On Interpretation of T-product Extensions of Possibility Distributions

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Abstract

T-product extensions (where T is a continuous t-norm) of a system of low-dimensional possibility distributions form an important class of solutions of the so-called possibilistic marginal problem. Nevertheless, they can differ from each other, e.g., from the viewpoint of inference. Therefore the need for their interpretation is obvious. To find it, we identify any possibility distribution with a set of probability distributions dominated by it and find a probability interpretation of models based on Gödel's, product and Łukasiewicz's t-norms.

Keywords: possibility theory, extension, triangular norm, conditional independence, sets of probability distributions

1 Introduction

The marginal problem — which addresses the question of whether or not a common extension exists for a given set of marginal distributions — is one of the most challenging types of problems in probability theory. The challenges lie not only in a wide range of relevant theoretical problems, but also in its applicability to various problems in statistics.

If an extension exists, it is usually not unique, i.e., the problem has an infinite number of solutions. Therefore the problem of existence of an extension is usually solved together with the problem of choosing one of them, in a sense an optimal representative from within the set of all possible solutions. Among them, the so-called product extensions, adopting the assumption of (conditional) independence, offer a closed form of extension.

Nevertheless, in the recent decades new mathematical tools have emerged as alternatives to probability theory. They are used in situations whose nature of uncertainty does not meet the requirements of probability theory, or those in which probabilistic criteria are too strict (e.g., additivity). On the other hand, probability theory has always served as a source of inspiration for the development of these non-probabilistic calculi and they have been continually confronted with probability theory and mathematical statistics from various points of view.

In this paper we will deal with T-product extensions of systems of possibility distributions, which are, in a sense, counterparts of the above-mentioned product extensions of probability distributions. Nevertheless, these models can differ from each other from the viewpoint of the inference, therefore the need for their interpretation is obvious. For this purpose we will identify any possibility distribution with a set of probability distributions dominated by it and obtain interpretations of T-product extensions for Gödel's and product, and partially also for Łukasiewicz's, t-norms.

The paper is organised as follows: after a brief overview of basic concepts (Section 2), in Section 3 we will recall a possibilistic marginal problem and its T-product solutions. In Section 4 we summarize a generalisation of T-product extensions, so-called compositional models, and in Section 5 we present the problem of inference from these models. Finally, Section 6 is devoted to probabilistic interpretation of T-product models based on specific continuous t-norms.

2 Basic Concepts

The purpose of this section is to give, as briefly as possible, an overview of basic notions of De Cooman's measure-theoretical approach to possibility theory [7], necessary for understanding the paper. Special attention will be paid to conditioning, independence and conditional independence [8, 9, 18, 19]. We will start with the notion of a triangular norm, since it is the crucial concept of this paper.

2.1 Triangular Norms

A *triangular norm* (or a *t-norm*) T is a binary operator on $[0,1]$ (i.e., $T : [0,1]^2 \to [0,1]$) satisfying the following four conditions:

(i) *boundary condition:* for any $a \in [0,1]$

$$T(1,a) = a;$$

(ii) *isotonicity:* for any $a_1, a_2, b \in [0,1]$ such that $a_1 \leq a_2$

$$T(a_1,b) \leq T(a_2,b);$$

(iii) *associativity:* for any $a, b, c \in [0,1]$

$$T(T(a,b),c) = T(a,T(b,c)),$$

(iv) *commutativity:* for any $a, b \in [0,1]$

$$T(a,b) = T(b,a).$$

A t-norm T is called *continuous* if T is a continuous function. Within this paper, we will only deal with continuous t-norms, namely the following three:

(i) *Gödel's t-norm:* $T_G(a,b) = \min(a,b)$;

(ii) *product t-norm:* $T_\Pi(a,b) = a \cdot b$;

(iii) *Łukasiewicz's t-norm:* $T_L(a,b) = \max(0, a+b-1)$.

Let us take $a, b \in [0,1]$ and let T be a t-norm. We will call an element $x \in [0,1]$ T*-inverse* of a w.r.t. b if

$$T(x,a) = T(a,x) = b. \tag{1}$$

It is obvious that if $a < b$ then the equation (1) admits no solution, i.e., there are no T-inverses of a w.r.t. b. On the other hand, if a T-inverse exists, it need not be unique. Least specific solution of (1) is offered by a T-residual.

Let us take $a, b \in [0,1]$. The *T-residual* $b \triangle_T a$ of b by a is defined as

$$b \triangle_T a = \sup\{x \in [0,1] : T(x,a) \leq b\}.$$

The following lemma, taken from [7], expresses the relationship between T-inverses and T-residuals for continuous t-norms.

Lemma 1 *Let T be a continuous t-norm and $a, b \in [0,1]$. If the equation $T(x,a) = b$ in x admits a solution, then $b \triangle_T a$ is its greatest solution.*

2.2 Possibility Measures, Distributions and Variables

Let N be a finite index set and $\{\mathbf{X}_i\}_{i \in N}$ be a system of finite sets. We will deal with the Cartesian-product space

$$\mathbf{X} = \times_{i \in N} \mathbf{X}_i,$$

called a *universe of discourse* and its subspaces

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i$$

for $K \subset N$.

A *possibility measure* Π is a mapping from the power set $\mathcal{P}(\mathbf{X})$ of $\mathbf{X}$ to the real unit interval $[0,1]$ satisfying the following two requirements:

(i) $\Pi(\emptyset) = 0$;

(ii) for any family $\{A_j, j \in J\}$ of elements of $\mathcal{P}(\mathbf{X})$

$$\Pi(\bigcup_{j \in J} A_j) = \max_{j \in J} \Pi(A_j)^1.$$

Π is called *normal* if $\Pi(\mathbf{X}) = 1$. Within this paper we will always assume that Π is normal. The importance of this assumption will be apparent namely in Section 6.

For any Π there exists a mapping $\pi : \mathbf{X} \to [0,1]$ (a possibilistic counterpart of a density function in probability theory), called a *distribution* of Π, such that for any $A \in \mathcal{P}(\mathbf{X})$, $\Pi(A) = \max_{x \in A} \pi(x)$. It is evident that (in the finite case) Π is normal iff there exists at least one $x \in \mathbf{X}$ such that

[1] max must be substituted by sup if $\mathbf{X}$ is not finite.

