COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **14**, Number **5**, September **2015**

doi:10.3934/cpaa.2015.14.1685

pp. **1685–1704**

FINITE-DIMENSIONAL GLOBAL ATTRACTORS FOR PARABOLIC NONLINEAR EQUATIONS WITH STATE-DEPENDENT DELAY

Igor Chueshov

Department of Mechanics and Mathematics
Karazin Kharkov National University, Kharkov, 61022, Ukraine

Alexander Rezounenko

Department of Mechanics and Mathematics
Karazin Kharkov National University, Kharkov, 61022, Ukraine, and
Institute of Information Theory and Automation, Academy of Sciences
of the Czech Republic, P.O. Box 18, 182 08 Praha, CR

(Communicated by Alain Miranville)

Abstract. We deal with a class of parabolic nonlinear evolution equations with state-dependent delay. This class covers several important PDE models arising in biology. We first prove well-posedness in a certain space of functions which are Lipschitz in time. This allows us to show that the model considered generates an evolution operator semigroup S_t on a certain space of Lipschitz type functions over delay time interval. The operators S_t are closed for all $t \geq 0$ and continuous for t large enough. Our main result shows that the semigroup S_t possesses compact global and exponential attractors of finite fractal dimension. Our argument is based on the recently developed method of quasi-stability estimates and involves some extension of the theory of global attractors for the case of closed evolutions.

1. **Introduction.** Differential equations with different types of delay attract much attention during last decades. Including delay terms in differential equations is a natural step of taking into account that many of real-world problems depend on the pre-history of the evolution. Delay terms in an equation reflect a well-understood phenomenon that evolution of a state of a system depends not only on this state but rather on the states during some previous interval of time (memory of the system). This leads to infinite-dimensional dynamics even in the case of ordinary differential equations. The general theory of delay differential equations was initially developed for the simplest case of constant delays. We cite just classical monographs [2, 13, 18] on ordinary differential equations (ODEs) and milestone articles [16, 37] on partial differential equations (PDEs) with constant delays. On the other hand it is clear that the constancy of the delay is just an extra assumption made to simplify the study, but it is not really well-motivated by real-world models. To describe a process more naturally a new class of *state-dependent delay* models was introduced and

2000 *Mathematics Subject Classification.* Primary: 35R10, 35B41; Secondary: 93C23.

Key words and phrases. Parabolic evolution equations, state-dependent delay, global attractor, finite-dimension, exponential attractor.

intensively studied during last decades. We mention works on ODEs [14, 20, 22, 38] and on PDEs [12, 29, 30, 32, 33] with state-dependent delays.

The simplest case of a state-dependent delay is a delay explicitly given by a real-valued function $\eta : \mathbb{R} \to \mathbb{R}_+$ which depends on the value $x(t)$ at the reference time t but not on previous values of the solution $\{x(\tau), \tau \leq t\}$. This leads to terms of the form $f(x(t - \eta(x(t)))$ in the model considered. Even in this case the non-uniqueness could appear (see the scalar ODE example constructed by R.Driver [14] in 1963 for initial data from the space of continuous functions on the delay interval). The standard way for general models to avoid non-uniqueness in the case of infinite-dimensional dynamics is to consider smoother (narrower) classes of solutions. However in this case the existence problem may become critical. The main task is to find a good balance between these two issues.

In this paper we deal with a certain abstract parabolic problem with the state dependent delay term of a rather general structure. Our considerations are motivated by several biological models, see the discussion and the references in [3], [17] and [33]. Our main goal in this paper is to find appropriate phase spaces in which we can establish the well-posedness of our model and study its long-time (qualitative) dynamics.

Our first result (Theorem 3.2) states well-posedness of the problem and allows us to define an evolution semigroup S_t of closed mappings on a certain Banach space of functions on the delay time interval with values in an appropriate Hilbert space. In some sense this result extends the well-posedness statements in [30, 32, 33] to more general delay terms. The main result of the paper (Theorem 4.2) states the existence of a global *finite-dimensional* attractor, the object which is responsible for long-time dynamics. We also show that the model possesses an exponential fractal global attractor (see the definition in the Appendix).

Although for some parabolic problems with state-dependent delay the existence of compact global attractors was established earlier in [30, 33], to the best of our knowledge, results on *finite-dimensional* behavior for parabolic state-dependent delay problems were not known before. The main difficulty is related to the fact that the corresponding delay term is not Lipschitz on the natural energy balance space. We also mention that our Theorem 4.2 can be applied in the situation considered in [33] and gives the finite-dimensionality of the global attractor constructed in that paper.

We note that the evolution operators S_t we construct are not continuous mapping on the phase space for t small enough. Therefore to prove the existence of a compact global attractor we use the extension of the standard theory suggested in [28]. As for dimension issues we apply the idea of the method of quasi-stability estimates developed earlier in [6, 7, 8, 9] for the second order in time evolution models which generate continuous evolution semigroups. This is possible in our case due to the continuity of evolution operator for large times. We note that in the delay case the quasi-stability method was applied earlier in [10, 11, 12] for second order models, see also [5, Chapter 6].

2. **Model description.**

We deal with well-posedness and long-time dynamics of abstract evolution equations with delay of the form

$$\dot{u}(t) + Au(t) + F(u_t) + G(u(t)) = h, \quad t > 0, \tag{1}$$

in some Hilbert space H. Here the dot over an element means time derivative, A is linear and F, G are nonlinear operators, $h \in H$. The term $F(u_t)$ represents (nonlinear) delay effect in the dynamics. As usually for delay equations, the history segment (the state) is denoted by $u_t \equiv u_t(\theta) \equiv u(t+\theta)$ for $\theta \in [-r, 0]$. Here and below r represents the (maximal) delay. The analysis is carried out for an arbitrary (but fixed) $0 < r < \infty$. The case of systems with infinite time memory ($r = +\infty$) is beyond the scope of the theory developed.

