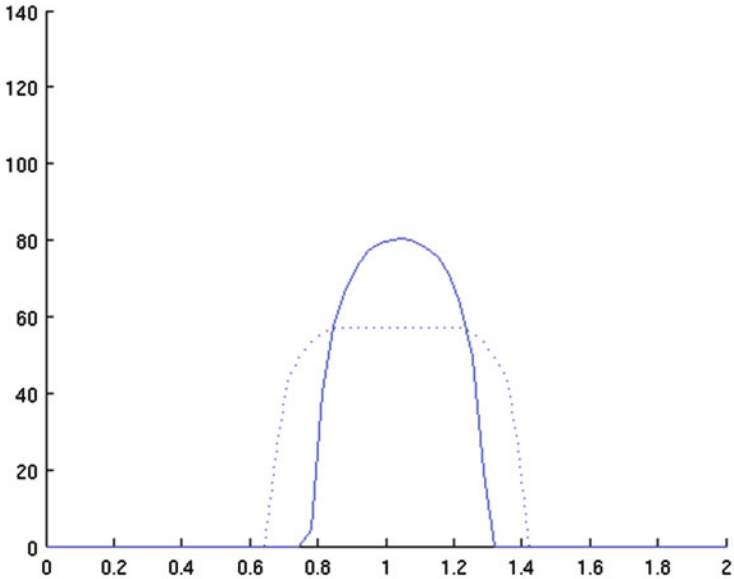

Fig. 9 Example 2; normal stress distribution on $\Omega(\overline{\boldsymbol{\alpha}}_{\mathrm{opt}})$

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