$\pi(x) = 1$. In the rest of this paper we will deal with distributions rather than with measures.

Let us consider an arbitrary possibility distribution π defined on a universe of discourse $\mathbf{X}$. The *marginal possibility distribution* on $\mathbf{X}_K$ $(K \subset N)$ is defined by the expression

$$\pi_{X_K}(x_K) = \max_{z \in \mathbf{X}_{N \setminus K}} \pi(x_K, z) \tag{2}$$

for any $x_K \in \mathbf{X}_K$ (notice that (x_K, z) is an element of $\mathbf{X}_N$).

Let $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$ denote two finite universes of discourse provided with possibility distributions π_1 and π_2, respectively. Possibility distributions π_1 and π_2 are *projective* if they coincide on overlapping subspaces, i.e. if

$$\pi_1(x_{K_1 \cap K_2}) = \pi_2(x_{K_1 \cap K_2}). \tag{3}$$

It is evident, that if $K_1 \cap K_2 = \emptyset$, then any possibility distributions π_1 and π_2 defined on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, are projective.

Let, furthermore $K_1 \cap K_2 = \emptyset$. The possibility measure Π on $\mathbf{X}_{K_1} \times \mathbf{X}_{K_2}$ is called the *T-product possibility measure* if, for the corresponding possibility distributions for any $(x_{K_1}, x_{K_2}) \in \mathbf{X}_{K_1} \times \mathbf{X}_{K_2}$

$$\pi(x_{K_1}, x_{K_2}) = T(\pi_1(x_{K_1}), \pi_2(x_{K_2})). \tag{4}$$

In Subsection 3.2 will generalize this concept to projective distributions defined on overlapping subspaces (in Subsection 4.1 even the projectivity requirement will be relaxed).

Let us consider a finite *basic space* Ω, provided by a possibility measure Π_Ω with distribution π_Ω. A mapping $X : \Omega \longrightarrow \mathbf{X}$ is called a *(possibilistic) variable* [2] in $\mathbf{X}$. The *induced* (or *transformed*) possibility measure Π_X on $\mathbf{X}$ is determined by

$$\Pi_X(A) = \Pi_\Omega(X^{-1}(A))$$

for any $A \in \mathcal{P}(\mathbf{X})$ and its distribution is

$$\pi_X(x) = \max_{\omega : X(\omega) = x} \pi_\Omega(\omega)$$

for any $x \in \mathbf{X}$.

[2] This definition corresponds to that introduced by De Cooman in [7], but it is simplified due to the assumption that possibility measures are defined on power sets instead of general ample fields.

2.3 Conditioning

Let T be a continuous t-norm on $[0,1]$ and $\mathcal{G}(\Omega) = \{h : \Omega \to [0,1]\}$. For any possibility measure Π on $\mathbf{X}$ with distribution π, we define in accordance with [7] the following binary relation $\stackrel{(\Pi,T)}{=}$ on $\mathcal{G}(\mathbf{X})$: for h_1 and h_2 in $\mathcal{G}(\mathbf{X})$ we say that h_1 and h_2 are (Π, T)-*equal almost everywhere* (and write $h_1 \stackrel{(\Pi,T)}{=} h_2$) if, for any $x \in X$,

$$T(h_1(x), \pi(x)) = T(h_2(x), \pi(x)).$$

This notion is very important for the definition of a *conditional possibility distribution* $\pi_{X_K|_T X_L}$ $(K \cap L = \emptyset)$ which is defined (in accordance with [8]) as *any* solution of the equation

$$\pi_{X_K X_L}(x_K, x_L) = T(\pi_{X_L}(x_L), \pi_{X_K|_T X_L}(x_K|_T x_L)), \tag{5}$$

for any $(x_K, x_L) \in \mathbf{X}_K \times \mathbf{X}_L$. Continuity of a t-norm T guarantees the existence of a solution of this equation. This solution is not unique (in general), but the ambiguity vanishes when almost-everywhere equality is considered. We are able to obtain a representative of these conditional possibility distributions (if T is a continuous t-norm) by taking the residual $\pi_{X_K X_L}(x_K, \cdot) \triangle_T \pi_{X_L}(\cdot)$, as

$$\pi_{X_K|_T X_L}(x_K|_T \cdot) \stackrel{(\Pi_{X_L},T)}{=} \pi_{X_K X_L}(x_K, \cdot) \triangle_T \pi_{X_L}(\cdot), \tag{6}$$

i.e., the maximal solution of the equation (5) (cf. Lemma 1).

Let us note that the right-hand side of (6) is the upper envelope of the set of all solutions of equation (5).

The importance of this way of conditioning from the theoretical viewpoint is obvious: as mentioned in [8, 18], it brings a unifying view on several conditioning rules, and is also used in the definition of conditional T-independence (see also below). On the other hand, its practical meaning is not so substantial. Although De Cooman [8] claims that conditional distributions are never used *per se*, the notion of almost everywhere equality does not solve the non-uniqueness problem, while the choice of T-residual does; for more details see [20].

Let us also note that the problem of conditioning in possibility theory was also solved (from a different viewpoint) in [6].

2.4 Independence

Two variables X_K and X_L ($K \cap L = \emptyset$) (taking their values in $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively) are *T-independent* [9] if, for any $F_{X_K} \in X_K^{-1}(\mathcal{P}(\mathbf{X}_K))$, $F_{X_L} \in X_L^{-1}(\mathcal{P}(\mathbf{X_L}))$,

$$\begin{aligned}
\Pi(F_{X_K} \cap F_{X_L}) &= T(\Pi(F_{X_K}), \Pi(F_{X_L})), \\
\Pi(F_{X_K} \cap F_{X_L}^C) &= T(\Pi(F_{X_K}), \Pi(F_{X_L}^C)), \\
\Pi(F_{X_K}^C \cap F_{X_L}) &= T(\Pi(F_{X_K}^C), \Pi(F_{X_L})), \\
\Pi(F_{X_K}^C \cap F_{X_L}^C) &= T(\Pi(F_{X_K}^C), \Pi(F_{X_L}^C)),
\end{aligned}$$

where A^C denotes the complement of A.

