On Marginal Problem in Evidence Theory

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Abstract. Marginal problem in the framework of evidence theory is introduced in a way analogous to probabilistic one, to address the question of whether or not a common extension exists for a given set of marginal basic assignments. Similarities between these two problem types are demonstrated, concerning necessary condition for the existence of an extension and sets of all solutions. Finally, product extension of the set of marginal basic assignments is introduced as a tool for the expression of a representative in a closed form.

Keywords: Marginal problem, extension, product extension.

1 Introduction

The marginal problem – which addresses the question of whether or not a common extension exists for a given set of marginal distributions – is one of the most challenging problem types in probability theory. The challenges lie not only in a wide range of the relevant theoretical problems (probably the most important among them is to find conditions for the existence of a solution to this problem), but also in its applicability to various problems of statistics [4], computer tomography [7], and artificial intelligence [12]. Recently it has also been studied in other frameworks, for example, in possibility theory [10] and quantum mathematics [8].

In this paper we will introduce an evidential marginal problem analogous to that encountered in the probabilistic framework. We will demonstrate the similarities between these frameworks concerning necessary conditions, and sets of solutions; finally we will also introduce product extension of the set of marginal basic assignments.

The paper is organised as follows: after a brief overview of necessary concepts and notation (Section 2), we will introduce the evidential marginal problem, necessary condition, and the set of solutions in Section 3; and in Section 4 we will deal with product extension.

2 Basic Concepts and Notation

In this section we will, as briefly as possible, recall basic concepts from evidence theory [9] concerning sets and set functions.

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For an index set $N = \{1, 2, \ldots, n\}$, let $\{X_i\}_{i \in N}$ be a system of variables, each X_i having its values in a finite set $\mathbf{X}_i$. In this paper we will deal with a *multidimensional frame of discernment* $\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n$, and its *subframes* (for $K \subseteq N$)

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i.$$

Throughout this paper, X_K will denote a group of variables $\{X_i\}_{i \in K}$ when dealing with groups of variables on these subframes.

For $M \subset K \subseteq N$ and $A \subset \mathbf{X}_K$, we denote by $A^{\downarrow M}$ a *projection* of A into $\mathbf{X}_M$:

$$A^{\downarrow M} = \{y \in \mathbf{X}_M \mid \exists x \in A : y = x^{\downarrow M}\},$$

where, for $M = \{i_1, i_2, \ldots, i_m\}$,

$$x^{\downarrow M} = (x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in \mathbf{X}_M.$$

In addition to the projection, in this text we will also need its inverse operation that is usually called a cylindrical extension. The *cylindrical extension* of $A \subset \mathbf{X}_K$ to $\mathbf{X}_L$ $(K \subset L)$ is the set

$$A^{\uparrow L} = \{x \in \mathbf{X}_L : x^{\downarrow K} \in A\} = A \times \mathbf{X}_{L \setminus K}.$$

A more complex instance is to make a common extension of two sets, which will be called a join [1]. By a *join* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ $(K, L \subseteq N)$, we will understand a set

$$A \bowtie B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \;\;\&\;\; x^{\downarrow L} \in B\}.$$

Let us note that, for any $C \subseteq \mathbf{X}_{K \cup L}$, it naturally holds $C \subseteq C^{\downarrow K} \bowtie C^{\downarrow L}$, but generally $C \neq C^{\downarrow K} \bowtie C^{\downarrow L}$.

Let us also note that if K and L are disjoint, then the join of A and B is just their Cartesian product, $A \bowtie B = A \times B$, and if $K = L$ then $A \bowtie B = A \cap B$. If $K \cap L \neq \emptyset$ and $A^{\downarrow K \cap L} \cap B^{\downarrow K \cap L} = \emptyset$ then $A \bowtie B = \emptyset$ as well. Generally, $A \bowtie B = A^{\uparrow K \cup L} \cap B^{\uparrow K \cup L})$, i.e., a join of two sets is the intersection of their cylindrical extensions.