Assumption 2.1 (Basic Hypotheses)**.** In our study we assume that:

(A) A is a positive operator with a discrete spectrum in a separable Hilbert space H with a dense domain $D(A) \subset H$. Hence there exists an orthonormal basis $\{e_k\}$ of H such that

$$Ae_k = \lambda_k e_k, \quad \text{with } 0 < \lambda_1 \le \lambda_2 \le \dots, \qquad \lim_{k\to\infty} \lambda_k = \infty.$$

We define the spaces H_α which are $D(A^\alpha)$ for $\alpha \ge 0$ (the domain of A^α) and the completions of H with respect to the norm $\|A^\alpha \cdot\|$ when $\alpha < 0$ (see, e.g., [25]). Here and below, $\|\cdot\|$ is the norm of H, and $\langle \cdot, \cdot \rangle$ is the corresponding scalar product. For $r > 0$, we denote for short $C_\alpha = C([-r, 0]; H_\alpha)$ which is a Banach space with the norm

$$|v|_{C_\alpha} \equiv \sup\{\| v(\theta) \|_\alpha : \theta \in [-r, 0]\},$$

where $\|v\|_\alpha = \|A^\alpha v\|$ is the norm in H_α for $\alpha \in \mathbb{R}$. We also write $C = C_0$.

(F) The delay term $F(u_t)$ has the form $F(u_t) \equiv F_0(u(t - \eta(u_t)))$, where **(a)** $F_0 : H_\alpha \mapsto H_\alpha$ is globally Lipschitz for $\alpha = 0$ and $\alpha = -1/2$, i.e., there exists $L_F > 0$ such that

$$\|F_0(v) - F_0(u)\|_\alpha \le L_F \|v - u\|_\alpha, \ v, u \in H_\alpha, \ \alpha = 0, -1/2; \tag{2}$$

and **(b)** $\eta : C \equiv C([-r, 0]; H) \mapsto [0, r] \subset \mathbb{R}$ is globally Lipschitz:

$$|\eta(\phi) - \eta(\psi)| \le L_\eta |\phi - \psi|_C, \ \phi, \psi \in C([-r, 0]; H). \tag{3}$$

(G) $G : H_{1/2} \mapsto H$ is locally Lipschitz, i.e.

$$\|G(v) - G(u)\| \le L_G(R) \|v - u\|_{1/2}, \ v, u \in H_{1/2}, \ \|v\|_{1/2}, \|u\|_{1/2} \le R, \tag{4}$$

where $L_G : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. We also assume that G is a potential mapping, the latter means that there exists a (Frechét differentiable) functional $\Pi(u) : H_{1/2} \to \mathbb{R}$ such that $G(u) = \Pi'(u)$ in the sense

$$\lim_{\|v\|_{1/2}\to 0} \|v\|_{1/2}^{-1} \big[\Pi(u+v) - \Pi(u) + \langle G(u), v\rangle\big] = 0.$$

Moreover, we assume that **(a)** there exist positive constants c_1 and c_2 such that

$$\langle G(u), Au\rangle \ge -c_1 \|A^{\frac{1}{2}} u\|^2 - c_2, \quad u \in D(A); \tag{5}$$

and **(b)** there exist $\delta > 0$ and $m \ge 0$ such that $G : H_{1/2-\delta} \mapsto H_{-m}$ is continuous.

Our main motivating example of a system with discrete state-dependent delay is the following one:

$$\frac{\partial}{\partial t} u(t,x) - \Delta u(t,x) + b\left(B[u(t - \eta(u_t), \cdot)](x)\right) + g(u(t,x)) = h(x), \quad x \in \Omega, \ t > 0, \tag{6}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $B : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator and $b : \mathbb{R} \to \mathbb{R}$ stands for a Lipschitz map. The function $\eta : C([-r, 0]; L^2(\Omega)) \to [0, r] \subset \mathbb{R}_+$ denotes *a state-dependent discrete delay*. The Nemytskii operator $u \mapsto g(u)$ with C^1 function g represents a nonlinear non-delayed reaction term and $h(x)$ describes sources. The form of the delay term is motivated by models in population dynamics where function b is a birth function (could be $b(s) = c_1 s \cdot e^{-c_2 s}$, with $c_1, c_2 > 0$) and the delay η represents the maturity age. For more detailed discussion and further examples (the diffusive Nicholson blowflies equation, Mackey-Glass equation - the diffusive model of Hematopoiesis - blood cell production, the Lasota-Wazewska-Czyzewska model in hematology) with state-dependent delay we refer to [17] and [33] and to the references therein. We note that several special cases of the model in (6) were studied in [30, 31, 32, 33]). For instance it was assumed in [33] that $g(s) \equiv 0$, $b(s)$ is a bounded function, and B is an integral compact linear operator. This leads to nonlocal (in space) models. Our assumptions cover the non-compact case. We can take $b(s) = s$ and $B = Id$, for instance. We also note that if we equip (6) with the Dirichlet boundary condition, then the dissipativity property in (5) holds provided $g \in C^1(\mathbb{R})$, $g(0) = 0$ and the derivative $g'(s)$ is bounded from below. This follows by the standard integration by parts. Thus population dynamics models with nonlinear sink/source feedback terms can be included in consideration. For this kind of biological models, but with state-*independent* delay, we refer to [39].

We equip the equation (1) with the initial condition

$$u(\theta) = \varphi(\theta), \;\; \theta \in [-r, 0], \tag{7}$$

and for initial data φ consider the space

$$\mathcal{C}L \equiv \left\{ \varphi \in C([-r,0]; H) \;\middle|\; \mathrm{Lip}_{[-r,0]}(A^{-\frac{1}{2}}\varphi) < +\infty; \;\; \varphi(0) \in D(A^{\frac{1}{2}}) \right\}, \tag{8}$$

where

$$\mathrm{Lip}_{[a,b]}(\varphi) \equiv \sup_{s \neq t} \left\{ \frac{\|\varphi(s) - \varphi(t)\|}{|s-t|} \; : \; s,t \in [a,b], \;\; s \neq t \right\}$$

denotes the corresponding Lipschitz constant. One can show that the space $\mathcal{C}L$ consists of continuous functions φ on $[-r, 0]$ with values in H such that $\varphi(0) \in H_{1/2}$ and which are absolutely continuous in $H_{-1/2}$. The latter means that there exists the derivative $\dot{\varphi} \in L^\infty(-r, 0; H_{-1/2})$ such that

$$\varphi(s) = \varphi(0) - \int_s^0 \dot{\varphi}(\xi) d\xi, \;\; s \in [-r, 0],$$

and

$$\mathrm{Lip}_{[-r,0]}(A^{-\frac{1}{2}}\varphi) = \operatorname{ess\,sup} \left\{ \|A^{-\frac{1}{2}}\dot{\varphi}(s)\| \; : \; s \in [-r,0] \right\} \equiv |\dot{\varphi}|_{L^\infty(-r,0;H_{-1/2})}.$$