From this definition it immediately follows that the independence concept is parameterised by T. This fact is reflected in some definitions and assertions that follow and, above all, in the main results of this paper.

The following theorem, an immediate consequence of Proposition 2.6. of the above-mentioned paper [9], led to the definition of conditional possibilistic T-independence [17] recalled bellow.

Theorem 1 *Let us assume that a t-norm T is continuous and $K \cap L = \emptyset$. Then the following propositions are equivalent.*

(i) X_K *and* X_L *are T-independent.*

(ii) For any $x_K \in \mathbf{X}_K$ *and* $x_L \in \mathbf{X}_L$

$$\pi_{X_{K\cup L}}(x_{K\cup L}) = T(\pi_{X_K}(x_K), \pi_{X_L}(x_L)).$$

(iii) For any $x_K \in \mathbf{X}_K$ *and* $x_L \in \mathbf{X}_L$

$$\begin{aligned}
T(\pi_{X_K}(x_K), \pi_{X_L}(x_L)) &= T(\pi_{X_K|_T X_L}(x_K|_T x_L), \pi_{X_L}(x_L)) \\
&= T(\pi_{X_L|_T X_K}(x_L|_T x_K), \pi_{X_K}(x_K)).
\end{aligned}$$

Given a possibility measure Π on $\mathbf{X}_K \times \mathbf{X}_L \times \mathbf{X}_M$ (K, L, M mutually disjoint) with the respective distribution$\pi(x_K, x_L, x_M)$, variables X_K and X_L are *conditionally T-independent* [3] given X_M (with respect to π) if, for any pair$(x_K, x_L) \in \mathbf{X}_K \times \mathbf{X}_L$,

$$\pi_{X_{K\cup L}|_T X_M}(x_{K\cup L}|_T \cdot) \overset{(\Pi_{X_M}, T)}{=} T(\pi_{X_K|_T X_M}(x_K|_T \cdot), \pi_{X_L|_T X_M}(x_L|_T \cdot)). \quad (7)$$

[3]Let us note that a similar definition of conditional independence can be found in [13].

Let us stress once more that we do not deal with the pointwise equality but with the *almost everywhere equality*, in contrast to the conditional noninteractivity introduced by Fonck [12]. Clearly, this is not the only way how to define conditional independence in possibility theory, other approaches can be found e.g. in [2, 3] or in [4, 11].

The following theorem, proven in [18], is a "conditional counterpart" of Theorem 1.

Theorem 2 *For a continuous t-norm T and K, L, M mutually disjoint subsets of N, the following propositions are equivalent:*

(i) *X_K and X_L are T-independent given X_M with respect to π.*

(ii) *For any $x_K \in \mathbf{X}_K$, $y \in \mathbf{X}_L$ and $x_M \in \mathbf{X}_M$*

$$\pi_{X_K|_T X_{L\cup M}}(x_K|_T x_{L\cup M}) \stackrel{(\Pi_{X_{L\cup M}}, T)}{=} \pi_{X_K|_T X_M}(x_K|_T x_M). \tag{8}$$

3 Possibilistic Marginal Problem

In this section we recall the definition of the possibilistic marginal problem, a necessary condition for the existence of its solution, sets of all solutions and, finally, T-product extensions [20].

3.1 Definition

Using the procedure of marginalisation (2) we can always uniquely restrict a possibility distribution π defined on $\mathbf{X}$ to the distribution π_K defined on $\mathbf{X}_K$ for $K \subset N$ (for $K = \emptyset$ let us set $\pi_K \equiv 1$). However, the opposite process, the procedure of an *extension* of a system of distributions π_{K_i}, $i = 1, \ldots, m$ defined on $\mathbf{X}_{K_i}$ to a distribution π_K on $\mathbf{X}_K$ ($K = K_1 \cup \cdots \cup K_m$), is much harder.

The possibilistic marginal problem can be (analogous to probability theory) understood as follows: Let us assume that $\mathbf{X}_i$, $i \in N$, $1 \leq |N| < \infty$ are finite universes of discourse, $\mathcal{K}$ is a system of nonempty subsets of N and

$$\mathcal{S} = \{\pi_K, K \in \mathcal{K}\} \tag{9}$$

is a family of possibility distributions, where each π_K is a distribution on a product space $\mathbf{X}_K$.

The problem we are interested in is the existence of an *extension*, i.e., a distribution π on $\mathbf{X}$ whose marginals are distributions from $\mathcal{S}$; or, more generally, the set

$$\mathcal{P} = \{\pi(x) : \pi(x_K) = \pi_K(x_K), K \in \mathcal{K}\} \tag{10}$$

is of interest.[4]

Let us note that possibilistic marginal problem can also be viewed as a special case of checking coherence of possibility assessments, see [1], where a different characterization of this problem can be found.

It has been shown in [20] that we will not be able to find any distribution with prescribed marginals if they do not satisfy projectivity condition (3), nevertheless this condition is not sufficient, as also shown in [20].

Within the probabilistic framework, projectivity is a necessary condition for the existence of an extension, too, and becomes a sufficient condition if the index sets of the marginals can be ordered in such a way that it satisfies a special property called the running intersection property (see, e.g., [15]). At the end of the next section we will recall this notion and present an analogous result within the possibilistic framework.

3.2 T-product Extensions

If a solution of a possibilistic marginal problem exists, it is (usually) not unique. This fact is fully analogous to the probabilistic framework. However, contrary to the probabilistic marginal problem, the set of extensions of a set of possibility distributions is closed under maximisation, but (generally) not convex, as proven in [20]. It is evident that it is difficult (even more so than within the probabilistic framework) to handle the whole set of extensions and therefore an additional requirement is necessary to enable us to choose one representative of this set. The most natural requirement seems to be that of (conditional) T-independence.

