Projection Operators and Moment Invariants to Image Blurring

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Abstract—In this paper we introduce a new theory of blur invariants. Blur invariants are image features which preserve their values if the image is convolved by a point-spread function (PSF) of a certain class. We present the invariants to convolution with an arbitrary N-fold symmetric PSF, both in Fourier and image domain. We introduce a notion of a *primordial image* as a canonical form of all blur-equivalent images. It is defined in spectral domain by means of projection operators. We prove that the moments of the primordial image are invariant to blur and we derive recursive formulae for their direct computation without actually constructing the primordial image. We further prove they form a complete set of invariants and show how to extent their invariance also to translation, rotation and scaling. We illustrate by simulated and real-data experiments their invariance and recognition power. Potential applications of this method are wherever one wants to recognize objects on blurred images.

Index Terms—Blurred image, *N*-fold rotation symmetry, projection operators, image moments, moment invariants, blur invariants, object recognition

1 Introduction

Automatic object recognition, which is based on invariant features, has become an established discipline in image analysis. Among numerous descriptors used for this purpose, moments and moment invariants play a very important role and often serve as a reference state-of-the-art method for performance evaluation (interested readers can find a comprehensive survey of moment invariants in [1]).

1.1 Brief History of Moment Invariants

In the long history of moment invariants, one can identify a few milestones that substantially influenced further development. The first one was in 1962, when Hu [2] employed the results of the theory of algebraic invariants, which was thoroughly studied in 19th century by Hilbert [3], and derived his seven famous invariants to rotation of 2D objects. This was the date when moment invariants were introduced to broader pattern recognition and image processing community. The second landmark dates in 1991 when Reiss [4] and Flusser and Suk [5] independently discovered and corrected a mistake in so-called Fundamental Theorem and derived first correct sets of moment invariants to general affine transformation. The third turning point was in 1996-98 when Flusser and Suk [6], [7] introduced a new class of moment-based image descriptors which are invariant to convolution of an image with an arbitrary centrosymmetric kernel. They offered another theoretical view on moment invariants and also opened the door to new application areas. For the first time, moment invariants were able to handle not only geometric distortions of the images as before but also blurring and filtering in intensity domain.

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1.2 Motivation to Blur Invariants

Assuming the image acquisition time is so short that the blurring factors do not change during the image formation and also assuming that the blurring is of the same kind for all pixels and all colors/gray-levels, we can describe the observed blurred image $g(x,y)$ of a scene $f(x,y)$ as a convolution

$$g(x,y) = (f * h)(x,y), \tag{1}$$

where the kernel $h(x,y)$ stands for the point-spread function (PSF) of the imaging system. The model (1) is a frequently used compromise between universality and simplicity — it is general enough to describe many practical situations such as out-of-focus blur of a flat scene, motion blur of a flat scene in case of translational motion, motion blur of a 3D scene caused by camera rotation around x or y axis, and media turbulence blur. At the same time, its simplicity allows reasonable mathematical treatment.

In many cases we do not need to know the whole original image the restoration of which may be ill-posed, time consuming or even impossible; we only need, for instance, to localize or recognize some objects on it (typical examples are matching of a blurred template against a database and a feature-based registration of blurred frames, see Fig. 1). In such situations, the knowledge of a certain incomplete but robust representation of the image is sufficient. However, such a representation should be independent of the imaging system and should actually describe those features of the original image, which are not affected by the degradations. We are looking for a functional I that is invariant to the degradation (1), i.e.

$$I(f) = I(f * h) \tag{2}$$

(a) (b)

Fig. 1. Blurred template (a) to be matched against a database (b). A typical situation where the convolution invariants can be employed.

must hold for any admissible $h(x, y)$. Descriptors satisfying the condition (2) are called *blur invariants* or *convolution invariants*.

1.3 Current State of the Art

Although the PSF is supposed to be unknown, we still have to accept certain assumptions about it to find invariants (for an unconstrained PSF no non-trivial blur invariants exist[1]). In our fundamental paper [7] we supposed that the PSF is brightness-preserving, i.e.

$$\int\int h(x, y)\, dxdy = 1,$$

which is a non-restrictive natural assumption, and that it is *centrosymmetric*, which means $h(x, y) = h(-x, -y)$. Under these assumptions, we derived in [7] a system of blur invariants which are recursive functions of standard (geometric) moments.

These invariants, along with the centrosymmetry assumption, have been adopted by numerous researchers. They (as well as their equivalent counterparts in Fourier domain) have become very popular image descriptors and have found a number of applications, namely in matching and registration of satellite and aerial images [7], [9], [10], [11], [12], in medical imaging [13], [14], [15], in face recognition on out-of-focus photographs [6], in normalizing blurred images into canonical forms [16], [17], in blurred digit and character recognition [18], in robot control [19], in image forgery detection [20], [21], in traffic sign recognition [22], [23], in fish shape-based classification [24], in wood industry [25], [26], in weed recognition [27], in cell recognition [28] and in focus/defocus quantitative measurement [29]. In the last few years yet another broad application area of blur invariants has appeared. When performing multichannel deconvolution and/or superresolution of still images or video, registration of blurred low-resolution input frames is always the necessary preprocessing step (see [30], [31] for a state-of-the-art survey). Having registration methods which are particularly suitable for blurred images is of great demand and the blur invariants are one of the possible solutions.

Several authors have further developed the theory of blur invariants. Since image blurring is in practice often coupled with spatial transformations of the image, an effort has been put into developing so-called *combined invariants* that are invariant simultaneously to convolution and to certain transformations of spatial coordinates. Although a few attempts to construct combined invariants from geometric moments can be found already in [7], only the introduction of complex moments into the blur invariants and a consequent understanding of their behavior under geometric transformations made this possible in a systematic way. Combined invariants to convolution and to rotation were introduced by Zitová and Flusser [32], who also reported their successful usage in satellite image registration [33] and in camera motion estimation [34]. Additional invariance to scaling and/or to contrast changes can be achieved by an obvious normalization (see [35]). The first attempt to find the combined invariants both to convolution and affine transform was published by Zhang et al. [17], who employed an indirect approach using image normalization. Later on, Suk derived combined affine invariants in explicit forms [36]. Their use for aircraft silhouette recognition [37], for sign language recognition [38], for the classification of winged insect [39] and for robust digital watermarking [40] was reported. A slightly different approach to the affine-blur invariant matching was presented in [41], where the combined invariants are constructed in Fourier domain.

The existence of imaging devices providing 3D data, namely in medical imaging, stimulated generalization of the blur invariants from 2D into higher dimensions. Boldys et al. first extended blur invariants into n-D [42] and then, for 3D case, they also added invariance to rotation [43]. Their latest paper [44] presents a general approach to constructing blur and affine invariants in arbitrary number of dimensions. Candocia [45] analyzed in detail the impact of discretization on the n-D blur invariants.

Some authors extended the blur invariants to other domains. Ojansivu and Heikkilä [41], [46] and Tang et al. [47] used equivalent blur-invariant properties of Fourier transform phase for image registration and matching. Makaremi and Ahmadi [48], [49] and Galighere and Swamy [50] observed that the blur invariants retain their properties even if wavelet or Radon transform is applied on the image. The Radon domain was also used by Xiao et al. [51] in order to transform 2D blur and rotation into 1D blur and a cyclic shift.

Several researchers attempted derivation of blur invariants which are functions of orthogonal (OG) moments rather than of the geometric moments. Legendre moments [52], [53], [54], [55], Zernike moments [56], [57], [58], and Chebyshev moments [59] were employed for this purpose. Zuo et al. [60] even combined moment blur invariants and SIFT features [61] into a single vector with weighted components but without a convincing improvement. However, as was proved by Kautsky and Flusser [62], moment invariants in any two different polynomial bases are mutually dependent and theoretically equivalent. He showed that, when knowing invariants from geometric moments, one can easily derive blur invariants in arbitrary polynomial basis.

In all papers quoted above, the invariance property was considered—exactly as in the original paper [7] — only to centrosymmetric PSF's. Few authors were apparently aware of this limitation which decreases the discrimination power (as is discussed later in this paper) and tried to

1. An attempt to construct invariants to arbitrary PSF was published in [8] but that method was designed in a discrete domain only. It does not have a continuous-domain counterpart in principle. The assumption of the PSF symmetry was replaced by the limitation of its support size. However the paper [8] lacks a convincing analysis of the recognition power.

construct invariants to more specific blurs. Flusser et al. derived invariants to motion blur [63], to axially symmetric blur in case of two axes [64], to circularly symmetric blur [65], and to Gaussian blur [1]. Similar motion-blur invariants were later proposed by Stern et al. [66]. Peng and Jun used the motion blur invariants for weed recognition from a camera moving quickly above the field [27] and for classification of wood slices on a moving belt [25] (these applications were later enhanced by Flusser et al. [26], [67]). Zhong used the motion blur invariants for recognition of reflections on a waved water surface [68], although the motion-blur model used there is questionable. Some other authors [69], [70] used a least-square moment matching of motion blurred and clear images instead of an explicit usage of motion blur invariants which might have some advantages in numeric computation.[2] Invariants to circularly symmetric blur equivalent to [65] but expressed in terms of Fourier-Mellin moments were proposed in [71]. Tianxu and Zhang [72] realized without a deeper analysis that the complex moments, one index of which is zero, are invariant to Gaussian blur. Xiao et al. [51] seemingly derived invariants to Gaussian blur and rotation but, since he did not employ the parametric Gaussian form explicitly, he actually did not narrow the class of centrosymmetric PSF's. Gaussian parametric shape was employed by Zhang et al. [73] who proposed a blur-invariant similarity measure between two images without deriving blur invariants explicitly.