We equip the space $\mathcal{C}L$ with the natural norm

$$|\varphi|_{\mathcal{C}L} \equiv \max_{s \in [-r,0]} \|\varphi(s)\| + \mathrm{Lip}_{[-r,0]}(A^{-\frac{1}{2}}\varphi) + \|A^{\frac{1}{2}}\varphi(0)\|. \tag{9}$$

We note that the delay term $F(\varphi) \equiv F_0(\varphi(-\eta(\varphi)))$ in (1) is well-defined for every $\varphi \in C$ and possesses the property (see (2) for $\alpha = 0$) $\|F(\varphi) - F(0)\| \leq L_F \|\varphi(-\eta(\varphi))\| \leq L_F |\varphi|_C$, hence

$$\|F(\varphi)\| \leq c_1 + c_2 |\varphi|_C, \;\; \varphi \in C, \tag{10}$$

with $c_1 = \|F(0)\|$ and $c_2 = L_F$. However it is not Lipschitz on the space C. One can only show that the delay term F satisfies the inequality

$$\|F(\varphi) - F(\psi)\|_{-1/2} \leq L_F \left(1 + L_\eta \mathrm{Lip}_{[-r,0]}(A^{-\frac{1}{2}}\varphi)\right) |\varphi - \psi|_C \tag{11}$$

for every $\varphi \in \mathcal{C}L$ and $\psi \in C$. Using the terminology of [26] we can call this mapping F "almost Lipschitz" from C into $H_{-1/2}$, see also a discussion in [20].

Remark 1. We can also include in (1) a delay term $M(u_t)$ which is defined by a globally Lipschitz function from $C([-r,0];H_{1/2})$ into H. We will not pursue this generalization because our main goal is state-dependent delay models.

3. Well-posedness.

In this section we prove the existence and uniqueness theorem and study properties of solutions. Then we use these results to construct the corresponding evolution semigroup and describe its dynamical properties.

We introduce the following definition.

Definition 3.1 (Strong solution)**.** A vector-function

$$u(t) \in C([-r,T];H) \cap C([0,T];H_{1/2}) \cap L^2(0,T;H_1) \tag{12}$$

is said to be a (strong) solution to the problem defined by (1) and (7) on $[0,T]$ if

- **(a)** $u(\theta) = \varphi(\theta)$ for $\theta \in [-r,0]$;
- **(b)** $\forall v \in L^2(0,T;H)$ such that $\dot v \in L^2(0,T;H_{-1})$ and $v(T)=0$ we have that

$$-\int_0^T \langle u(t), \dot v(t)\rangle\, dt + \int_0^T \langle Au(t), v(t)\rangle\, dt + \int_0^T \langle F(u_t) + G(u(t)), v(t)\rangle\, dt = \langle \varphi(0), v(0)\rangle + \int_0^T \langle h, v(t)\rangle\, dt. \tag{13}$$

Remark 2. Let $u(t)$ be a strong solution on an interval $[0,T]$ with some initial data $\varphi \in C$. Then it follows from (12) and also from (4) and (10) that

$$F(u_t) + G(u(t)) - h \in L^\infty(0,T;H).$$

This allows us to conclude from (12) and (13) that

$$\dot u(t) \in L^\infty(0,T;H_{-1/2}) \cap L^2(0,T;H). \tag{14}$$

Moreover, the relation in (13) implies that $u(t)$ satisfies (1) for almost all $t \in [0,T]$ as an equality in H. We also note that relations (12) and (14) yield

$$u_t \in \mathcal{C}L \text{ for every } t \in [0,T] \text{ and } \max_{[0,T]} |u_t|_{\mathcal{C}L} < +\infty \tag{15}$$

for every strong solution u with initial data φ from the space $\mathcal{C}L$ defined in (8).

Our first result is the following theorem on the existence and uniqueness of solutions.

Theorem 3.2. *Let Assumption 2.1 be in force. Assume that $\varphi \in \mathcal{C}L$, see (8). Then the initial-value problem defined by (1) and (7) has a unique strong solution on any time interval $[0,T]$. This solution possesses the property*

$$\dot u(t) \in C([0,T];H_{-1/2}) \cap L^2(0,T;H) \tag{16}$$

and satisfies the estimate

$$\|A^{-1/2}\dot u(t)\|^2 + \|A^{1/2}u(t)\|^2 + \int_0^t \left[\|\dot u(\tau)\|^2 + \|Au(\tau)\|^2\right] d\tau \le C_T(R) \tag{17}$$

for all $t \in [0,T]$ and $\|A^{1/2}\varphi(0)\|^2 + |\varphi|_C^2 \le R^2$. Moreover, for every two strong solutions u^1 and u^2 with initial data φ^1 and φ^2 from $\mathcal{CL}$ we have that

$$\sup_{\tau\in[0,t]} \|u^1(\tau)-u^2(\tau)\|^2+\int_0^t \|A^{1/2}(u^1(\tau)-u^2(\tau))\|^2\,d\tau \le C_R(T)|\varphi^1-\varphi^2|_C^2,\ \forall\, t\in[0,T], \tag{18}$$

for all φ^i such that $|\varphi^i|_{\mathcal{CL}} \le R$.

Proof. To prove the existence we use the standard compactness method [24] based on Galerkin approximations with respect to the eigen-basis $\{e_k\}$ of the operator A (see Assumption 2.1 (A)).