There exists a special class of solutions to a marginal problem, namely the class of T-product distributions, defined in Section 2.2. If K_1 and K_2 are disjoint, the resulting distribution is just a T-productof the given distributions, i.e.,

$$\tilde{\pi}_T(x_{K_1 \cup K_2}) = \tilde{\pi}(x_{K_1}, x_{K_2}) = T(\pi_1(x_{K_1}), \pi_2(x_{K_2})). \tag{11}$$

[4]Let us stress that the introduced problem is different from those solved by De Campos and Huete in [2, 3]; the reader is referred to [20] for more details concerning this difference.

It follows from Theorem 1 that the equality (11) holds iff X_{K_1} and X_{K_2} are T-independent, therefore we obtain different T-product distributions for different t-norms.

The generalisation of a T-product distribution to a general set of marginal distributions with pairwise disjoint index sets is straightforward.

If the index sets K_1 and K_2 are not disjoint, the situation is somewhat more complicated. Let us assume π_1 and π_2 to be projective distributions of X_{K_1} and X_{K_2}, respectively. Then the *T-product extension* of these distributions can be defined by the equality

$$\tilde{\pi}_T(x_{K_1 \cup K_2}) = T(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})). \qquad (12)$$

Clearly, T-product extensions of projective possibility distributions are a generalization of T-product distributions as (12) collapses to (11) if $K_1 \cap K_2 = \emptyset$.

The following theorem, proven in [20] and expressing the relationship between T-product extensions and conditional independence, is of great importance from the viewpoint of this paper.

Theorem 3 *Let T be a continuous t-norm and π_1 and π_2 be projective possibility distributions of X_{K_1} and X_{K_2}, respectively. Then the distribution $\tilde{\pi}_T$ of $X_{K_1 \cup K_2}$*

$$\begin{aligned} \tilde{\pi}_T(x_{K_1 \cup K_2}) &= T(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})) \qquad (13) \\ &= T(\pi_1(x_{K_1}) \triangle_T \pi_1(x_{K_1 \cap K_2}), \pi_2(x_{K_2})), \end{aligned}$$

if and only if $X_{K_1 \setminus K_2}$ and $X_{K_2 \setminus K_1}$ are conditionally T-independent given $X_{K_1 \cap K_2}$.

As the residuals are upper envelopes of sets of all solutions of the equation (5) (cf. Subsection 2.3), this result can be reformulated also in terms of upper envelopes.

A generalisation of this approach to a more general system $\mathcal{S}$ of marginal possibility distributions will be at the centre of our attention in the next section.

4 Compositional Models

This section is devoted to a twofold generalisation of the above-mentioned ideas to general sets of (not necessarily projective) possibility distributions.

It is based on operators of composition of possibility distributions introduced within the probabilistic framework in [14] and within the possibilistic framework in [16].

4.1 Operators of Composition

Considering a continuous t-norm T, two (not necessarily disjoint) subsets K_1, K_2 of N and two normal possibility distributions [5] $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$ of variables X_{K_1} and X_{K_2}, respectively, we define the *operator of right composition* of these possibility distributions by the expression

$$\pi_1(x_{K_1}) \rhd_T \pi_2(x_{K_2}) \quad = \quad T\left(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})\right),$$

and analogously the *operator of left composition* by the expression

$$\pi_1(x_{K_1}) \lhd_T \pi_2(x_{K_2}) \quad = \quad T\left(\pi_1(x_{K_1}) \triangle_T \pi_1(x_{K_1 \cap K_2}), \pi_2(x_{K_2})\right).$$

It is evident that both $\pi_1 \rhd_T \pi_2$ and $\pi_1 \lhd_T \pi_2$ are (generally different from each other) possibility distributions of variables $\{X_i\}_{i \in K_1 \cup K_2}$.

Now, we will present a lemma proven in [16], expressing basic properties of these operators. This lemma also reveals that the operators of composition are generalisations of T-product extensions in such a sense, that they are able to produce "extensions" of non-projective distributions.

Lemma 2 *Let T be a continuous t-norm and $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$ be two distributions. Then*

(i)

$$(\pi_1 \rhd_T \pi_2)(x_{K_1}) = \pi_1(x_{K_1}),$$

(ii)

$$(\pi_1 \lhd_T \pi_2)(x_{K_2}) = \pi_2(x_{K_2}),$$

(iii)

$$(\pi_1 \rhd_T \pi_2)(x_{K_1 \cup K_2}) = (\pi_1 \lhd_T \pi_2)(x_{K_1 \cup K_2})$$

iff π_1 and π_2 are projective.

[5] Let us stress that for the definition of these operators we do not require projectivity of distributions π_1 and π_2.

4.2 Generating Sequences

In this subsection we will show how to apply the operators iteratively. Consider a sequence of distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_m(x_{K_m})$ and the expression

$$\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright_T \pi_m.$$

Before beginning a discussion of its properties, we have to explain how to interpret it. Although we did not mention it explicitly, the operator $\triangleright_T$ (as well as $\triangleleft_T$) is neither commutative nor associative.[6] Therefore, generally

$$(\pi_1 \triangleright_T \pi_2) \triangleright_T \pi_3 \neq \pi_1 \triangleright_T (\pi_2 \triangleright_T \pi_3).$$

For this reason, let us note that in the part that follows, we always apply the operators from left to right, i.e.,

$$\pi_1 \triangleright_T \pi_2 \triangleright_T \pi_3 \triangleright_T \ldots \triangleright_T \pi_m \quad = \quad (\ldots((\pi_1 \triangleright_T \pi_2) \triangleright_T \pi_3 \triangleright_T) \ldots) \triangleright_T \pi_m.$$

This expression defines a multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$. Therefore, for any permutation $i_1, i_2, \ldots, i_m$ of indices $1, \ldots, m$, the expression

$$\pi_{i_1} \triangleright_T \pi_{i_2} \triangleright \ldots \triangleright_T \pi_{i_m}$$

determines a distribution of the same family of variables; however, these distributions can differ from one another for different permutations. In the following subsection, we will deal with special generating sequences (or their special permutations) which seem to possess the most advantageous properties.