1.4 Contribution of this Paper: N-fold Blur Invariants

As one can see from the literature review above, blur invariants have formed a well-established research area during last 15 years with many application-oriented papers published. However, there has been only a little development of the theory since 1996, although the need for a progress on this field is evident.

All methods reviewed in the previous section either assume the knowledge of the parametric form of the blurring function (such as motion or Gaussian blur), which is too restrictive, or suppose centrosymmetric blur with $h(x, y) = h(-x, -y)$. This assumption is usually justified in practice but it is too weak — a vast majority of real blurring functions have a higher degree of symmetry. For instance the PSF of out-of-focus blur is determined by the shape of the aperture. As it is formed by the diaphragm blades (common cameras use to have from 5 to 11 straight or slightly curved blades) it often takes a form similar to a polygon. If the aperture is fully open, then the PSF approaches circular symmetry $h(x, y) = h(\sqrt{x^2 + y^2})$. This case includes the well-known ideal "pillbox" out-of-focus model but it is not limited to it — some objectives may exhibit circular PSF's which are far from being constant on a circle and rather resemble a ring (see Fig. 2 for some real examples of out-of-focus blur). Also the diffraction blur, if present, is given by the aperture shape. For circular aperture it takes a well-known form of Airy function, however for a polygonal aperture the corresponding PSF is more complicated If the blur invariants were designed specifically for a particular symmetry of the blurring PSF, they should exhibit better performance than the invariants to centrosymmetric blur.

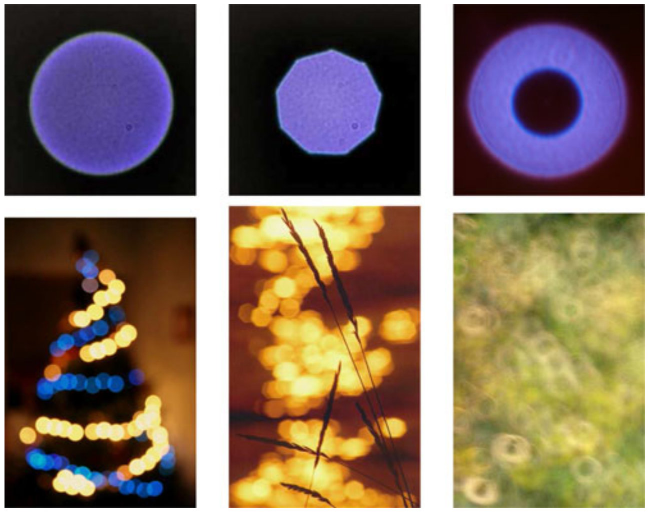

Fig. 2. Top row: Examples of the real out-of-focus blur PSF at circular aperture (left), at polygonal aperture formed by the diaphragm blades (middle), and the ring-shaped PSF of a catadioptric objective (right). Bottom row: images degraded by out-of-focus blur. The shape of the respective PSF can be observed as an image of bright points in the out-of-focus background (this effect is in photography called "bokeh").[3]

In this paper, we present a new general theory of blur invariants with respect to PSF's having N-fold rotation symmetry, for N ranging from one to infinity. The main contributions of the paper are the following.

- For any N we present a specific system of blur invariants. They are defined equivalently both in image domain (based on complex moments) as well as in frequency domain by means of projection operators. We prove that these invariants form a complete set. We introduce a notion of a primordial image as an equivalence class of all images which differ from one another by a convolution with an arbitrary N-fold symmetric PSF.
- We analyze the nullspace and the discrimination power of each set of the invariants. We demonstrate how they depend on N.
- We demonstrate that the new blur invariants can easily be made invariant also to translation, rotation, and scaling.
- We show the original centrosymmetric blur invariants [7] are just a special case of the new theory for $N = 2$.

Summarizing, the papers provides the readers with a two-fold benefit. For theoreticians it gives an insight into the construction and structure of blur invariants while to practically oriented researchers it offers powerful features for object recognition in explicit forms along with instructions on how to select a proper set for the given task.

2. Moment matching is theoretically equivalent to the invariants whenever the invariants exist but can also be applied in some cases when the relation between blurred and clear image moments is known but the blurring does not form a group.

3. The photographs in the bottom row of Fig. 2 were provided by courtesy of Wikipedia (`http://en.wikipedia.org/wiki/Bokeh`) and PALADIX (`http://www.paladix.cz/clanky/bokeh.html`).

2 Blur Invariants for N-fold Symmetric PSF's

In this section we first define basic terms needed along with their important properties. Then we define N-fold blur invariants and discuss their independence, completeness and recognition power.

2.1 Preliminaries on Moments

Definition 1. *By an* image function *(or* image*) we understand any absolutely integrable real function $f(x,y)$ defined on a compact support*[4] $D \subset \mathbb{R} \times \mathbb{R}$ *and having a nonzero integral.*

Definition 2. *Let $f(x,y)$ be an image function and p,q be two non-negative integers. Then the functionals*

$$m_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)dxdy \tag{3}$$

and

$$c_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^p(x-iy)^q f(x,y)dxdy, \tag{4}$$

where i is imaginary unit, are called geometric moment *and* complex moment *of order $p+q$.*

It follows from the definition that only the indices $p \geq q$ provide independent complex moments because $c_{pq} = c_{qp}^*$ (the asterisk denotes complex conjugate). Both geometric and complex moments carry the same amount of information about the image. To see this, we express each complex moment in terms of geometric moments of the same order as

$$c_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{q-j}\cdot i^{p+q-k-j}\cdot m_{k+j,p+q-k-j} \tag{5}$$

and vice versa

$$m_{pq} = \frac{1}{2^{p+q}i^q}\sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{q-j}\cdot c_{k+j,p+q-k-j}. \tag{6}$$

Characterization of the image by means of complex (as well as by geometric) moments is complete and unambiguous in the following sense. The moments of all orders of any image function exist and are finite. The image function can be exactly reconstructed from the set of all its moments.[5]

We use complex moments in this paper instead of more common geometric moments because the complex moments reflect various symmetries of the image (and of the PSF's) in a transparent way. This is implied by their favorable behavior under image rotation, as will be discussed below. The use of complex moments is theoretically not necessary but allows a simple and elegant mathematical treatment of the problem. Thanks to the equivalence of all polynomial bases, one could derive equivalent blur invariants in terms of arbitrary moments [62]. However, in all other bases such derivation would be much more laborious (the earlier invariants to centrosymmetric blur [7] were derived as functions of geometric moments which is practically impossible for a general N-fold symmetry).

Lemma 1. *Let f' be a rotated version (around the origin) of f by an angle α. Let us denote the complex moments of f' as c'_{pq}. Then*

$$c'_{pq} = e^{-i(p-q)\alpha}\cdot c_{pq}. \tag{7}$$

To prove this lemma, we express the image and its moments in polar coordinates (r,θ):

$$c_{pq} = \int_0^{\infty}\int_0^{2\pi} r^{p+q+1}e^{i(p-q)\theta}f(r,\theta)\, drd\theta. \tag{8}$$

Since rotation reduces in polar coordinates to a shift $f'(r,\theta) = f(r,\theta+\alpha)$, the rest of the proof is straightforward. Eq. (7) says that under rotation the moment magnitude $|c_{pq}|$ is preserved while the phase is shifted by $(p-q)\alpha$ (we recall the clear analogue with the Fourier Shift Theorem).

Another useful property of complex moments is their simple transformation if the image is convolved with another image function.

Lemma 2. *Let $f(x,y)$ and $h(x,y)$ be two arbitrary image functions and let $g(x,y) = (f*h)(x,y)$. Then $g(x,y)$ is also an image function and it holds for its moments*

$$c_{pq}^{(g)} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}c_{kj}^{(h)}c_{p-k,q-j}^{(f)}$$

for any p and q.

This lemma can be easily proven just using the definitions of complex moments and of 2D convolution. It holds for geometric moments too in exactly the same form.