We define a Galerkin approximate solution of order m by the formula

$$u^m = u^m(t) = \sum_{k=1}^m g_{k,m}(t)e_k,$$

where the functions $g_{k,m}$ are defined on $[-r,T]$, absolutely continuous on $[0,T]$ and such that the following equations are satisfied

$$\left\{\begin{array}{l} \langle \dot u^m + Au^m + F(u_t^m) + G(u^m) - h, e_k\rangle = 0,\ t>0, \\ \langle u^m(\theta), e_k\rangle = \langle \varphi(\theta), e_k\rangle,\ \forall \theta\in[-r,0],\ \forall k=1,\dots,m. \end{array}\right. \tag{19}$$

The equation in (19) is a system of (ordinary) differential equations in $\mathbb{R}^m$ with a concentrated (discrete) state-dependent delay[1] for the unknown vector function $U(t) \equiv (g_{1,m}(t),\dots,g_{m,m}(t))$

The condition $\varphi \in \mathcal{CL}$ implies that the function $U(\cdot)|_{[-r,0]} \equiv P_m\varphi(\cdot)$, which defines initial data, is Lipschitz continuous as a function from $[-r,0]$ to $\mathbb{R}^m$. Here P_m is the orthogonal projection onto the subspace $\mathrm{Span}\,\{e_1,\dots,e_m\}$. Hence, we can apply the theory of ODEs with discrete state-dependent delay (see e.g. [20]) to get the *local* existence of solutions to (19).

Next, we derive an a priori estimate which allows us to extend solutions u^m to (19) on an arbitrary time interval $[0,T]$. We also use it for the compactness of the set of approximate solutions.

We multiply the first equation in (19) by $\lambda_k g_{k,m}$ and sum for $k=1,\dots,m$ to get

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u^m(t)\|^2 + \|Au^m(t)\|^2 + \langle F(u_t^m) + G(u^m(t)) - h, Au^m(t)\rangle = 0.$$

Due to (10) and (5) this implies that

$$\begin{aligned} \frac{d}{dt}\left[\|A^{1/2}u^m(t)\|^2 + \int_0^t \|Au^m(\tau)\|^2 d\tau\right] &\le c\big[1 + |u_t^m|_C^2 + \|A^{1/2}u^m(t)\|^2\big] \\ &\le c_0\big[1+|\varphi|_C^2\big] + c_1 \max_{\tau\in[0,t]} \|A^{1/2}u^m(\tau)\|^2 \end{aligned}$$

Integrating the last inequality we can easily see that the function

$$\Psi(t) = \max_{\tau\in[0,t]} \|A^{1/2}u^m(\tau)\|^2 + \int_0^t \|Au^m(\tau)\|^2 d\tau$$

satisfies the inequality

$$\Psi(t) \le 2\|A^{1/2}\varphi(0)\|^2 + 2tc_0\big[1+|\varphi|_C^2\big] + 2c_1\int_0^t \Psi(\tau)d\tau.$$

[1] For the corresponding ODE theory see [38] and also the survey [20].

Therefore Gronwall's lemma gives us the a priori estimate

$$\|A^{1/2}u^m(t)\|^2 + \int_0^t \|Au^m(\tau)\|^2\,d\tau \le 2e^{at}\left[\|A^{1/2}\varphi(0)\|^2 + bt\big[1+|\varphi|_C^2\big]\right], \quad (20)$$

for all t from an existence interval, where a and b are positive constants. This a priori estimate allows us to extend approximate solutions on every time interval $[0,T]$ such that (20) remains true for every $t\in[0,T]$.

Now we establish additional a priori bounds. Using (20), (4) and (10) from the first equation in (19) we obtain that

$$\|\dot u^m(t) + Au^m(t)\| \le \|F(u_t^m)\| + \|G(u^m(t))\| + \|h\| \le C(R,T),\ t\in[0,T],$$

provided $\|A^{1/2}\varphi(0)\|^2 + |\varphi|_C^2 \le R^2$. Thus by (20) we obtain the estimate

$$\|A^{1/2}u^m(t)\|^2 + \int_0^t \left[\|\dot u^m(\tau)\|^2 + \|Au^m(\tau)\|^2\right]\,d\tau \le C_T(R) \quad (21)$$

for all $t\in[0,T]$ and $\|A^{1/2}\varphi(0)\|^2 + |\varphi|_C^2 \le R^2$. It also follows from (19) that

$$\sup_{t\in[0,T]} \|A^{-1/2}\dot u^m(t)\|^2 \le C_T(R). \quad (22)$$

Thus

$$\{u^m\}_{m=1}^\infty \text{ is a bounded set in } W_1 \equiv L^\infty(0,T;H_{1/2})\cap L^2(0,T;D(A))$$

and

$$\{\dot u^m\}_{m=1}^\infty \text{ is a bounded set in } W_2 \equiv L^\infty(0,T;H_{-1/2})\cap L^2(0,T;H).$$

Hence, there exist a subsequence $\{(u^k;\dot u^k)\}$ and an element $(u;\dot u)\in Z_1\equiv W_1\times W_2$ such that

$$\{(u^k;\dot u^k)\} \text{ *-weakly converges to } (u;\dot u) \text{ in } Z_1.$$

By the Aubin-Dubinski theorem [34, Corollary 4] we also have

$$u^k \to u \text{ in } C(0,T;H_{1/2-\delta}))\cap L^2(0,T;H_{1-\delta}) \text{ as } k\to+\infty.$$

Now the proof that any *-weak limit $u(t)$ is a solution is standard. To make the limit transition in the nonlinear terms F and G we use relation (11) and Assumption 2.1**(Gb)**.

The property $u(t)\in C([0,T];H_{1/2}))$ follows from the well-known continuous embedding (see [25, Theorem 1.3.1] or [35, Proposition 1.2]):

$$\{u\in L^2(0,T;H_1) : \dot u\in L^2(0,T;H)\} \subset C([0,T];H_{1/2}).$$

The continuity of $\dot u$ in $H_{-1/2}$ follows from equation (1) and from continuity of u in $H_{1/2}$. Thus the existence of strong solutions is proved. It is easy to see from (21) and (22) that the strong solution constructed satisfies (17).

Now we prove the uniqueness.