4.3 T-perfect Sequences

An ordered sequence of possibility distributions $\pi_1, \pi_2, \ldots, \pi_m$ is said to be *T-perfect* if

$$\begin{aligned}
\pi_1 \triangleright_T \pi_2 &= \pi_1 \triangleleft_T \pi_2, \\
\pi_1 \triangleright_T \pi_2 \triangleright_T \pi_3 &= \pi_1 \triangleleft_T \pi_2 \triangleleft_T \pi_3, \\
&\vdots \\
\pi_1 \triangleright_T \cdots \triangleright_T \pi_m &= \pi_1 \triangleleft_T \cdots \triangleleft_T \pi_m.
\end{aligned}$$

[6] Counterexamples can be found in [16].

The notion of T-perfectness suggests that a sequence perfect with respect to one t-norm need not be perfect with respect to another t-norm, which is analogous to (conditional) T-independence.

The following lemma (an immediate consequence of (iii) in Lemma 2) will be used in Section 6.

Lemma 3 *Let T be a continuous t-norm. The sequence $\pi_1, \pi_2, \ldots, \pi_m$ is T-perfect if and only if the pairs of distributions $(\pi_1 \triangleright_T \cdots \triangleright_T \pi_{k-1})$ and π_k are projective for all $k = 2, 3, \ldots, m$.*

The following characterisation theorem, proven in [20], expresses one of the most important results concerning T-perfect sequences. It says they can be composed into multidimensional distributions that are extensions of all the distributions from which the respective joint distribution is composed, i.e. if there exists an ordering of low-dimensional distributions from (9) such that they form a T-perfect sequence, then the resulting model is a solution of the possibilistic marginal problem.

Theorem 4 *The sequence $\pi_1, \pi_2, \ldots, \pi_m$ is T-perfect iff all the distributions $\pi_1, \pi_2, \ldots, \pi_m$ are marginal to distribution $\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright_T \pi_m$.*

Now we are ready to approach the formulation of the result concerning sufficient conditions for the existence of an extension of the given set of low-dimensional distributions, as we promised in Section 3.2. Before doing that, we need to recall what the running intersection property means.

A sequence of sets $K_1, K_2, \ldots, K_n$ is said to meet *running intersection property* (RIP) if

$$\forall i = 2, \ldots, n \;\; \exists j (1 \leq j < i) \qquad (K_i \cap (K_1 \cup \ldots \cup K_{i-1})) \subseteq K_j.$$

The following theorem, proven in [20], reveals the relationship between RIP and T-perfectness.

Theorem 5 *Let $\mathcal{S} = \{\pi_{K_i}, K_i \in \mathcal{K}\}$ be a system of pairwise projective low-dimensional possibility distributions defined by (9). If there exists a permutation $i_1, \ldots, i_m$ of indices $1, \ldots, m$ such that $K_{i_1}, \ldots, K_{i_m}$ meets RIP, then, for any continuous T, there exists a T-product extension*

$$\pi_{i_1} \triangleright_T \pi_{i_2} \triangleright \ldots \triangleright_T \pi_{i_m}$$

of these distributions.

In other words, RIP guarantees the existence of T-product extension of a system of pointwise projective possibility distribution (regardless of the underlying continuous t-norm).

5 Influence of a t-norm on Inference

The usual goal of the construction of multidimensional models in artificial intelligence is to use them for inference. As we presented an approach to the construction of multidimensional possibility distributions parameterised by a continuous t-norm, we should also note that the resulting models (based on different t-norms) may substantially differ from each other, from the viewpoint of inference.

To demonstrate it, let us present a simple example showing that, using different t-norms, we can obtain multidimensional models yielding different inferential rules. The example is the simplest (nontrivial) case described by Theorem 5; the index sets satisfy RIP and therefore one can construct a compositional model as a T-product extension of its marginals for any continuous t-norm T.

Example 1 Consider two binary variables X_1, X_2 and a ternary variable X_3 having two-dimensional possibility distributions contained in Table 1.

$\pi_1(x_1, x_3)$	X_3	0	1	2
$X_1 = 0$		1	1	1
$X_1 = 1$		.5	.7	.9

$\pi_2(x_2, x_3)$	X_3	0	1	2
$X_2 = 0$		1	1	1
$X_2 = 1$		.5	.4	.3

Table 1: Example 1 — two possibility distributions

It is evident that inference about variable X_3 based on values of variable X_1 is different from that based on values of X_2. Our aim is to make inference based on both X_1 and X_2.

Therefore we have to construct a joint possibility distribution of X_1, X_2 and X_3 on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$. Using operators $\triangleright_T$ for three distinct t-norms (Gödel's, product and Łukasiewicz's) mentioned in Section 2.1 we will obtain the three different possibility distributions summarised in Table 2.[7]

[7]Let us mention at this moment that the marginal of π_2 on X_3 is identically equal to 1 and therefore (for any continuous t-norm) conditional possibility distribution $\pi_2(X_2|X_3)$ coincides with the joint one $\pi_2(X_2, X_3)$.

X_1	X_2	X_3	$\pi_1 \rhd_T \pi_2(X_1, X_2, X_3)$		
			G	Π	L
0	0	0	1	1	1
0	0	1	1	1	1
0	0	2	1	1	1
0	1	0	.5	.5	.5
0	1	1	.4	.4	.4
0	1	2	.3	.3	.3
1	0	0	.5	.5	.5
1	0	1	.7	.7	.7
1	0	2	.9	.9	.9
1	1	0	.5	.25	0
1	1	1	.4	.28	.1
1	1	2	.3	.27	.2

Table 2: Example 1 — three-dimensional possibility distributions

Now, let us use these 3-dimensional possibility distributions for inference: knowing values of X_1 and X_2 we have to choose the "most possible" value of X_3. It is evident that the most interesting case is that when both X_1 and X_2 equal 1. In this case, the model based on Gödel's t-norm chooses $X_3 = 0$, while the model based on product t-norm $X_3 = 1$ and that based on Łukasiewicz's t-norm $X_3 = 2$; see Table 3, where the respective conditional possibilities are summarised. ♢

X_3	$\pi_1 \rhd_T \pi_2(X_3 \vert X_1 = 1, X_2 = 1)$		
	G	Π	L
0	1	25/28	.8
1	.4	1	.9
2	.3	27/28	1

Table 3: Example 1 — conditional possibilities

Therefore, it is obvious that an interpretation of the resulting models is needed.