For derivation of blur invariants we will employ also the connection between complex moments and Fourier transform. This is however not as straightforward as in the case of geometric moments. Let us use the traditional definition of Fourier transform of f:

$$\begin{aligned}\mathcal{F}(f)(u,v) &\equiv F(u,v)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot e^{-2\pi i(ux+vy)}dxdy.\end{aligned}$$

Since $f \in L_1$, its Fourier transform always exists. After expansion of the exponential function into a power series we obtain the well-known formula

$$F(u,v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2\pi i)^{j+k}}{j!k!}u^j v^k m_{jk}, \tag{9}$$

which tells us that geometric moments of an image are Taylor coefficients (up to a constant factor) of its Fourier transform. We can find a similar meaning for complex moments, too. Let us make a substitution $U = u+v, V = i(u-v)$. Then

$$F(U,V) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2\pi i)^{j+k}}{j!k!}u^j v^k c_{jk}. \tag{10}$$

4. Assumption of the compact support could be omitted if we consider functions of fast decay, so their moments would be well-defined in $\mathbb{R}^2$. However, in practice the images are always of a finite extent.

5. More general moment problem is well known in statistics: can a given sequence be a set of moments of some compactly-supported function? The answer is yes if the sequence is completely monotonic.

2.2 The Space S_N

Now we define the *N–fold rotation symmetry* (N-FRS), which is the central term of this paper. Function f is said to have N-FRS if it is "rotation periodic", i.e. if it repeats itself when it rotates around the origin by $\alpha_j = 2\pi j/N$ for all $j = 1, \ldots, N$. In polar coordinates this means that

$$f(r, \theta) = f(r, \theta + \alpha_j) \qquad j = 1, \ldots, N.^6$$

Particularly, $N = 1$ means no symmetry in a common sense and $N = 2$ denotes a the central symmetry $f(-x, -y) = f(x, y)$. We use this definition not only for finite N but also for $N = \infty$. Thus, in our terminology, a function having *circular symmetry* $f(r, \theta) = f(r)$ is said to have ∞-FRS. Later in this paper, the assumption of N–fold rotation symmetry will be imposed on the blurring PSF's.

We denote a set of all functions with N–fold rotation symmetry as S_N. For any N, the set S_N is closed under addition, multiplication and convolution. Considering pointwise addition and convolution, S_N forms a (commutative) ring.[7] Later on, we will employ particularly the closure property of S_N with respect to convolution.

In this paper we do not deal explicitly with axial (reflection) symmetry but it is worth mentioning that axial and rotational symmetries are related in the following way. If f has N axes of symmetry then it belongs to S_N. On the other hand, if $f \in S_N$ then it has either none or N symmetry axes. Hence, axial symmetry cannot exist without rotation symmetry (such compound symmetry is called *dihedral* symmetry) but the opposite case is possible.

The N–fold rotation symmetry implies vanishing of certain moments. This property allows the existence of blur invariants.

Lemma 3. *If $f \in S_N$, N finite, and if $(p - q)/N$ is not an integer, then $c_{pq} = 0$.*

Proof. Let us rotate f around the origin by $\alpha = 2\pi/N$. Due to its symmetry,

$$f'(r, \theta) \equiv f(r, \theta + \alpha) = f(r, \theta).$$

In particular, it must hold $c'_{pq} = c_{pq}$ for any p and q. On the other hand, as follows from eq. (7),

$$c'_{pq} = e^{-2\pi i(p-q)/N} \cdot c_{pq}.$$

Since $(p - q)/N$ is assumed not to be an integer, this equation can be fulfilled only if $c_{pq} = 0$. □

Lemma 3a. *If $f \in S_\infty$ and if $p \neq q$, then $c_{pq} = 0$.*

Proof. The proof is similar to that of Lemma 3 with α being an arbitrary angle. We get

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq} = c_{pq},$$

which can be fulfilled only if $c_{pq} = 0$. □

Note that these two lemmas do not hold the other way round – an integer $(p - q)/N$ (or $p = q$) does not necessarily imply a non-zero c_{pq}.

The space S_N is closed also with respect to Fourier transform: If $f \in S_N$ then also $F \in S_N$ (if it exists). This follows immediately from the rotation property of Fourier transform. However, note that F is real-valued only for even N.

Interesting relations hold among the sets S_N for various N. Assuming N can be factorized as

$$N = \prod_{i=1}^{L} N_i,$$

where $N_i = k_i^{n_i}$ and k_i are mutually different primes. Then

$$S_N = \bigcap_{i=1}^{L} S_{N_i}.$$

For S_{N_i} we have a nested sequence of the sets

$$S_{k_i} \supsetneqq S_{k_i^2} \supsetneqq \cdots \supsetneqq S_{k_i^{n_i}} \equiv S_{N_i}.$$

Particularly,

$$S_\infty = \bigcap_{k=1}^{\infty} S_k.$$

2.3 Projection Operators

In this section we introduce *projection operators* onto S_N. These operators decompose any function into its N-fold symmetric part and "the rest", similarly as in 1-D one can decompose any function into an even and an odd parts.

Definition 3. *Projection operator P_N is for a finite N defined as*

$$(P_N f)(r, \theta) = \frac{1}{N} \sum_{j=1}^{N} f(r, \theta + \alpha_j),$$

where $\alpha_j = 2\pi j/N$, and

$$(P_\infty f)(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta)\, d\theta.$$

Operator P_N rotates f repeatedly by $2\pi/N$ and calculates an average. The following properties are valid for any f and N.

- $P_N f \in S_N$ (i.e. P_N projects f onto S_N).
- If $f \in S_N$ then $P_N f = f$ and vice versa.
- Operator P_N is linear:

$$P_N(af + g) = aP_N f + P_N g.$$

- Operator P_N is idempotent, i.e. $P_N(P_N f) = P_N f$.
- Any function f can be expressed as $f = P_N f + f_A$, where f_A is its N-fold *antisymmetric* part.[8] Clearly, $P_N f_A = 0$.

6. We can also define exact N–fold rotation symmetry where N is the highest number with this property. However, such a definition is not very useful because we lose the closure property of S_N.

7. Since the validity of this assertion is intuitive, we skip a formal proof which requires some algebraic manipulations.

8. The word "antisymmetric" here means "anything but for N-fold symmetry", so f_A may include terms with other symmetries as well as asymmetric terms.

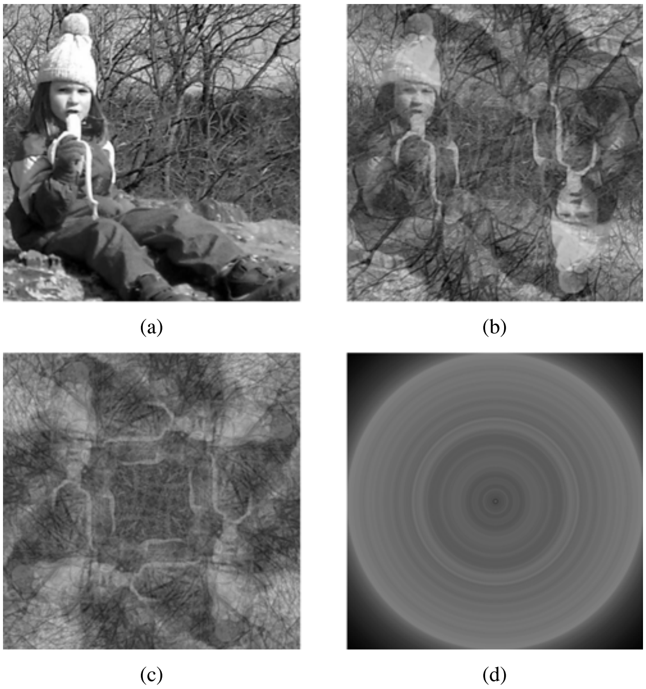

Fig. 3. Performance of the projection operators. (a) original image f, its projections (b) $P_2 f$, (c) $P_4 f$ and (d) $P_\infty f$.

- Operator P_N commutes with Fourier transform:

$$\mathcal{F}(P_N f) = P_N F.$$

- Complex moments of a function f are either preserved or zeroed by the projection operator P_N. For a finite N it holds

$$c_{pq}^{(P_N f)} = c_{pq}^{(f)} \qquad \text{iff} \quad (p-q)/N \quad \text{is an integer},$$
$$c_{pq}^{(P_N f)} = 0 \qquad \text{otherwise}.$$

For $N = \infty$ we get

$$c_{pp}^{(P_\infty f)} = c_{pp}^{(f)},$$
$$c_{pq}^{(P_\infty f)} = 0 \qquad \text{for} \quad p \neq q.$$

The last property follows from the properties of complex moments of N-fold symmetric functions and will play an important role later when the invariants will be constructed.

A visual example of the projection operators is shown in Fig. 3.

2.4 Definition of N-fold Blur Invariants

Let us consider an image which was blurred according to (1) with an unknown PSF $h(x,y)$ which has an N-fold rotation symmetry.