Let u^1 and u^2 be two solutions (at this point we do not assume that they have the same initial data). Then the difference $z = u^1 - u^2 \in C([0,T];H_{1/2})\cap L^2(0,T;H_1)$ is a strong solution to the linear parabolic type (non-delay) equation

$$\dot z(t) + Az(t) = f(t),\ t>0,\ \text{with } f(t)\equiv F(u_t^2) - F(u_t^1) + G(u^2(t)) - G(u^1(t)). \quad (23)$$

By Remark 2 $f\in L^\infty(0,T;H)$. From (4) and (11) using (15) we also have that

$$\|G(u^2(t)) - G(u^1(t))\| \le L_G(\varrho)\|z(t)\|_{1/2},\ t\in[0,T],$$

and

$$\|A^{-1/2}(F(u_t^2) - F(u_t^1))\| \le L_F(1+L_\eta\varrho)|z_t|_C,\ t\in[0,T],$$

for every $\varrho \geq \max_{[0,T]}\left\{|u_t^1|_{\mathcal{C}L}+|u_t^2|_{\mathcal{C}L}\right\}$. Therefore

$$\begin{aligned}|\langle f(t),z(t)\rangle| \leq & L_F(1+L_\eta\varrho)|z_t|_C\|z(t)\|_{1/2}+L_G(\varrho)\|z(t)\|_{1/2}\|z(t)\| \\ \leq & \frac{1}{2}\|z(t)\|_{1/2}^2+C(\varrho)|z_t|_C.\end{aligned}$$

Thus using the standard multiplier z in (23) we obtain that

$$\frac{d}{dt}\|z(t)\|^2+\|A^{1/2}z(t)\|^2 \leq C(\varrho)\|z_t\|_C^2 \leq C(\varrho)\left[|\varphi^1-\varphi^2|_C^2+\sup_{\tau\in[0,t]}\|z(\tau)\|^2\right]$$

for every $\varrho \geq \max_{[0,T]}\left\{|u_t^1|_{\mathcal{C}L}+|u_t^2|_{\mathcal{C}L}\right\}$. Applying Gronwall's lemma we obtain

$$\sup_{\tau\in[0,t]}\|u^1(\tau)-u^2(\tau)\|^2+\int_0^t\|A^{1/2}(u^1(\tau)-u^2(\tau))\|^2d\tau \leq C(\varrho)|\varphi^1-\varphi^2|_C^2,\ \forall\, t\in[0,T], \tag{24}$$

This implies uniqueness of strong solutions.

As a by-product the uniqueness yields that *any* strong solution satisfies (17). Therefore we can apply (24) with some $\varrho=\varrho(R,T)$ to obtain (18).

Thus the proof of Theorem 3.2 is complete. □

Theorem 3.2 allows us to define an evolution semigroup S_t on the space $\mathcal{C}L$ (see (8)) by the formula

$$S_t\varphi \equiv u_t, \quad t\geq 0, \tag{25}$$

where $u(t)$ is the unique solution to the problem (1) and (7). We note that (18) implies that S_t is *almost* locally Lipschitz on C, i.e.,

$$|S_t\varphi^1-S_t\varphi^2|_C \leq C_R(T)|\varphi^1-\varphi^2|_C \text{ for every } \varphi^i\in\mathcal{C}L,\ |\varphi^i|_{\mathcal{C}L}\leq R,\ t\in[0,T].$$

However, it seems that a similar bound is not true in the space $\mathcal{C}L$. We can only guarantee that $\varphi\mapsto S_t\varphi$ is a continuous mapping on $\mathcal{C}L$ for all $t>r$. Moreover, the following assertion shows that the mapping $\varphi\mapsto S_t\varphi$ is even $\frac{1}{2}$-Hölder on $\mathcal{C}L$ with respect to φ when $t>r$.

Proposition 1 (Dependence on initial data in the space $\mathcal{C}L$)**.** *Assume that the hypotheses of Theorem 3.2 are in force. Let u^1 and u^2 be two solutions on $[0,T]$ with initial data φ^1 and φ^2 from $\mathcal{C}L$. Then the difference $z=u^1-u^2$ satisfies the estimate*

$$\begin{aligned}(t-r)&\left[\|A^{-1/2}\dot z(t)\|^2+\|A^{1/2}z(t)\|^2\right] \\ &+\int_r^t(\tau-r)\left[\|\dot z(\tau)\|^2+\|Az(\tau)\|^2\right]d\tau \leq C_T(R)|\varphi^1-\varphi^2|_C\end{aligned} \tag{26}$$

for all $t\in[r,T]$ and for all initial data φ^i such that $|\varphi^i|_{\mathcal{C}L}\leq R$. This implies that for every $t>r$ the evolution semigroup S_t is $\frac{1}{2}$-Hölder continuous in the norm of $\mathcal{C}L$. In the case when $t\in(0,r]$ we can guarantee the closedness of the evolution operator S_t only. This means[2] *(see, e.g., [28]) that the properties $\varphi_n\to\varphi$ and $S_t\varphi_n\to\psi$ in the norm of $\mathcal{C}L$ as $n\to\infty$ imply that $S_t\varphi=\psi$.*

[2]We refer to the Appendix for a discussion of closed evolutions. Here we only mention that any continuous mapping is closed and a mapping can be closed but not continuous, see examples in [28] and also in [5, Sect.1.1].

Proof. Multiplying (23) by Az and using (17) and (4) we obtain that

$$\frac{d}{dt}\|A^{1/2}z(t)\|^2 + \|Az(t)\|^2 \le \|F(u_t^2) - F(u_t^1)\|^2 + C_R(T)\|A^{1/2}z(t)\|^2,\ t > 0.$$

From (17), (2) and (3) we also have that

$$\begin{aligned}\|F(u_t^2) - F(u_t^1)\|^2 &\le 2L_F^2\left[\left|\int_{t-\eta(u_t^1)}^{t-\eta(u_t^2)} \|\dot u^2(\xi)\|d\xi\right|^2 + |u_t^2 - u_t^1|_C^2\right]\\ &\le 2L_F^2\left[\left|\eta(u_t^1) - \eta(u_t^2)\right|\int_{t-r}^{t}\|\dot u^2(\xi)\|^2 d\xi + |u_t^2 - u_t^1|_C^2\right]\\ &\le C_T(R)|u_t^2 - u_t^1|_C^2 \end{aligned}\tag{27}$$

for every $t \ge r$. Therefore

$$\frac{d}{dt}\|A^{1/2}z(t)\|^2 + \|Az(t)\|^2 \le C_T(R)\left[\max_{[0,t]}\|z(s)\|^2 + \|A^{1/2}z(t)\|^2\right]^{1/2},\ t \ge r.$$