6 Interpretation of T-perfect Sequences

In this section we will deal with probabilistic interpretation of the compositional models, namely those based on Gödel's, product and Łukasiewicz's t-norm.

6.1 Upper Envelopes of Sets of Probability Distributions

With any possibility measure Π on $\mathbf{X}$, one can associate a class of probability measures usually called a *credal set* $\mathcal{M}(\Pi)$ on $\mathbf{X}$ dominated by it, i.e.,

$$\mathcal{M}(\Pi) = \{P : P(A) \leq \Pi(A) \quad \forall A \in \mathcal{P}(\mathbf{X})\}.$$

As proven in [10], the normality of Π is equivalent not only to the fact that $\mathcal{M}(\Pi) \neq \emptyset$, but also that Π is an *upper envelope* of $\mathcal{M}(\Pi)$, i.e., that for any $A \in \mathcal{P}(\mathbf{X})$ there exists $P \in \mathcal{M}(\Pi)$ such that $P(A) = \Pi(A)$.

Nevertheless, since we are interested in distributions rather than in measures, the results in this section will be formulated in terms of distributions. It should be noted that the results need not be valid for possibility measures (although this fact is not so important from the viewpoint of this paper).

Example 2 Let $\mathbf{X} = \{0, 1\}$ and $\pi(0) = 0.6, \pi(1) = 1$ a distribution on $\mathbf{X}$. Then π is the upper envelope of

$$\mathcal{M}(\pi) = \{p : p(0) \in [0, 0.6], p(1) = 1 - p(0)\},$$

which is a convex set of probability distributions. ♢

From the results obtained by De Cooman and Aeyels [10] one can conclude that Gödel's extension (i.e., T-product extension for $T(a, b) = \min(a, b)$) of two one-dimensional possibility distributions is the upper envelope of the set of all extensions of probability distributions from the corresponding sets. In the next subsection we present an analogous result for a product t-norm, then generalise it to a product extension of distributions on overlapping subspaces and, finally, to product-perfect sequences.

6.2 Product Extensions

Let us start with the simplest case, as suggested above.

Lemma 4 *Let X_{K_1} and X_{K_2} be two (groups of) variables ($K_1 \cap K_2 = \emptyset$) with values in $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, and π_1 and π_2 be their possibility distributions. Then the distribution defined by the equality*

$$\pi_\Pi(x_{K_1}, x_{K_2}) = \pi_1(x_{K_1}) \cdot \pi_2(x_{K_2})$$

for any $(x_{K_1}, x_{K_2}) \in \mathbf{X}_{K_1} \times \mathbf{X}_{K_2}$ is the upper envelope of the set of product extensions of probability distributions from $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$.

Proof. First, let us note that $p_1(x_{K_1}) \leq \pi_1(x_{K_1})$ for all $x_{K_1} \in \mathbf{X}_{K_1}$ and $p_2(x_{K_2}) \leq \pi_2(x_{K_2})$ for all $x_{K_2} \in \mathbf{X}_{K_2}$. Let

$$\mathcal{M}(\pi_\Pi) = \{p(x_{K_1}, x_{K_2}) = p_1(x_{K_1}) \cdot p_2(x_{K_2}); p_1 \in \mathcal{M}(\pi_1), p_2 \in \mathcal{M}(\pi_2)\}$$

be the set of product extensions of probability distributions from $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$. From this and from the above-presented inequalities it follows that

$$p(x_{K_1}, x_{K_2}) = p_1(x_{K_1}) \cdot p_2(x_{K_2}) \leq \pi_1(x_{K_1}) \cdot \pi_2(x_{K_2})$$

is satisfied for any $(x_{K_1}, x_{K_2}) \in \mathbf{X}_{K_1} \times \mathbf{X}_{K_2}$. To prove that π_Π is the upper envelope of $\mathcal{M}(\pi_\Pi)$, it is enough to show that for any $(x_{K_1}, x_{K_2}) \in \mathbf{X}_{K_1} \times \mathbf{X}_{K_2}$ there exist probability distributions p_1 and p_2 such that $p_1(x_{K_1}) \cdot p_2(x_{K_2}) = \pi_1(x_{K_1}) \cdot \pi_2(x_{K_2})$. But it is evidently the case since both π_1 and π_2 are upper envelopes of $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$, respectively. □

According to the results of Walley and De Cooman [21] this assertion can be generalised as follows:

Lemma 5 *Let X_{K_1} and X_{K_2} be two (groups of) variables with values in $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, and π_1 and π_2 be their projective possibility distributions. Then the distribution defined by*

$$\pi_\Pi(x_{K_1 \cup K_2}) = \pi_1(x_{K_1}) \triangleright_\Pi \pi_2(x_{K_2})$$

for any $x_{K_1 \cup K_2} \in \mathbf{X}_{K_1 \cup K_2}$ is the upper envelope of the set of product extensions of projective pairs of probability distributions from $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$.

Proof. Walley and De Cooman [21] proved that Dempster's conditioning rule, i.e., the conditioning rule based on product-residual, produces (from joint possibility distribution π) a conditional possibility distribution which is an upper envelope of the set of conditional probabilities obtained from $\mathcal{M}(\pi)$. Due to the definition of the operator $\triangleright_T$, the desired result follows directly from Lemma 4. □

Finally, we obtain the following theorem.

Theorem 6 *Let $\pi_1, \pi_2, \ldots, \pi_m$ be a product-perfect sequence of possibility distributions. Then*

$$\pi_1 \triangleright_\Pi \pi_2 \triangleright_\Pi \cdots \triangleright_\Pi \pi_m$$

is the upper envelope of the probability distributions

$$p_1 \triangleright p_2 \triangleright \cdots \triangleright p_m,$$

where $p_1, p_2, \ldots p_m$ form perfect sequences of probability distributions from $\mathcal{M}(\pi_1), \mathcal{M}(\pi_2), \ldots, \mathcal{M}(\pi_m)$.