Now we formulate the central theorem of this paper.

Theorem 1. *Let f be an arbitrary image function, then*

$$I_N(u,v) = \frac{\mathcal{F}(f)(u,v)}{\mathcal{F}(P_N f)(u,v)}$$

is an N-fold blur invariant, i.e. $I_N^{(f)} = I_N^{(f*h)}$ for any N-fold symmetric $h(x,y)$.

Proof.

$$\begin{aligned} I_N^{(f*h)} &= \frac{\mathcal{F}(f*h)}{\mathcal{F}(P_N(f*h))} = \frac{F\cdot H}{P_N(F\cdot H)} \\ &= \frac{N\cdot F\cdot H}{\sum_{j=1}^{N} F(r,\theta+\alpha_j)H(r,\theta+\alpha_j)}. \end{aligned}$$

Since $H \in S_N$, we have $H(r,\theta+\alpha_j) = H(r,\theta)$ for any $j = 1,\ldots,N$. Consequently,

$$I_N^{(f*h)} = \frac{F\cdot H}{H\cdot P_N F} = \frac{F}{P_N F} = I_N^{(f)}.$$

□

The above theorem holds also for $N = \infty$, the proof is similar.

The Fourier-domain blur invariant I_N is a ratio of two functions, that can be expressed by absolutely convergent power series the coefficients of which are geometric moments. As we have shown in Section 2.1, after the substitution $U = u+v, V = i(u-v)$ we obtain expressions in terms of complex moments. Hence, we can express I_N as a convergent power series

$$I_N(U,V) = \frac{F}{P_N F}(U,V) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2\pi i)^{j+k}}{j!k!}A_N(j,k)u^j v^k.$$

The coefficients $A_N(j,k)$ can be easily obtained from the constraint

$$\begin{aligned} &\left(\sum_{\substack{j=0 \\ (j-k)/N}}^{\infty}\sum_{\substack{k=0 \\ \text{is integer}}}^{\infty}\frac{(-2\pi i)^{j+k}}{j!k!}c_{jk}^{(f)}u^j v^k\right) \\ &\quad\cdot\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2\pi i)^{j+k}}{j!k!}A_N(j,k)u^j v^k\right) \\ &\quad= \left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2\pi i)^{j+k}}{j!k!}c_{jk}^{(f)}u^j v^k\right). \end{aligned} \tag{11}$$

The first factor is an expansion of $P_N F(U,V)$. Recalling the relation between the complex moments of a function and those of its projection derived in Section 2.3, we can see why the summation goes over indices with an integer value of $(j-k)/N$ only — all other complex moments of $P_N f$ are zero, while the non-zero ones equal $c_{jk}^{(f)}$. Comparing the coefficients of the same powers we get, after some manipulation,

$$A_N(p,q) = \frac{c_{pq}^{(f)}}{c_{00}^{(f)}} - \sum_{\substack{j=0 \\ 0<j+k \\ (j-k)/N \text{ is integer}}}^{p}\sum_{k=0}^{q}\binom{p}{j}\binom{q}{k}\frac{c_{jk}^{(f)}}{c_{00}^{(f)}}A_N(p-j,q-k). \tag{12}$$

Since I_N was proven to be invariant to blur, all coefficients $A_N(p,q)$ must also be blur invariants. If we further assume that the blurring is brightness-preserving, i.e. $c_{00}^{(h)} = 1$, then c_{00} is also blur invariant and by excluding it from

(12) we obtain the final form for moment invariants to N-fold symmetric blur[9]

$$K_N(p,q) = c_{pq} - \frac{1}{c_{00}} \sum_{j=0}^{p} \sum_{\substack{k=0 \\ 0<j+k \\ (j-k)/N \text{ is integer}}}^{q} \binom{p}{j}\binom{q}{k} K_N(p-j, q-k) \cdot c_{jk}. \tag{13}$$

The assumption of overall brightness preservation is not necessary for construction of blur invariants. However, it is mostly fulfilled in real imaging systems and we also keep it because of the consistency with [7].

There is another (seemingly different but actually equivalent) instructive approach leading to the same result. It is natural to express problems related to rotation symmetry using the circular harmonics basis. Applied in the Fourier domain, where convolution becomes multiplication, it leads to the following conclusion. The linear space spanned by all the harmonics can be divided into N independent subspaces, where each of them stays invariant under convolution with an N-fold symmetric kernel. Thus, it can be expected, that an invariant to N-fold symmetric convolution should be based on circular harmonics from one of these subspaces. Looking for a relative invariant, it is then easy to find the expressions identical with the invariants defined in Theorem 1 by projection operator and with those defined by moments (13).

Let us discuss some important properties of the blur invariants $K_N(p,q)$.

- *Complex values.* The invariants (13) are generally complex. If we prefer working with real-valued invariants, we should use their real and imaginary parts separately. On the other hand, for any invariant (13) $K_N(p,q) = K_N(q,p)^*$, so it is sufficient to consider the cases $p \geq q$ only.
- *Zero-order invariant.* For any N (finite or infinite) it holds $K_N(0,0) = c_{00}$. This is a "singular" invariant, the invariance of which comes from the fact that the PSF is assumed to have a unit integral and not, contrary to all other invariants, from its symmetry.
- *First-order invariants.* For any N, invariant $K_N(1,0) = c_{10}$ is non-trivial. However, since in practical applications c_{10} uses to be employed for image normalization to shift, $K_N(1,0)$ becomes useless for image recognition.
- *Non-symmetric PSF.* Invariance property formally holds also for $N = 1$ (i.e. PSF without any symmetry) but all invariants except $K_1(0,0) = c_{00}$ are identically zero.
- *Factorization of N.* Let N be a product of L integers, $N = k_1 k_2 \cdots k_L$. Then the PSF has also k_n-fold symmetry for any $n = 1, 2, \ldots, L$. Thus, any $K_{k_n}(p,q)$ is also a blur invariant. However, using these invariants together with $K_N(p,q)$'s is useless since they are all dependent on $K_N(p,q)$'s. (Note that the dependence here does not mean equality, generally $K_N(p,q) \neq K_{k_n}(p,q)$).
- $N = \infty$. In the case of a circularly symmetric PSF, Theorem 1 obtains much simpler form because the summation goes over the indices $j = k$ only. Hence, provided that $p \geq q$, we get

$$K_\infty(p,q) = c_{pq} - \frac{1}{c_{00}} \sum_{k=1}^{q} \binom{p}{k}\binom{q}{k} K_\infty(p-k, q-k) \cdot c_{kk}.$$

  Circularly symmetric PSF's appear in imaging as out-of-focus blur and diffraction blur on a circular aperture, and describe also a long-term atmospheric turbulence blur.
- *Relation to earlier work:* $N = 2$. The invariants to centrosymmetric blur introduced in [7] are nothing but a particular case of Eq (13). The invariants in [7] have formally exactly the same form as those in Eq (13) with complex moments replaced by geometric moments (this "direct" substitution is possible only for $N = 2$). To see that both systems are actually equivalent, one can substitute from (5).

2.5 The Null-Space and the Discrimination Power of the Invariants

Unlike many geometric invariants, the invariants (13) do not have a straightforward "physical" interpretation. However, understanding what image properties they reflect is important for their practical application. Two mutually connected key questions pertain to the null-space of the invariants (i.e. to the set of images having, for a given N, all invariants $K_N(p,q)$ zero) and to their discrimination power.

First of all, note that certain invariants (13) are *always* zero for any object and are of course completely useless for recognition. For any N, p, q such that $(p-q)/N$ is integer (except $p = q = 0$) we get $K_N(p,q) = 0$. A formal proof can easily be done by induction, we present only the core idea here.

If $(p-q)/N$ is integer and the summation in (13) goes over $(j-k)/N$ integer only, then also $((p-j)-(q-k))/N$ is always an integer. In other words, if $(p-q)/N$ is integer, then on the right-hand side of (13) we have only K_N's of the same property. Now we look what always happens with the highest-order term. Since $(p-q)/N$ is integer, then the last term in the summation is $K_N(0,0)c_{pq} \equiv c_{00}c_{pq}$ which cancels with the c_{pq} standing outside the sum. Knowing that, is easy to prove by induction that all other terms are zero because they contain the K_N of orders less than $p+q$ and of the same nature. The existence of only trivial invariants for $(p-q)/N$ being integer is unavoidable. The moment $c_{pq}^{(h)}$ of the blurring PSF is non-zero and there is no way how to eliminate it. This is for instance the reason why for a centrosymmetric blur $(N = 2)$ cannot exist valid invariants of even orders. On the other hand, for $(N = \infty)$ the only trivial invariants are $K_\infty(p,p)$'s (for $p > 0$), all others are valid invariants.