In evidence theory [9], two dual measures are used to model the uncertainty: belief and plausibility measures. Each of them can be defined with the help of another set function called a *basic (probability* or *belief) assignment* m on $\mathbf{X}_N$, i.e., $m : \mathcal{P}(\mathbf{X}_N) \longrightarrow [0, 1]$, where $\mathcal{P}(\mathbf{X}_N)$ is the power set of $\mathbf{X}_N$, and $\sum_{A \subseteq \mathbf{X}_N} m(A) = 1$. Furthermore, we assume that $m(\emptyset) = 0$.[1] A set $A \in \mathcal{P}(\mathbf{X}_N)$ is a *focal element* if $m(A) > 0$.

For a basic assignment m on $\mathbf{X}_K$ and $M \subset K$, a *marginal basic assignment* of m on $\mathbf{X}_M$ is defined (for each $A \subseteq \mathbf{X}_M$) by the equality

$$m^{\downarrow M}(A) = \sum_{\substack{B \subseteq \mathbf{X}_K \\ B^{\downarrow M} = A}} m(B). \tag{1}$$

[1] This assumption is not generally accepted, e.g., in [2] it is omitted.

3 Marginal Problem

Let $\{X_i\}_{i \in N}$ be a finite system of finite-valued variables with values in $\{\mathbf{X}_i\}_{i \in N}$. Using the procedure of marginalisation (1) one can always uniquely restrict a basic assignment m on $\mathbf{X}_N$ to the basic assignment m_K on $\mathbf{X}_K$ for $K \subset N$. However, the opposite process, the procedure of an *extension* of a system of basic assignments m_{K_i}, $i = 1, \ldots, m$ on $\mathbf{X}_{K_i}$ to a basic assignment m_K on $\mathbf{X}_K$ ($K = K_1 \cup \cdots \cup K_m$), is not unique (if it exists) and can be done in many ways.

Let us demonstrate this fact with two simple examples.

Example 1. Consider, for $i = 1, 2$, two basic assignments m_i on $\mathbf{X}_i = \{a_i, b_i\}$, specified in the left-hand side of Table 1. Our task is to find a basic assignment m

Table 1. Example 1: basic assignments m_1 and m_2 and m and m'

$A \subseteq \mathbf{X}_1$	$m_1(A)$
$\{a_1\}$	0.2
$\{b_1\}$	0.3
$\mathbf{X}_1$	0.5

$A \subseteq \mathbf{X}_2$	$m_2(A)$
$\{a_2\}$	0.6
$\{b_2\}$	0
$\mathbf{X}_2$	0.4

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m(A)$
$\{a_1a_2\}$	0.2
$\{b_1a_2\}$	0.3
$\mathbf{X}_1 \times \{a_2\}$	0.1
$\{a_1a_2, b_1b_2\}$	0.4

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m'(A)$
$\{a_1\} \times \mathbf{X}_2$	0.2
$\{b_1\} \times \mathbf{X}_2$	0.2
$\{b_1a_2\}$	0.1
$\mathbf{X}_1 \times \mathbf{X}_2$	0.5

on $\mathbf{X}_1 \times \mathbf{X}_2$ satisfying these marginal constraints. It is easy to realise that, e.g., m or m' contained in the left-hand side of Table 1 is a solution to this problem. It is obvious that one can find numerous different solutions to this problem. ◇

The following example is devoted to a (more interesting) case of overlapping marginals.

Example 2. Consider two basic assignments m_i (for $i = 1, 2$) on $\mathbf{X}_i \times \mathbf{X}_3$ ($\mathbf{X}_i = \{a_i, b_i\}$, $i = 1, 2, 3$) specified in Table 2. It is again easy to realise that both

Table 2. Example 2: basic assignments m_1 and m_2.