Integrating over interval $[\tau, t]$ with $\tau \ge r$ and using (18) we obtain that

$$\|A^{1/2}z(t)\|^2 + \int_\tau^t \|Az(\xi)\|^2 d\xi \le \|A^{1/2}z(\tau)\|^2 + C_T(R)|\varphi^1 - \varphi^2|_C,\ t \ge \tau \ge r. \tag{28}$$

Now we integrate (28) with respect to τ over $[r, t]$, change the order of integration, and use (18) again to get

$$(t-r)\|A^{1/2}z(t)\|^2 + \int_r^t (\xi - r)\|Az(\xi)\|^2\, d\xi \le C_T(R)|\varphi^1 - \varphi^2|_C,\ t \ge r.$$

Using the expression for $\dot z$ from (23) and also the bounds in (18) and (27) we have that

$$\|\dot z(t) + Az(t)\|^2 + \|A^{-1/2}\dot z(t)\|^2 \le C_T(R)\left[\|A^{1/2}z(t)\|^2 + |\varphi^1 - \varphi^2|_C\right],\ t \ge r.$$

This implies (26).

The $\frac{1}{2}$-Hölder continuity of the evolution semigroup S_t in the norm of $\mathcal{C}L$ follows from (26).

The closedness of S_t for $t \in (0, r]$ easily follows from (18). □

Remark 3. As it follows from (27) we can obtain a $\frac{1}{2}$-Hölder continuity relation like (26) *for all* $t \ge 0$ if we assume in addition that one of initial data φ^i possesses the property $\dot\varphi^i \in L_2(-r, 0; H)$. In this case the argument above leads to the relation

$$\begin{aligned}\|A^{-1/2}\dot z(t)\|^2 + \|A^{1/2}z(t)\|^2 + \int_0^t \left[\|\dot z(\tau)\|^2 + \|Az(\tau)\|^2\right] d\tau&\\ \le C_T(R)\left[\|A^{1/2}(\varphi^1(0) - \varphi^2(0))\| + |\varphi^1 - \varphi^2|_C\right]&\end{aligned}\tag{29}$$

for all $t \in [0, T]$ and for all initial data φ^i such that $|\varphi^i|_{\mathcal{C}L} + |\dot\varphi^i|_{L^2(-r,0;H)} \le R$. Moreover, one can also see that the set

$$\mathcal{C}L_0 = \left\{\varphi \in \mathcal{C}L\ :\ \dot\varphi \in L^2(-r, 0; H)\right\} \tag{30}$$

is forward invariant with respect to S_t. Thus $\varphi \mapsto S_t\varphi$ is a $\frac{1}{2}$-Hölder continuous mapping for each $t \ge 0$ on the Banach space $\mathcal{C}L_0$ endowed with the norm $|\varphi|_{\mathcal{C}L_0} = |\varphi|_{\mathcal{C}L} + |\dot\varphi|_{L^2(-r,0;H)}$. Hence the dynamical (in the classical sense, see, e.g., [1, 4, 36]) system $(\mathcal{C}L_0, S_t)$ arises. However we prefer to avoid property $\dot\varphi \in L_2(-r, 0; H)$ in the description of the phase space. The point is that our goal is long-time dynamics and it is well-known (see, e.g., [1, 4, 36]) that the existence of limiting objects requires some compactness properties. Unfortunately we cannot guarantee these properties in the space $\mathcal{C}L_0$ without serious restrictions concerning the delay term. This is why we prefer to use the observation made in [28] concerning long-time dynamics of closed evolutions.

Remark 4. A similar problem as above we have with *time* continuity of evolution operator S_t. It is clear from (12) and (16) that $t \mapsto S_t\varphi$ is continuous for every $\varphi \in \mathcal{C}L$ when $t > r$. To guarantee the continuity $t \mapsto S_t\varphi$ *for all* $t \geq 0$ we need to make further restriction[3] on initial data. The main restriction is a compatibility condition at time $t = 0$. To describe this condition we introduce the following (complete) metric space

$$X \equiv \left\{ \varphi \in C^1([-r,0]; H_{-1/2}) \cap C([-r,0]; H) \;\middle|\; \begin{array}{l} \varphi(0) \in H_{1/2}; \\ \dot{\varphi}(0) + A\varphi(0) + F(\varphi) + G(\varphi(0)) = h \end{array} \right\} \tag{31}$$

Here the compatibility condition $\dot{\varphi}(0) + A\varphi(0) + F(\varphi) + G(\varphi(0)) = h$ is understood as an equality in $H_{-1/2}$. The distance in X is given by the relation

$$\mathrm{dist}_X(\varphi, \psi) = \max_{[-r,0]} \|\varphi(\theta) - \psi(\theta)\| + Lip_{[-r,0]}\|A^{-1/2}(\varphi - \psi)\| + \|A^{1/2}(\varphi(0) - \psi(0))\|. \tag{32}$$

One can see that X is a closed subset in the Banach space $\mathcal{C}L$ and the topology generated by the metric dist_X coincides with the induced topology of $\mathcal{C}L$, see (9).

In the following assertion we collect several dynamical properties of the evolution semigroup S_t which are direct consequences of Theorem 3.2 and Proposition 1 and Remark 4.

Proposition 2. *Under the conditions of Theorem 3.2 problem (1) generates an evolution semigroup S_t of closed mappings on $\mathcal{C}L$ such that*

- ***(a)*** *$S_t\mathcal{C}L \subset X$ for every $t \geq r$ and the set S_tB is bounded in X for each $t \geq r$ when B is bounded in the space $\mathcal{C}L$;*
- ***(b)*** *the set X is forward invariant: $S_tX \subset X$;*
- ***(c)*** *the mapping $\varphi \mapsto S_t\varphi$ is a $\frac{1}{2}$-Hölder continuous on $\mathcal{C}L$ (and hence on X) for all $t > r$;*
- ***(d)*** *the trajectories $t \mapsto S_t\varphi$ are continuous for $t > r$ and $\varphi \in \mathcal{C}L$. If $\varphi \in X$, then these trajectories are continuous for all $t \geq 0$.*

4. **Long-time dynamics.** This section is central for the whole paper. Here we study long-time dynamics of the delay model generated by (1) and (7). The main result stated below in Theorem 4.2 deals with finite-dimensional global and exponential attractors. We refer to the Appendix for the corresponding definitions and the auxiliary facts which we use in our argument.