The number of valid (i.e. non-trivial) invariants up to the given order r increases as N increases, reaching its maximum for all $N > r$ (see Table 1). This is in accordance with our intuitive expectation — the more we know about the

9. Thanks to the projection operators, the proof of invariance of I_N is very elegant and overcomes the necessity of proving invariance of $K_N(p,q)$ directly. This is of course possible by induction but such proof is long and tedious, see [7] for the case $N = 2$ and geometric moments.

TABLE 1
The Number of Non-Trivial N-Fold Blur Invariants Up to the Order r

N r	1	2	3	4	5	6
2	1	1	3	3	6	6
3	1	2	3	5	7	9
4	1	2	4	5	8	10
5	1	2	4	6	8	11
6	1	2	4	6	9	11
$7-\infty$	1	2	4	6	9	12

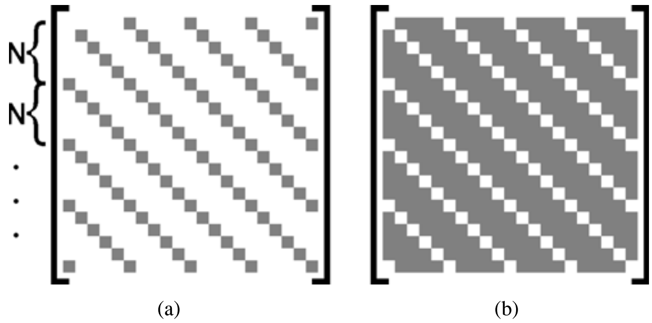

Fig. 4. The structure (a) of the invariant matrix and (b) of the PSF moment matrix. The gray elements are zero for any f and h. The white elements stand for non-trivial invariants in (a) and non-zero moments of the PSF in (b). Note the complementarity of both matrices (except the $(0,0)$ element).

PSF (i.e. the less "degrees of freedom" the PSF has), the more invariants are available. The structure of invariant matrix $\mathbf{K}_{ij} = K_N(i-1, j-1)$ is shown in Fig. 4a. There are always zero-stripes on the main diagonal regardless of N and also on all minor diagonals for $(p-q)/N$ being an integer. As one can see, the invariant matrix of the image is complementary to the moment matrix of the PSF in terms of zero and non-zero elements (see Fig. 4b). The discrimination power of the set of the invariants up to the given order thus increases as N increases. This is also a property we could expect—the discrimination power goes always against the invariance and here the "maximum invariance" is provided by $N = 2$.

What is the null-space of the *non-trivial* blur invariants? Let us denote Z_N the joint null-space of all (non-trivial) invariants $K_N(p,q), \quad p \geq q, \quad p > 0$. For any N it holds $S_N \subset Z_N$. This is clear because any N-fold symmetric function f can be considered as a blurring PSF acting on a delta-function, which always lies in Z_N. Hence, also $f \in Z_N$. On the other hand, if $f \in Z_N$, we can reconstruct its moments. The reconstruction is ambiguous—the moments with an integer value of $(p-q)/N$ may be chosen arbitrary[10] while the moments with a non-integer $(p-q)/N$ are zero. Thus, the reconstructed function is always from S_N. Summarizing, we proved an important equivalence

$$S_N = Z_N.$$

This is one of the intrinsic limitations of discriminative power of the invariants. Any invariant to convolution with an N-fold symmetric PSF cannot distinguish different N-fold symmetric objects because it gives zero responses on all such images.

Let us conclude this section by a remark concerning the proper choice of N. It is the only user defined parameter of the method and its correct estimation might be a tricky problem. Ideally, it should be deduced from the physical model of the blurring source or from other prior information. In fact, we assume that in most situations N is given by the aperture shape and is known. If this is not the case, we may try to estimate N directly from the blurred image by analyzing its spectral patterns or the response to an ideal bright point or a spot of a known shape, if available. If none of the above is applicable and we overestimate N in order to have more invariants available then we lose the invariance property of some (or even all) of them. On the other hand, some users might rather underestimate it to be on a safe side (they might choose for instance $N = 2$ instead of correct $N = 16$). This would always lead to the loss of discriminability (note that $Z_2 \supsetneqq Z_{16}$) and sometimes could also violate the invariance, depending on the actual and chosen N. Obviously, the worst combination occurs if the true and estimated fold numbers are small coprime integers such as 2 and 3 or 3 and 4. If h is not exactly N-fold symmetric, then the more the ratio $H/P_N H$ differs from a constant 1, the more violated the invariance of the K_N's.

10. Provided they satisfy complete monotonicity constraint.

2.6 Completeness of the Invariants

A crucial question concerns the general discrimination power—what are the "equivalence classes" of images having the same values of all invariants? We may ask an equivalent question about the reconstruction possibility. Knowing, for a given N, all invariants $K_N(p,q)$ (or, equivalently, I_N), what image can we reconstruct? What are the degrees of freedom of this reconstruction? The previous section gave us the answer for the case of $K_N(p,q) = 0$, but how is it in a general case? Apparently, any shape descriptor invariant to a certain group of transformations cannot, in principle, distinguish objects that differ from one another only by transformations from this group. Complete descriptors are able to distinguish all other cases, incomplete descriptors do not have this ability. Below we demonstrate the completeness of the blur invariants.

The frequency domain provides us with a good insight. I_N is a ratio of two Fourier transforms which may be interpreted as a deconvolution. Having an image f, we seemingly "deconvolve" it by the kernel $P_N f$. This "deconvolution" exactly eliminates the symmetric part of f (more precisely, it transfers $P_N f$ to δ-function) and acts on the antisymmetric part:

$$I_N = \frac{\mathcal{F}(f)}{\mathcal{F}(P_N f)} = \frac{P_N F + \mathcal{F}(f_A)}{P_N F} = 1 + \frac{\mathcal{F}(f_A)}{P_N F} \equiv 1 + \Psi_N.$$

Hence, I_N can be viewed as a Fourier transform of a *primordial image* (although such an image may not exist in a common sense) and then $K_N(p,q)$ are its complex moments. In other words, this is a kind of normalization. We seemingly calculate a blind deconvolution with an N-fold symmetric kernel which is chosen in such a way that it eliminates the N-fold symmetric component of the image. Hence, the primordial image, which plays the role of a canonical form of

f, is the "maximally deconvolved" antisymmetric part. Moments of the primordial image are obviously invariants to N-fold convolution and the trick is that we are able to calculate these moments directly using (13) without actually performing the deconvolution. Now we can again see why those $K_N(p,q)$ where $(p-q)/N$ is integer must be trivial. Note that the primordial image depends only on the antisymmetric part of the image. Hence, $K_N(p,q)$ can be also viewed as a measure of antisymmetry of f. This corresponds to our intuitive expectation that the antisymmetric part can be modified by a symmetric convolution kernel only such that certain significant features are always preserved while the symmetric part may be changed to any other symmetric function.

Taking this into consideration, we can understand the limitation of the recognition power: two images having the same primordial image cannot be distinguished. This conclusion is incorporated into the following theorem, which at the same time says that in all other cases the images are distinguishable, i.e. the blur invariants are complete.

Theorem 2. *Let f and g be arbitrary image functions and N be an integer or $N=\infty$. Then*

$$K_N(p,q)^{(f)} = K_N(p,q)^{(g)} \qquad p,q = 0,1,\ldots$$

if and only if there exist functions h_1 and h_2 from S_N such that

$$f * h_1 = g * h_2.$$

Proof. The backward implication follows directly from the invariance property. To prove the forward implication, let us realize that if all invariants equal then $I_N^{(f)} = I_N^{(g)}$ which means

$$\frac{F}{P_N F} = \frac{G}{P_N G}$$

and, consequently,

$$f * P_N g = g * P_N f.$$

Since both $P_N f$ and $P_N g$ are from S_N regardless of f and g, the proof is completed. □

Note that the completeness is guaranteed only if an infinite set of invariants for all p,q is used. In practice we always work with a finite (sometimes very small) subset, so the actual discriminability is influenced by this factor.

2.7 Combined Invariants

In practice, image blurring is often coupled with spatial transformations of the image. To handle these situations, having *combined invariants* that are invariant simultaneously to convolution and to certain transformations of spatial coordinates is of great demand.

Shift invariance is achieved easily by using central moments and scale invariance is provided by normalizing each invariant $K_N(p,q)$ by $c_{00}^{(p+q)/2+1}$, which is equivalent to the use of normalized complex moments in (13) (note that the invariance to convolution is not violated by normalization because c_{00} itself is a convolution invariant for any N).

Achieving rotation invariance is harder but still can be accomplished in an elegant way. Note that the N-fold symmetry of the blurring PSF is not violated by an image rotation (applied either before or after the blurring), so it makes sense to look for combined invariants. When investigating the behavior of the invariants $K_N(p,q)$ under rotation, we observe that they change in the same way as the complex moments themselves, i.e.

$$K'_N(p,q) = e^{-i(p-q)\alpha} \cdot K_N(p,q). \tag{14}$$

Hence, the simplest way is to take the magnitudes $|K_N(p,q)|$ which provide combined invariants but create only an incomplete system. A more sophisticated method of creating a complete set of combined invariants is based on phase cancellation by multiplication of proper invariants. It is very similar to the construction of pure rotation invariants from the complex moments [74] (including selection of the independent set and symmetry problems) where we just replace the moments by the invariants.