Proof of this theorem follows directly from the definition of a perfect sequence, Lemma 3 and iterative application of Lemma 5. □

Loosely speaking, if conditional stochastic independence relations among (groups of) variables appearing in the model are valid (or at least expected), then it is suitable to use a compositional model based on product t-norm (as the relationship between probabilistic operator of composition and stochastic conditional independence is analogous to that between $\triangleright_T$ and conditional T-independence.

6.3 Gödel's Extensions

Unfortunately, it is not possible to use a similar approach for the case of Gödel's t-norm, since Walley and De Cooman showed that Dubois and Prade's conditioning rule, i.e., the conditioning rule based on min-residual, produces — from a joint possibility distribution π — a conditional distribution which is *not* an upper envelope of the set of conditional probabilities obtained from $\mathcal{M}(\pi)$; see [21] for more details.

Nevertheless, it is possible to prove an analogy of Lemma 5 — the following assertion generalises the above-mentioned result obtained by De Cooman and Aeyels.

Lemma 6 *Let X_{K_1} and X_{K_2} be two (groups of) variables with values in $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, and π_1 and π_2 be their projective possibility distributions. Then the distribution defined by the equality*

$$\pi_G(x_{K_1\cup K_2}) = \pi_1(x_{K_1}) \triangleright_G \pi_2(x_{K_2}) \tag{14}$$

for any $x_{K_1\cup K_2} \in \mathbf{X}_{K_1\cup K_2}$ is the upper envelope of the set of all extensions of projective probability distribution pairs from $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$.

Proof. Let us recall that $p_1(x_{K_1}) \leq \pi_1(x_{K_1})$ for all $x_{K_1} \in \mathbf{X}_{K_1}$ and $p_2(x_{K_2}) \leq \pi_2(x_{K_2})$ for all $x_{K_2} \in \mathbf{X}_{K_2}$.

First let us prove that (14) dominates the set of all extensions of projective probability distribution pairs from $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$, i.e., for any p from $\mathcal{M}(\pi_G)$ the inequality

$$p(x_{K_1\cup K_2}) \leq \min(\pi_1(x_{K_1}), \pi_2(x_{K_2})\triangle_{\min}\pi_2(x_{K_1\cap K_2})) \tag{15}$$

is satisfied for any $(x_{K_1\cup K_2}) \in \mathbf{X}_{K_1\cup K_2}$. Since the equalities

$$\begin{aligned} p_1(x_{K_1}) &= \sum_{(x_{K_2\setminus K_1})\in \mathbf{X}_{K_2\setminus K_1}} p(x_{K_1\cup K_2}), \\ p_2(x_{K_2}) &= \sum_{(x_{K_1\setminus K_2})\in \mathbf{X}_{K_1\setminus K_2}} p(x_{K_2\cup K_1}) \end{aligned}$$

must hold for any $p \in \mathcal{M}(\pi_G)$, it is obvious that

$$p(x_{K_1\cup K_2}) \leq \min(p_1(x_{K_1}), p_2(x_{K_2})) \leq \min(\pi_1(x_{K_1}), \pi_2(x_{K_2}))$$

from which (15) immediately follows due to the inequalities presented above and the simple fact that $a \leq a\triangle_T b$ for any $0 \leq a \leq b \leq 1$ and any continuous t-norm T.

To prove that π_G defined by (14) is also the upper envelope, it is enough to prove that for any $x \in \mathbf{X}_{K_1\cup K_2}$ there exists a distribution $p \in \mathcal{M}(\pi_G)$ such that $p(x) = \pi(x)$.

Let $\bar{x}$ be an arbitrary fixed element of $\mathbf{X}_{K_1\cup K_2}$. As π_1 and π_2 are upper envelopes of $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$, respectively, there must exist probability distributions $p_1^* \in \mathcal{M}(\pi_1)$ on $\mathbf{X}_{K_1}$ and $p_2^* \in \mathcal{M}(\pi_2)$ on $\mathbf{X}_{K_2}$ such that $p_1^*(\bar{x}^{\downarrow K_1}) = \pi_1(\bar{x}^{\downarrow K_1})$ and $p_2^*(\bar{x}^{\downarrow K_2}) = \pi_2(\bar{x}^{\downarrow K_2})$.

Now, we have to distinguish between two situations: either $\pi_G(\bar{x}) = \pi_1(\bar{x}^{\downarrow K_1})$ or $\pi_G(\bar{x}) = \pi_2(\bar{x}^{\downarrow K_2})$.[8] In the first case we define

$$p^*(x^{\downarrow K_1}, y) = \begin{cases} p_1^*(x^{\downarrow K_1}) & \text{if} \quad y = \bar{x}^{\downarrow K_2 \setminus K_1}, \\ 0 & \text{otherwise,} \end{cases}$$

and, similarly, in the second one

$$p^*(y, x^{\downarrow K_2}) = \begin{cases} p_2^*(x^{\downarrow K_2}) & \text{if} \quad y = \bar{x}^{\downarrow K_1 \setminus K_2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$p^*(\bar{x}) = \min(\pi_1(\bar{x}^{\downarrow K_1}), \pi_2(\bar{x}^{\downarrow K_2} \triangle_T \pi_2(\bar{x}^{\downarrow K_1 \cap K_2})),$$

as desired. □

Theorem 7 *Let $\pi_1, \pi_2, \ldots, \pi_m$ be a min-perfect sequence of possibility distributions. Then*

$$\pi_1 \triangleright_G \pi_2 \triangleright_G \cdots \triangleright_G \pi_m$$

is the upper envelope of the set of all extensions of projective probability distributions from $\mathcal{M}(\pi_1), \mathcal{M}(\pi_2), \ldots, \mathcal{M}(\pi_m)$.

Proof of this theorem follows directly from the definition of a perfect sequence, Lemma 3 and the iterative application of Lemma 6. □

In other words, if one does not know anything about (in)dependence relations among involved variables, a compositional model based of Gödel's t-norm is the most appropriate, as it is the upper envelope of all probabilistic models with marginals from the corresponding credal sets.