We first impose the standard hypotheses (see, e.g., [36]) concerning the nonlinear (non-delayed) sink/source term G.

Assumption 4.1. The nonlinear mapping $G : H_{1/2} \to H$ has the form $G(u) = \Pi'(u)$. Here $\Pi(u) = \Pi_0(u) + \Pi_1(u)$, where $\Pi_0(u) \geq 0$ is bounded on bounded sets in $H_{1/2}$ and $\Pi_1(u)$ satisfies the property

$$\forall \eta > 0 \; \exists C_\eta > 0 : \; |\Pi_1(u)| \leq \eta \left(\|A^{1/2}u\|^2 + \Pi_0(u) \right) + C_\eta, \qquad u \in H_{1/2}. \tag{33}$$

[3] We refer to some discussion in [31, 33] for the related PDE models.

Moreover, we assume that
(a) there are constants $\eta \in [0,1), c_4, c_5 > 0$ such that

$$-\langle u, G(u)\rangle \le \eta \|A^{1/2}u\|^2 - c_4\Pi_0(u) + c_5, \quad u \in H_{1/2}; \tag{34}$$

(b) for every $\widetilde{\eta} > 0$ there exists $C_{\widetilde{\eta}} > 0$ such that

$$\|u\|^2 \le C_{\widetilde{\eta}} + \widetilde{\eta}\left(\|A^{1/2}u\|^2 + \Pi_0(u)\right), \quad u \in H_{1/2}. \tag{35}$$

In the case of parabolic models like (6) examples of functions $g(u)$ such that the corresponding Nemytskii operator satisfies Assumptions 2.1(G) and 4.1 can be found in [1] and [36]. The simplest one is $g(u) = u^3 + a_1u^2 + a_2u$ with arbitrary $a_1, a_2 \in \mathbb{R}$ in the case when Ω is a 3D domain.

Our main result is the following assertion.

Theorem 4.2. *Let Assumptions 2.1 and 4.1 be in force. Suppose that S_t is the evolution semigroup generated in $\mathcal{CL}$ by (1) and (7). Then there exists $\ell_0 > 0$ such that this semigroup possesses a compact connected global attractor $\mathfrak{A}$ provided $m_F r < \ell_0$, where r is the delay time and m_F is the linear growth constant for F_0 in H defined by the relation*

$$m_F = \limsup_{\|u\|\to+\infty} \frac{\|F_0(u)\|}{\|u\|}. \tag{36}$$

Moreover, for every $0 < \beta \le 1$ and $\alpha < \min\{\beta, 1/2\}$ this attractor belongs to the set

$$D^R_{\alpha,\beta} = \left\{ \varphi \in X \;\middle|\; \begin{aligned} &|A^{1-\beta}\varphi|_C + |A^{-\beta}\dot\varphi|_C + \mathrm{Hold}_\alpha(A^{1-\beta}\varphi) + \mathrm{Hold}_\alpha(A^{-\beta}\dot\varphi) \\ &\qquad + \left[\int_{-r}^0 \left(\|A\varphi(\theta)\|^2 + \|\dot\varphi(\theta)\|^2\right) d\theta\right]^{1/2} \le R \end{aligned} \right\} \tag{37}$$

for some $R = R(\alpha,\beta)$, where the Hölder seminorm $\mathrm{Hold}_\alpha(\psi)$ is given by

$$\mathrm{Hold}_\alpha(\psi) = \sup\left\{ \frac{\|\psi(t_1) - \psi(t_2)\|}{|t_1 - t_2|^\alpha} \;:\; t_1 \ne t_2,\; t_1, t_2 \in [-r, 0] \right\}.$$

Assume in addition that there exist $\gamma, \delta \in (0, 1/2]$ such that **(a)** *the mapping F_0 is globally Lipschitz from $H_{-\gamma}$ into $H_{-1/2+\delta}$, i.e.,*

$$\|F_0(u) - F_0(v)\|_{-1/2+\delta} \le c\|u - v\|_{-\gamma}, \; u, v \in H_{-\gamma}; \tag{38}$$

and **(b)** *the mapping G is almost locally Lipschitz from $H_{1/2-\gamma}$ into $H_{-1/2+\delta}$ in the sense that*

$$\|G(u) - G(v)\|_{-1/2+\delta} \le c(R)\|u - v\|_{1/2-\gamma}, \; u, v \in H_{1/2}, \; \|u\|_{1/2}, \|v\|_{1/2} \le R. \tag{39}$$

Then

(A) *The global attractor $\mathfrak{A}$ has finite fractal dimension.*
(B) *There exists a fractal exponential attractor $\mathfrak{A}_{\mathrm{exp}}$.*

We devote the remaining subsections to the proof of Theorem 4.2.

4.1. **Existence of a global attractor.** To prove the existence of a global attractor it is sufficient to show that the evolution operator possesses a compact absorbing set. In this case we can apply the standard existence result in the form given in [28] for closed semigroups (see the Appendix for more details).

We start with the existence of a bounded absorbing set.

Proposition 3 (Bounded dissipativity)**.** *Assume that $u(t)$ solves (1) and (7) with $\varphi \in \mathcal{C}L$. Then one can find $\ell_0 > 0$ such that for every delay time r such that $m_F r < \ell_0$ the following property holds: there exists R_* such that for every bounded set B in $\mathcal{C}L$ there is t_B such that*

$$\|A^{-1/2}\dot{u}(t)\|^2 + \|A^{1/2}u(t)\|^2 + \int_t^{t+1} \left[\|\dot{u}(\tau)\|^2 + \|Au(\tau)\|^2\right] d\tau \le R_*^2 \tag{40}$$

for all $t \ge t_B$ and for all initial data $\varphi \in B$. This means that the evolution semigroup S_t is dissipative on both $\mathcal{C}L$ and X provided $m_F r < \ell_0$.