3 Implementation and Practical Issues

In this section we answer several questions that could be raised by anyone who is thinking about the implementation of the convolution invariants.

- *How many invariants shall we use in practice?* There is no general answer and it is not true that the more the better. The noise sensitivity, complexity and other numerical problems increase with an increasing order (this is a well-known property of moments which we do not investigate here; the noise sensitivity of convolution invariants for $N=2$ was studied in [7] and it is the same for other N). On the other hand, low-order invariants often do not provide sufficient discriminability. The "optimal" number of invariants always depends on the data and should be found by a feature selection procedure which optimizes the separability (measured for instance by Mahalanobis distance) of a particular training set/database.
- *Shall we preferably use Fourier-domain or moment-based invariants?* Although theoretically equivalent, they may behave differently. In most applications the invariants are typically required for relatively small images or templates and a few of them are sufficient. In such a case moment-based invariants are the preferred choice. If the image in question is large (several hundreds or thousands of pixels in one dimension) then Fourier domain might be better because the moment values may overflow. On the other hand, when constructing I_N, we possibly divide by very small numbers which requires certain care. High frequencies of I_N are sensitive to noise so it is better to suppress them by a low-pass filter.

  Another reason why we prefer constructing combined invariants from moments in the image space is that we usually need only a few invariants and it is not necessary to calculate all of them.
- *What shall we do if the images to be compared are blurred with PSF's having different number of folds (say N_1 and*

N_2)? To ensure invariance to both and the best possible discriminability, we choose N as the greatest common divisor of N_1 and N_2. If N_1 and N_2 are coprime, the task is not correctly solvable. In such a case we choose either N_1 or N_2 depending on which blur is more severe.

- *How to avoid over/underflow of moment values?* If the image size is large, we may face overflow (if a pixel means 1 on a coordinate axes) or underflow (if the whole image is mapped into $(0,1) \times (0,1)$ domain). A prevention of this is using orthogonal polynomials and OG moments instead of the power basis and complex moments. The range of values of OG polynomials uses to be small so the OG moments are kept within a reasonable interval. Kautsky and Flusser [62] shows that there is no need to re-built the theory of convolution invariants when changing the basis; the formulas for convolution invariants in any polynomial basis can be obtained from (13) by simple matrix operations.
- *How to eliminate different dynamic range of different invariants?* If the dynamic range of the invariants is significantly different, then in terms of Euclidean metric the ones with high range are preferred which is not desirable. One way to suppress this is using OG moments as described above. Another (simpler) possibility is the value normalization into the same range or to the same standard deviation or, equivalently, using weighted Euclidean norm. All these solutions might be misleading when the different range actually reflects a different significance which is hard to recognize in advance. Since the scale-normalized versions of the invariants have much less range, scale normalization is often a good choice even if the images are not spatially scaled.
- *How to handle the boundary effect?* For discrete images of a finite support and non-zero background, the boundary effect limits the usability of the invariants. The convolution model is violated, the boundary stripes are affected also by pixels laying in the scene outside the visual field and the convolution invariants are no longer truly invariant. The influence of the boundary effect on template matching was studied thoroughly in [7] for $N = 2$ and in [1] also for motion blur. Since it is the same for any N, we do not repeat experiments of this kind here. It was shown that boundary effect cannot be removed but if the blur size is less than 15% of the template size most templates are still recognized correctly.
- *What is the complexity of (13)?* The complexity of evaluation of (13) is given solely by the complexity of moment calculation, other operations consume negligible time. For the review of fast algorithms for moment computation we refer to [1], Chapter 7, and to [75]. As soon as the moment values are calculated, we implement (13) directly from the definition because Matlab (as well as many other languages) supports recursive operations. It is of course possible to derive, by backward substitution, non-recursive definition of the $K_N(p,q)$'s but it does not lead to any measurable speed-up.

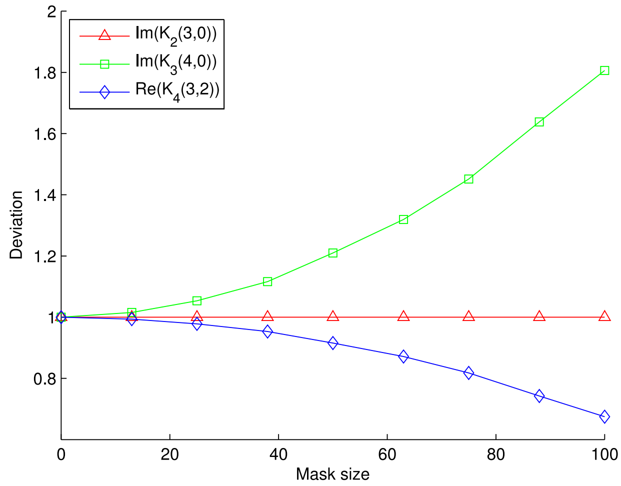

Fig. 5. The ratio between the invariants of the blurred image and of the original. Blur mask of $N = 2$ (elongated rectangle in this case). K_2 is a perfect invariant while K_3 and K_4 vary significantly. The size of the original was cca 1,000 $\times$ 1,000 pixels.

4 Experiments

In this section we demonstrate the performance and limitations of the convolution invariants. We focus on the experiments showing the specific behavior of the invariants $K_N(p,q)$ for various N.

4.1 Basic Experiment on Simulated Data

The aim of the first experiment was to illustrate the behavior of the invariants (13) in situations, where the convolution model and the assumption of the symmetry of the PSF are perfectly valid. The only source of errors could be image sampling (note that the theory was derived in a continuous domain only) and finite precision of the calculations, so the properties of the invariants can be precisely analyzed.

We took 1,000 images (common photographs), blurred them by convolution with masks of various sizes, coefficients, and symmetries (we used $N = 2, 3, 4, 6, 8$ and $N = \infty$). We prevented the boundary effect by zero-padding and calculated the invariants (13) up to the order 10 for every blurred image. The relative error of each individual invariant was about (10^{-10}) as one expects from the theory. Then we compared the values that according to the theory need not match—for instance $K_4(p,q)$ of the original with $K_4(p,q)$ of its blurred version but with the PSF's of $N = 2$. Here the relative errors ranged up to 100% depending on the size of the blur. This clearly shows the necessity of choosing a correct N namely in case of a heavy blur or, equivalently, the necessity of having special invariants for each N. See Fig. 5 for an illustration of this experiment.

4.2 Leaf Recognition

Automatic recognition of leafs has been studied thoroughly in the last decade. Many methods and several publicly available leaf databases have appeared, some of them implemented in a very user-friendly form in mobile devices (for instance the Leafsnap [76] running on Apple i-phones and i-pads). Several leaf recognition algorithms use moments and moment invariants to rotation and scale as features for the leaf description [77], [78], [79], [80]. Our

Fig. 6. The leaves used in the experiment—one sample per each species. Top row: *Aesculus hippocastanum* (leaflet of palmately compound leaf), *Ailanthus altissima* (leaflet of pinnately compound leaf), *Betula papyrifera*, *Chaenomeles japonica*, *Cornus alba*. Middle row: *Cotoneaster integerrimus*, *Deutzia scabra*, *Fagus sylvatica*, *Fraxinus excelsior* (leaflet of pinnately compound leaf). Bottom row: *Gymnocladus dioicus* (leaflet of pinnately compound leaf), *Juglans regia* (leaflet of pinnately compound leaf), *Lonicera involucrata*, *Prunus laurocerasus* and *Staphylea pinnata* (leaflet of pinnately compound leaf).

group maintains the largest leaf database of Central European wood species called MEW which contains the leafs of all domestic and most of the imported trees and shrubs growing in the Czech Republic (altogether 9,745 leaf items, representing 151 species) along with a web-based recognition system [81]. The classification algorithm used there [82] is based on moments and Fourier descriptors.

The aim of this experiment is twofold. First, we demonstrate the invariance and recognition power of the new blur invariants in situations when the blur model is exactly a convolution, N is known and we recognize blurred versions of the leaves from the training set. In this experiment the fact that the objects are leaves is not essential; the invariants behave in the same way for any object set.

The second experiment is more related to real leaf recognition. It is aimed to recognize the generic species from the blurred image of a leaf, which is *not* a member of the training set. Since here the degradation between the query image and the training template(s) includes not only the blur but namely the variations of the shape, color, size, etc., the task is much more challenging.