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_3$	$m_1(A)$
$\{a_1a_3\}$	0.5
$\{a_1a_3, b_1b_3\}$	0.3
$\mathbf{X}_1 \times \{a_3\}$	0.2

$A \subseteq \mathbf{X}_2 \times \mathbf{X}_3$	$m_2(A)$
$\{a_2a_3\}$	0.5
$\{a_2a_3, b_2b_3\}$	0.3
$\mathbf{X}_2 \times \{a_3\}$	0.2

joint basic assignments m and m' contained in Table 3 satisfy these constraints. And it is again obvious that one can find numerous different solutions to this problem. ◇

Table 3. Example 2: basic assignments m and m'

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$	$m(A)$
$\{a_1a_2a_3\}$	0.5
$\{a_1a_2a_3, b_1b_2b_3\}$	0.3
$\mathbf{X_1} \times \mathbf{X_2} \times \{a_3\}$	0.2

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$	$m'(A)$
$\{a_1a_2a_3\}$	0.3
$\{a_1a_2a_3, b_1b_2b_3\}$	0.3
$\{a_1\} \times \mathbf{X_2} \times \{a_3\}$	0.2
$\mathbf{X_1} \times \{a_2a_3\}$	0.2

The evidential marginal problem can be, analogous to probability theory, understood as follows: Let us assume that X_i, $i \in N$, $1 \leq |N| < \infty$ are finitely-valued variables, $\mathcal{K}$ is a system of nonempty subsets of N and

$$\mathcal{S} = \{m_K, K \in \mathcal{K}\} \tag{2}$$

is a family of basic assignments, where each m_K is a basic assignment on $\mathbf{X}_K$.

The problem we are interested in is the existence of an *extension*, i.e., a basic assignment m on $\mathbf{X}$ whose marginals are basic assignments from $\mathcal{S}$; or, more generally, the set

$$\mathcal{E} = \{m : m^{\downarrow K} = m_K, K \in \mathcal{K}\} \tag{3}$$

is of interest.

Let us note that we will not be able to find any basic assignment on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ with prescribed two-dimensional marginals in Example 2 if these marginals do not satisfy quite a natural condition called a projectivity (or compatibility) condition.

Having two basic assignments m_1 and m_2 on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively ($K, L \subseteq N$), we say that these assignments are *projective* if

$$m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L},$$

which occurs if and only if there exists a basic assignment m on $\mathbf{X}_{K\cup L}$ such that both m_1 and m_2 are marginal assignments of some m on $\mathbf{X}_{K\cup L}$ (cf. also Theorem 2).

This condition is clearly necessary, but not sufficient, as demonstrated in Example 3.

Example 3. Let $\mathbf{X}_i$ be the same as in Example 2, and m_1, m_2 and m_3 be defined as shown in Table 4.

Although these three basic assignments are projective, more exactly, $m_i(\{a_j\}) = 0.5$ and $m_i(\mathbf{X}_j) = 0.5$ for $i = 1, 2, 3$ and $j = i, i+1(mod3)$, no basic assignment m on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ exists that would have them as its marginals . From the first two marginals one can derive that the only focal elements of m are $\{a_1a_2a_3\}$ and $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$, but none of them is projected to any of the focal elements of m_3. ◇

Table 4. Example 3: basic assignments m_1, m_2 and m_3

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m_1(A)$
$\{a_1 a_2\}$	0.5
$\mathbf{X}_1 \times \mathbf{X}_2$	0.5

$A \subseteq \mathbf{X}_2 \times \mathbf{X}_3$	$m_2(A)$
$\{a_2 a_3\}$	0.5
$\mathbf{X}_2 \times \mathbf{X}_2$	0.5

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_3$	$m_1(A)$
$\{a_1\} \times \mathbf{X}_3$	0.5
$\mathbf{X}_1 \times \{a_3\}$	0.5

In the probabilistic framework, projectivity is a necessary condition for the existence of an extension, too, and becomes a sufficient condition if the index sets of the marginals can be ordered in such a way that it satisfies a special property called the running intersection property (see, e.g., [6]), or equivalently, if the model is decomposable. We conjure that a similar result also holds in evidential framework; nevertheless, it will remain a topic for our future research.