6.4 Łukasiewicz's Extensions

Due to the ordering of t-norms one would expect that Łukasiewicz's extension will dominate, in a sense, the minimal set of extensions of probability distributions. It is true, but only partially, as can be seen from the following theorem and example.

[8]Let us note that due to projectivity of π_1 and π_2 it cannot happen that

$$\pi_2(\bar{x}^{\downarrow K_1 \cap K_2}) = \pi_2(\bar{x}^{\downarrow K_2}) < \pi_1(\bar{x}^{\downarrow K_1}).$$

Theorem 8 *Let X_1 and X_2 be two binary variables with distributions π_1 and π_2, respectively. Then the distribution defined by the equality*

$$\pi_L(x_1, x_2) = \max(0, \pi_1(x_1) + \pi_2(x_2) - 1)$$

for any $(x_1, x_2) \in \mathbf{X}_1 \times \mathbf{X}_2$ dominates the set $\mathcal{M}(\pi_L)$, where at least one common extension exists for each pair $p_1 \in \mathcal{M}(\pi_1)$ and $p_2 \in \mathcal{M}(\pi_2)$. This distribution is the minimal one with this property.

Proof. Without loss of generality let us suppose that $\mathbf{X}_1 = \mathbf{X}_2 = \{0, 1\}$. Let π_1 and π_2 be possibility distributions on $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, and $p_1 \in \mathcal{M}(\pi_1)$ and $p_2 \in \mathcal{M}(\pi_2)$ be arbitrary probability distributions dominated by them. Since we only deal with normal possibility distributions, let us suppose, again without loss of generality, that $\pi_1(1) = \pi_2(1) = 1$. Let us denote $\alpha = \min(p_1(0), p_2(0), \max(0, \pi_1(0) + \pi_2(0) - 1))$. Then

$$\begin{aligned} p(0,0) &= \alpha, \\ p(0,1) &= p_1(0) - \alpha, \\ p(1,0) &= p_2(0) - \alpha, \\ p(1,1) &= 1 + \alpha - p_1(0) - p_2(0) \end{aligned}$$

is clearly dominated by possibility distribution

$$\begin{aligned} \pi_L(0,0) &= \max(0, \pi_1(0) + \pi_2(0) - 1), \\ \pi_L(0,1) &= \pi_1(0), \\ \pi_L(1,0) &= \pi_2(0), \\ \pi_L(1,1) &= 1 \end{aligned}$$

and simultaneously satisfies the marginal constraints. It only remains to check that p is a probability distribution, i.e., that all of its values are non-negative (the summation up to 1 is evident). But it follows directly from the definition of α (which also implies that $p_1(0) + p_2(0) - 1 \leq \alpha$).

Now we prove that π_L is the minimal distribution with this property. Let us suppose that there exists π' smaller than π_L. Then, due to boundary condition, $\pi'(i,j) = \pi_L(i,j)$ for $i + j \geq 1$ and therefore $\pi'(0,0) < \pi_L(0,0)$. Since π' must be nonnegative, $\pi_1(0) + \pi_2(0) > 1$. Let us set $p_1(0) = \pi_1(0)$ and $p_2(0) = \pi_2(0)$. Then the extension of p_1 and p_2 must satisfy $p(0,0) < \pi_1(0) + \pi_2(0) - 1$, and therefore $p(1,1) = 1 + p(0,0) - p_1(0) - p_2(0) < 0$. Hence, π_L is minimal. □

The condition that the variables must be binary is substantial, as can be seen from the following example.

Example 3 Let π_1 and π_2 be marginal distributions on $\mathbf{X}_1 = \{0, 1, 2\}$ and $\mathbf{X}_2 = \{0, 1\}$, respectively, and their Łukasiewicz's extension be defined as suggested in Table 4.

$\pi_L(x_1, x_2)$	X_1	0	1	2	$\pi_2(x_2)$
	$X_2 = 0$	0	0	.5	.5
	$X_2 = 1$	.5	.5	1	1
$\pi_1(x_1)$		.5	.5	1	

Table 4: Example 3 — Łukasziwicz's extension

Let us choose $p_1 \in \mathcal{M}(\pi_1)$ such that $p_1(0) = 0.5, p_1(1) = 0.5$ (therefore $p_1(2) = 0$) and $p_2 \in \mathcal{M}(\pi_2)$ such that $p_2(0) = 0.5$ (i.e., $p_2(1) = 0.5$). None of the extensions of these two probability distributions (cf. Table 5) is dominated by π_L, since both x and y should equal 0, but their sum should equal 0.5. $\Diamond$

$p(x_1, x_2)$	X_1	0	1	2	$p_2(x_2)$
	$X_2 = 0$	x	y	0	.5
	$X_2 = 1$	$.5 - x$	$.5 - y$	0	.5
$p_1(x_1)$		.5	.5	0	

Table 5: Example 3 — a set of dominated probabilities

7 Conclusions

We have recalled a possibilistic marginal problem introduced in [20], necessary conditions, and the sets of all solutions. A lot of attention was paid to T-product extensions — distributions that can be obtained from the marginals by adopting a (conditional) independence requirement. We also recalled a sufficient condition under which they exist and described the apparatus for their construction.

Since these models can differ from each other for different t-norms, the need for their interpretation is obvious. In this paper we confined ourselves to three distinguished t-norms, namely Gödel's, product and Łukasiewicz's, and identified any possibility distribution with the set of probability distributions dominated by it.

We proved that multidimensional possibility models based on product t-norm are upper envelopes of product extensions with marginals dominated by the marginal possibility distributions, and models based on Gödel's t-norm upper envelopes of any probability distributions with marginals dominated by the marginal possibility distributions.

The result for Łukasiewicz's t-norm is much weaker: two-dimensional possibility distributions of binary variables is the minimal one dominating a set of probability distributions. For more general models this assertion need not hold.

Nevertheless, both min-perfect sequences and product-perfect sequences have sensible probabilistic interpretation. This interpretation simultaneously gives a hint of which model should be used in which situation.

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