We selected 15 species (classes) of the MEW database. We intentionally chose the species with very similar leaves to make the recognition difficult even for humans. In the first experiment the training set consisted of only one leaf per class (see Fig. 6). We blurred each of these leaves ten times by a simulated out-of-focus blur of various size, so we obtained 150 blurred leaves to be recognized (see Fig. 7 for some examples). First we tried to recognize the blurred leaves by plain moments c_{11} and c_{22}, which are not invariant to blur. The space of these two features is depicted in Fig. 8. The clusters that correspond to the species are spread, overlap one another and cannot be separated (with two exceptions). The classification has failed completely, which illustrates the necessity of using blur invariants. Then the classification was done by means of the invariants $K_N(p,q)$ and the situation changed significantly. We achieved 100% success rate already for almost any couple of the invariants. The clusters are so compact that they look like single points

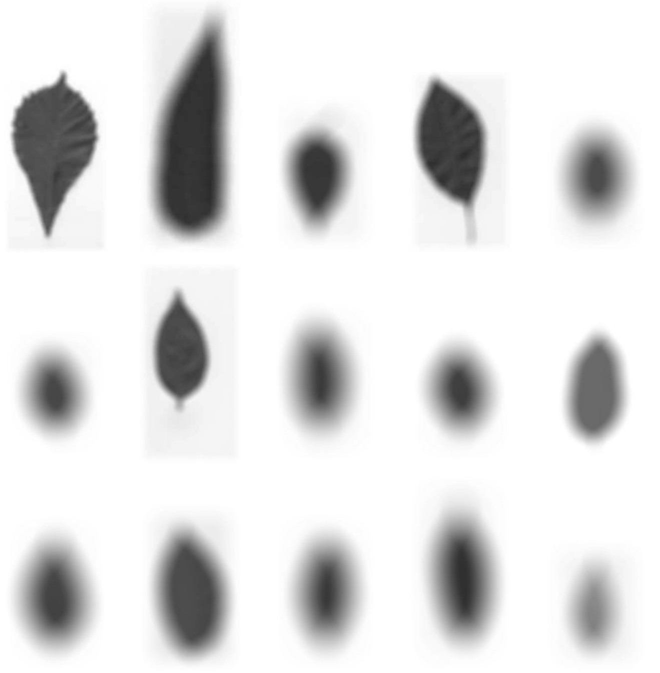

Fig. 7. Examples of the blurred leaves.

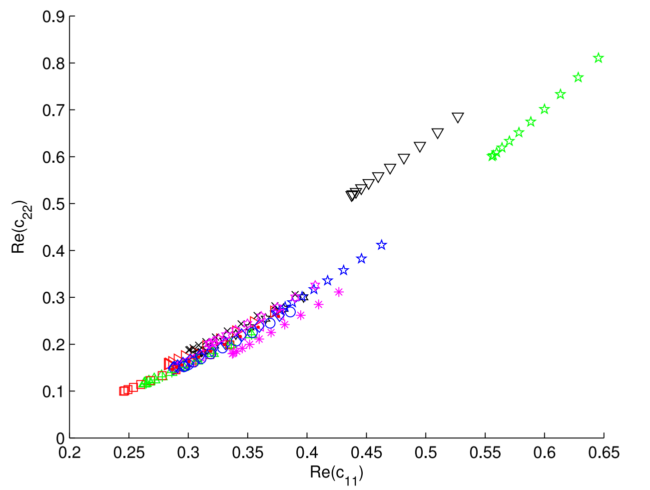

Fig. 8. The feature space of two non-invariant moments c_{11} and c_{22}. Legend: ✡– *Aesculus hippocastanum*, ☆– *Ailanthus altissima*, ▷– *Betula papyrifera*, ◁– *Cornus alba*, △– *Cotoneaster integerrimus*, ▽– *Deutzia scabra*, ◇– *Euonymus europaea*, □– *Fagus sylvatica*, ∗– *Fraxinus excelsior*, +– *Gymnocladus dioicus*, ×– *Chaenomeles japonica*, ○– *Juglans regia*, ·– *Lonicera involucrata*, ✡– *Prunus laurocerasus*, ☆– *Staphylea pinnata*.

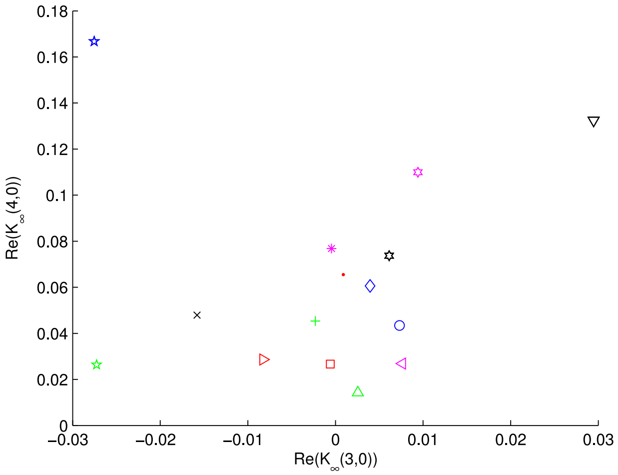

Fig. 9. The feature space of real parts of the invariant $\mathcal{R}eK_\infty(3,0)$ and $\mathcal{R}eK_\infty(4,0)$. Each cluster is formed by ten blurred images and one training image of the leaf. Legend: ✡– *Aesculus hippocastanum*, ☆– *Ailanthus altissima*, ▷– *Betula papyrifera*, ◁– *Cornus alba*, △– *Cotoneaster integerrimus*, ▽– *Deutzia scabra*, ◇– *Euonymus europaea*, □– *Fagus sylvatica*, ✳– *Fraxinus excelsior*, +– *Gymnocladus dioicus*, ×– *Chaenomeles japonica*, ○– *Juglans regia*, •– *Lonicera involucrata*, ✡– *Prunus laurocerasus*, ☆– *Staphylea pinnata*.

and are easy to separate (see Fig. 9 for an example). In this experiment the success rate almost did not depend on the particular invariants and, for $N \geq 4$, on the choice of N.

In the second experiment we used the same 15 classes but now each class was represented by 20 training leaves. Then we created an independent test set consisting of 10 leaves per class, each of them degraded by an out-of-focus blur. Each test leaf was classified by a minimum-distance rule in the space of the invariants $K_N(p,q)$ up to the order 16. The success rate was about 70% for any $N \geq 8$. Since in this case theoretically the true PSF fold number is infinity, the $K_N(p,q)$'s should be invariant for any choice of N. However, as we explained in Section 2, the lower N the less recognition power—if $K_2(p,q)$ were used, the success rate was only about 50%.

The success rate of course depends on the number of classes and the size of the training set. For bigger training sets and/or for classes which are more distinguishable, the success rate is between 80 and 90 percent. The achieved success rate 71 percent is surprisingly high. The moments were not specifically designed to tolerate the intra-class variations of the leaves of the same species and emphasize the differences between the species but apparently they exhibit this ability while providing invariance to blurring. As a concluding test, we classified the leaves by the system [81], which uses mainly Fourier descriptors as the features. The recognition rate was only 23 percent because this system was not designed to be robust to the image blurring. This illustrates one possible application of the blur invariants—they can be incorporated into existing recognition systems even if they use different principles in order to achieve robustness to low-pass filtering.

4.3 Matching of Blurred Templates

This experiment was performed on real data under challenging conditions where reaching a good matching score is difficult. We intentionally choose the scenario with a heavy blur and the scenes with many self-similar parts in order to make differences between the performance of various features apparent. We used the camera Nikon D5100 with seven diaphragm blades and adjusted the aperture such that the blades form a clear 7-fold symmetric PSF (see Fig. 10 for the image of a bright point). We took a pair of images, one sharp and the other one out of focus, of the size $2{,}048 \times 2{,}048$ pixels (see Fig. 11; note the PSF shape which is apparent on the blurred image).

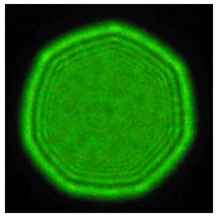

Fig. 10. The 7-fold PSF captured as the image of a LED diode.

We selected randomly 230 circular templates of the radius 200 pixels in the blurred image. Each template was matched against the sharp image by a full search over the whole scene, without using any prior information about its position. The matching criterion was the minimum distance in the space of blur invariants. We used the invariants K_2 and K_7 up to the sixth order. Since we know the ground truth, we can measure the matching error (i.e. the Euclidean distance between the correct and matched location). First, we compared the invariants by the absolute number of accurate matches. The match is considered accurate if the error is less than 10 pixels (we choose this threshold because multichannel processing algorithms such as [83] which, are in practice applied after the matching, are able to compensate for such registration errors). The invariants K_2 yielded 47 correct matches while K_7 yielded 64 correct matches. We used for comparison also plain moments, which is a well known matching technique, which lead to only 45 correct matches. The absolute number of correct matches might seem to be low even in the case of K_7 but one should keep in mind that the task was intentionally difficult and the impact of the boundary effect (which is given by the ratio between template and blur sizes) is extraordinary high.