If a solution of an evidential marginal problem exists, it is (usually) not unique, as we have already seen in Examples 1 and 2. This fact is completely analogous to the probabilistic framework. And the following theorem reveals another analogy in this respect.

Theorem 1. *The set $\mathcal{E}(\mathcal{S})$ is a convex set of basic assignments.*

Proof. Let $m_1, m_2 \in \mathcal{E}(\mathcal{S})$ and m be such that

$$m(C) = \alpha m_1(C) + (1-\alpha) m_2(C)$$

for any $C \subset \mathbf{X}_N$. Since $m_1^{\downarrow K}(C^{\downarrow K}) = m_2^{\downarrow K}(C^{\downarrow K}) = m_K^{\downarrow K}(C^{\downarrow K})$ for any $K \in \mathcal{K}$, we get

$$m^{\downarrow K}(C^{\downarrow K}) = \alpha m_1^{\downarrow K}(C^{\downarrow K}) + (1-\alpha) m_2^{\downarrow K}(C^{\downarrow K}) = m_K^{\downarrow K}(C^{\downarrow K})$$

for any $K \in \mathcal{K}$. Therefore, $m \in \mathcal{E}(\mathcal{S})$. □

A convex combination of basic assignments m and m' usually leads to a more complex basic assignment with a higher number of focal elements, as can be seen from the following simple example.

Example 1. *(Continued)* Combining m and m' with $\alpha = 0.5$, we obtain the basic assignment contained in Table 4. ♢

This fact is again analogous to a probabilistic framework, but contrary to the probabilistic case, where the number of focal elements is limited to the cardinality of $\mathbf{X}_N$, in evidence theory the increase of the number of focal elements may lead to intractable tasks.

4 Product Extensions

It is evident that it is rather hard to deal with the whole sets of extensions; hence it seems to be reasonable to look for a representative of each such set.

Table 5. Example 1: basic assignment m^*

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m^*(A)$
$\{a_1a_2\}$	0.1
$\{b_1a_2\}$	0.2
$\{a_1a_2, b_1b_2\}$	0.2

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m^*(A)$
$\{a_1\} \times \mathbf{X}_2$	0.1
$\{b_1\} \times \mathbf{X}_2$	0.1

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m^*(A)$
$\mathbf{X}_1 \times \{a_2\}$	0.05
$\mathbf{X}_1 \times \mathbf{X}_2$	0.25

Dempster's rule of combination [9] is a standard way to combine (in the framework of evidence theory) information from different sources. It has been frequently criticised since the time it first appeared. That is why many alternatives to it have been suggested by various authors.

From the viewpoint of this paper, the most important among them is the *conjunctive combination rule* [2], which is, in fact, a non-normalised Dempster's rule defined for m_1 and m_2 on the same space $\mathbf{X}_K$ by the formula

$$(m_1 \circledcirc m_2)(C) = \sum_{A,B \subseteq \mathbf{X}_K\, A \cap B = C} m_1(A) m_2(B).$$

The result of this rule is one of the examples of a non-normalised basic assignment.

It can easily be generalised [3] to the case when m_1 is defined on X_K and m_2 is defined on X_L $(K \neq L)$ in the following way (for any $C \in \mathbf{X}_{K \cup L}$):

$$(m_1 \circledcirc m_2)(C) = \sum_{\substack{A \subseteq \mathbf{X}_K, B \subseteq \mathbf{X}_L \\ A^{\uparrow L \cup K} \cap B^{\uparrow L \cup K} = C}} m_1(A) m_2(B). \tag{4}$$

Another possible way to solve this problem is to use the product extension of marginal basic assignments defined as follows:

Definition 1. *Let m_1 and m_2 be projective basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$ $(K, L \subseteq N)$, respectively. We will call basic assignment m on $\mathbf{X}_{K \cup L}$ product extension of m_1 and m_2 if for any $A = A^{\downarrow K} \bowtie A^{\downarrow L}$*

$$m(A) = \frac{m_1^{\downarrow K}(A^{\downarrow K}) \cdot m_2^{\downarrow L}(A^{\downarrow L})}{m_1^{\downarrow K \cap L}(A^{\downarrow K \cap L})}, \tag{5}$$

whenever the right-hand side is defined, and $m(A) = 0$ otherwise. ∎

Let us note that the definition is only seemingly non-commutative, as m_1 and m_2 are supposed to be projective. Therefore, it is irrelevant which marginal is used in the denominator.

In the following example we will show that a product extension is more appropriate than Dempster's rule of combination.

Table 6. Example 4: basic assignments m_1, m_2Z

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_3$	$m_1(A)$
$\mathbf{X}_1 \times \{b_3\}$	0.5
$\{(a_1b_3, b_1a_3)\}$	0.5

$A \subseteq \mathbf{X}_2 \times \mathbf{X}_3$	$m_2(A)$
$\mathbf{X}_2 \times \{b_3\}$	0.5
$\{(a_2b_3, b_2b_3)\}$	0.5

Example 4. Let $\mathbf{X}_i, i = 1, 2, 3$, be the same as in previous examples and m_1 and m_2 be two basic assignments defined as shown in Table 6.

Since their marginals are projective, as can easily been checked, there exists (at least one) common extension of both of them.

Applying the conjunctive combination rule to the marginals, one obtains values contained in the left-hand part of Table 7 with the marginal basic assignments different from the originals.

Table 7. Example 4: basic assignments obtained by Dempster's combination rule and product extension

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_3$	$m_1(A)$
$\mathbf{X}_1 \times \mathbf{X}_2 \times \{b_3\}$	0.25
$\{(a_1a_2b_3, b_1b_2a_3)\}$	0.25
$\mathbf{X}_1 \times \{a_2\} \times \{b_3\}$	0.25
$\{a_1\} \times \mathbf{X}_2 \times \{b_3\}$	0.25

$A \subseteq \mathbf{X}_2 \times \mathbf{X}_3$	$m_2(A)$
$\mathbf{X}_1 \times \mathbf{X}_2 \times \{b_3\}$	0.5
$\{(a_1a_2b_3, b_1b_2a_3)\}$	0.5

On the other hand, product extensions of basic assignments m_1 and m_2 contained in the right-hand side of Table 7 keep both marginals. ◇

The difference consists in assigning values to joins of focal elements of the marginal basic assignments. While in (4) the original basic assignments are used even in instances in which focal elements have different projections; at least one of the marginals is equal to zero in (5) in this case, which means that these sets cannot be focal elements of the joint basic assignment.

This result was not obtained by chance, as the following assertion implies.

Theorem 2. *Let m_1 and m_2 be two projective basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$ $(K, L \subseteq N)$, respectively, and m be their product extension. Then*

$$m^{\downarrow K}(B) = m_1(B),$$
$$m^{\downarrow L}(C) = m_2(C)$$

for any $B \in \mathbf{X}_K$ and $C \in \mathbf{X}_L$, respectively.

Proof. It follows directly from Theorem 1 in [11]. □

The next step would be to prove an analogous result for a more general system of basic assignments (as suggested in the previous section). Results form [5] indicate that it could be done.

5 Conclusions

We have introduced an evidential marginal problem in a way analogous to a probability setting, where marginal probabilities are substituted by marginal basic assignments.

We presented the necessary conditions for the existence of a solution to this problem and also dealt with the sets of all solutions. Finally, we introduced a so-called product extension, which enables us to express an extension of the problem in a closed form.

There are still many problems to be solved in the future, such as the structure of the set of extensions of the problem as well as a generalisation of the product extension to a more general index set of marginal basic assignments.

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