Since the evaluation by the number of correct matches does not take into account the errors of other trials, we evaluated the results also in a different way using all trials. Assuming the errors in horizontal and vertical directions are independent and identically normally distributed, then the Euclidean errors in 2D have Rayleigh distribution. This distribution has a single parameter σ which defines both

(a) (b)

Fig. 11. Sharp test image (left) and out-of-focus image of the same scene (right). The 7-fold PSF shape is apparent on the blurred image.

mean and variance and can be easily estimated from the data. Now the matching methods can be objectively compared by respective σ—the lower its value, the better the method. The σ-value for K_2 was 268 while for K_7 only 145, which again illustrates the superiority of K_7 invariants in this case.

We repeated this experiment with other images and another camera having circular aperture. Although particular values vary, the main trend was always the same—the invariants which correspond to the actual PSF shape perform better than the others.

4.4 Registration of Blurred Images

Image registration in general is a process of overlaying two or more images of the same scene. It is one of the most important and most frequently discussed topics of image processing in the literature (see [84] for a survey). In many applications, blurred frames are thrown away and not processed. This is why the traditional methods do not consider image blur, cannot handle it properly and usually fail when registering blurred images. However, there are situations where the images are inevitable blurred by camera shake, wrong focus, atmospheric turbulence, sensor imperfection, low resolution and other factors. A typical example is taking pictures from a hand by a cell-phone or a compact camera. In last few years new application areas have appeared, which inherently require registration of blurred low-resolution images—multichannel blind deconvolution [30] and/or superresolution [31] of still images or video frames. Hence, a development of special methods for registering blurred images is highly desirable.

Registration methods for blurred images can be, as well as the general-purpose registration methods, divided into two groups—global and landmark-based ones. Global methods do not search for particular landmarks in the images but rather try to estimate the between-image transformation directly. Most blur-invariant global methods were motivated by traditional phase correlation [85], where the blur-insensitivity was achieved by modifications of the cross-power spectrum [41], [46], [47], [86]. Global methods are fast and easy to implement, but their limiting requirements—simple between-frame distortion (most of them allow only translation and/or rotation), the need for a large overlap of the images and the assumption of the uniform blur—might be a drawback in more complicated situations.

Since their first appearance in 1996, invariants to a centrosymmetric blur ($N = 2$) have been several times successfully used as the features in landmark-based registration of remotely sensed [7], [10], [11], [12], [33], [87], medical [13], [14], [15], indoor [88] and outdoor [49], [55], [57] scenes. As we demonstrate, introducing new invariants to N-FRS blur with higher discrimination power broadens their registration capability.

In this experiment we register out-of-focus images that are shifted and rotated with respect to one another. Both blur and rotation are real and were intentionally introduced by the camera setting. We first studied the PSF of the camera and found it to be circularly symmetric (at least in the first-order approximation, see Fig. 12). Hence, we employed the invariants $K_\infty(p, q)$ or, more precisely, their rotation-

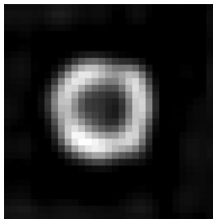

Fig. 12. The PSF of the camera used in the bookcase experiment.

invariant magnitudes. To show the differences in recognition power, we repeated the same experiment using $K_2(p, q)$. Since the images do not have a complete overlap, we apply the invariants locally as described below.

First, control point candidates (CPC's) are detected both in the reference and the sensed images. Significant corners and other corner-like dominant points are considered as the candidates. To detect them, a method developed particularly for blurred images [89] is employed. We detected 15 most prominent CPC's in each frame. To establish the correspondence between the CPC's, we calculated the blur invariants up to the order r over a circular neighborhood of radius 14 pixels of each CPC. The CPC's are matched in the space of the invariants by minimum-distance rule and two pairs of the most similar CPC's are found. This is the most complicated part which is influenced by the blur. Having found these two pairs, the rest of the procedure is obvious. We estimate the translation, rotation and scale parameters, transform one set of the CPC's over the other one and match the rest of the CPC's in the image domain. The invariants may serve as a consistency check.[11] After establishing the correspondence between all CPC's (those CPC's having no close-enough counterpart are rejected), we calculate the final affine mapping parameters via least-square fit and resample and overlay the sensed image over the reference one.

First we set $r = 5$. Since in that case both K_∞ and K_2 exhibit enough discrimination power, a correct match was found in each case as can be verified visually in Fig. 13 (note that the two initial matching pairs found by K_∞ and K_2 are different but both correct). The same situation occurs for $r > 5$. Now let us repeat the experiment with setting $r = 4$. The invariants K_∞ found a correct match again while K_2 failed (check Fig. 14). The reason is that K_∞ provide better discrimination because they are designed particularly for circularly symmetric blurs. If we restrict to low order invariants $r = 3$ only, both methods fail—the similarity of different CPC's is too high to be distinguished by so few features.

We finally registered (using K_∞ and $r = 4$) and subtracted the images (see Fig. 15 for the difference image). There are slight misregistration artifacts but in this experiment we do not want to measure the overall registration accuracy. We are only interested in the validity of the CPC matching step. If the matching by invariants failed, it would

11. Additional refinement step may be inserted here. For every pixel from a small neighborhood of the CP, its invariant vector is calculated. The point having the minimum distance to the invariant vector of the CP counterpart is set as the refined position of the CP. By an interpolation in the distance matrix, we can even achieve subpixel accuracy. Since this refinement has usually only slight impact on the transformation parameters, we did not apply it in this experiment.

Fig. 13. The bookcase, $r = 5$. Small circles shows the detected CPC's. The two initial matching pairs are denoted by "1" and "2". Correct match found by K_∞ (top) and K_2 (bottom).

result in heavy artifacts in the difference image, which is not the case here. Hence, all CPC's were matched correctly, even if their initial positions might be detected with slight errors.

The above described registration algorithm has, in addition to r and N the choice of which we already discussed, another user-defined parameter—the radius R of the neighborhood the invariants are calculated from. Its choice is influenced by the type of the scene and by the extent of the blur. There is no explicit relationship between R and other parameters, just heuristic conclusions based on our experiments can be made. Generally, the higher R the better discriminability and the higher computing complexity. In a "normal" clear image, R around 10 should be sufficient. If the image contains periodic structures and thus many similar CPC's, higher R is required. In a blurred image there is an additional constraint—to keep the boundary effect insignificant, R should be chosen such that the size of the blur does not exceed 10-15% of the size of the R-neighborhood. The order of the used invariants r and the radius R are "inversely proportional". Increasing one of them allows to decrease (up to some limit) the other one and vice versa. For the given image, the upper bound of R is set up by the homogeneity constraint—the blurring PSF can be different for different neighborhoods but should not vary within each of them.

Fig. 14. The bookcase, $r = 4$. Small circles shows the detected CPC's (they are the same as in Fig. 13). The two initial matching pairs are denoted by "1" and "2". Correct match found by K_∞ (top), incorrect match by K_2 (bottom).

Fig. 15. The difference of two registered bookcase images using K_∞ and $r = 4$. The slight misregistration artifacts are cased mainly by the fact that the actual transformation between the images includes also perspective projection and possibly also by an inaccurate CPC localization. The CPC matching itself was error-free.

5 Conclusion

In this paper we revised and substantially generalized the theory of blur invariants. We presented the invariants to convolution with an arbitrary N-fold symmetric PSF, both in Fourier and image domain. We introduced the notion of a *primordial image* as a canonical form of blurred images. This construct is defined in spectral domain by means of projection operators. We proved that the moments of the primordial image are invariant to blur and we derived recursive formulae for their direct computation without actually constructing the primordial image. We further proved they form a complete set of invariants. We also showed how to easily extent their invariance also to rotation. We discussed the properties of the proposed invariants and the implementation issues. We illustrated by experiments the recognition power of the new invariants and showed how it depends on N. We envisage the application of the new theory in tasks and systems where one has to recognize and/or register blurred images.

We can see future challenges on this field namely in constructing combined invariants to N-fold symmetric blur and affine transform, because we can meet affine-deformed images in practice often. Unlike translation, rotation and scaling, the affine transform does not preserve N-fold symmetries (except $N = 2$), so the extension is not straightforward even if both affine and blur invariants are known. Another direction is to look for invariants w.r.t. specific types of blur (Gaussian, motion, vibration, etc.). Knowing the parametric expressions of the PSF, we should be able to derive more invariants than under a general assumption of certain symmetry. Considering extensions to signals of other dimensions than two, probably only 3D case is meaningful because 1D is irrelevant (except $N = 2$) and higher dimensions are extremely complicated and of limited applicability. Even in 3D the set of possible symmetries of the PSF is much more rich than in 2D and a straightforward extension of the 2D theory is impossible.

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