Proof. We use the Lyapunov method to get the result. For this we consider the following functional

$$\widetilde{V}(t) \equiv \frac{1}{2}\left[\|u(t)\|^2 + \|A^{1/2}u(t)\|^2\right] + \Pi(u(t)) + \frac{\mu}{r}\int_0^r \left\{\int_{t-s}^t \|\dot{u}(\xi)\|^2 d\xi\right\} ds.$$

defined on strong solutions $u(t)$ for $t \ge r$. The positive parameter μ will be chosen later. We note that the main idea behind inclusion of an additional delay term in $\widetilde{V}$ is to find a compensator for the delay term in (1). This idea was already applied in [12] for second order in time models with state-dependent term, see also [8, p.480] and [10] for the case of a flow-plate interaction model which contains a linear constant delay term with the critical spatial regularity. The corresponding compensator is model-dependent.

One can see from (33) that there are $0 < c_0 < 1$ and $c, c_1 > 0$ such that

$$\begin{aligned} c_0\left[\|A^{1/2}u(t)\|^2 + \Pi_0(u(t))\right] - c \le \widetilde{V}(t) \\ \le c_1\left[\|A^{1/2}u(t)\|^2 + \Pi_0(u(t))\right] + \mu\int_0^r \|\dot{u}(t-\xi)\|^2 d\xi + c. \end{aligned} \tag{41}$$

We consider the time derivative of $\widetilde{V}$ along a solution. One can easily check that

$$\begin{aligned} \frac{d}{dt}\widetilde{V}(t) =& \langle u(t), \dot{u}(t)\rangle + \langle Au(t), \dot{u}(t)\rangle + \langle G(u(t)), \dot{u}(t)\rangle \\ &+ \frac{\mu}{r}\int_0^r \left\{\|\dot{u}(t)\|^2 - \|\dot{u}(t-s)\|^2\right\} ds \\ =& \langle \dot{u}(t) + Au(t) + G(u(t)), \dot{u}(t)\rangle - \langle \dot{u}(t), \dot{u}(t)\rangle + \langle u(t), \dot{u}(t)\rangle + \mu\|\dot{u}(t)\|^2 \\ &- \frac{\mu}{r}\int_0^r \|\dot{u}(t-\xi)\|^2 d\xi \\ =& -\langle F(u_t) - h, \dot{u}(t)\rangle - (1-\mu)\|\dot{u}(t)\|^2 - \frac{\mu}{r}\int_0^r \|\dot{u}(t-\xi)\|^2 d\xi \\ &- \|A^{1/2}u(t)\|^2 - \langle F(u_t) + G(u(t)) - h, u(t)\rangle. \end{aligned}$$

The last terms are due to (1):

$$\langle u(t), \dot{u}(t)\rangle = -\langle Au(t), u(t)\rangle - \langle F(u_t) + G(u(t)) - h, u(t)\rangle.$$

By the definition of m_F in (36) for any number M_F greater than m_F we can find $C(M_F)$ such that

$$\|F(u_t)\| = \|F_0(u(t-\eta(u_t)))\| \le M_F\|u(t-\eta(u_t))\| + C(M_F).$$

Therefore

$$\begin{aligned}\|F(u_t)\| \le& M_F\|u(t-\eta(u_t)) - u(t)\| + M_F\|u(t)\| + C(M_F)\\ =& M_F\left\|\int_{t-\eta(u_t)}^{t} \dot u(\theta)d\theta\right\| + M_F\|u(t)\| + C(M_F),\end{aligned}$$

and thus

$$\|F(u_t)\| \le M_F\cdot\left[\|u(t)\| + \int_0^r \|\dot u(t-\xi)\|d\xi\right] + C(M_F),\ t\ge r.$$

Since

$$\int_0^r \|\dot u(t-\xi)\|d\xi \le r^{1/2}\left(\int_0^r \|\dot u(t-\xi)\|^2 d\xi\right)^{1/2},$$

we have that

$$\begin{aligned}|\langle F(u_t)-h,\dot u(t)\rangle| \le& \frac{1}{2}\|\dot u(t)\|^2 + c_0\|h\|^2 + c_1M_F^2\|u(t)\|^2\\ &+ c_2M_F^2 r\int_0^r \|\dot u(t-\xi)\|^2 d\xi + C(M_F),\ t\ge r.\end{aligned}$$

In a similar way we also have that

$$|\langle F(u_t)-h,u(t)\rangle)| \le c_1M_F^2 r\int_0^r \|\dot u(t-\xi)\|^2 d\xi + C(M_F)(1+\|u(t)\|^2).$$

Thus

$$\begin{aligned}&|\langle F(u_t)-h,\dot u(t)\rangle| + |\langle F(u_t)-h,u(t)\rangle)|\\ &\qquad\le \frac{1}{2}\|\dot u(t)\|^2 + c_0M_F^2 r\int_0^r \|\dot u(t-\xi)\|^2 d\xi + c_1(M_F)(1+\|u(t)\|^2).\end{aligned}$$

The relations in (34) and (35) with small enough $\widetilde\eta>0$ (and $\eta\in[0,1)$) yield

$$c_1(M_F)(1+\|u\|^2) - \|A^{1/2}u\|^2 - \langle u, G(u)\rangle \le -a_0\left[\|A^{1/2}u\|^2 + \Pi_0(u)\right] + a_1(M_F)$$

for some $a_i>0$ with a_0 independent of M_F. Thus it follows from the relations above that

$$\begin{aligned}\frac{d}{dt}\widetilde V(t) \le& -\left(\frac{1}{2}-\mu\right)\|\dot u(t)\|^2\\ &- a_0\left[\|A^{1/2}u\|^2 + \Pi_0(u)\right] + a_1(M_F) + \left[-\frac{\mu}{r} + a_2M_F^2 r\right]\int_0^r \|\dot u(t-\xi)\|^2 d\xi\end{aligned}$$

for some a_i. Thus using the right inequality in (41) we arrive at the relation

$$\begin{aligned}\frac{d}{dt}\widetilde V(t) + \gamma\widetilde V(t) \le& -\left(\frac{1}{2}-\mu\right)\|\dot u(t)\|^2 - (a_0-\gamma c_1)\left[\|A^{1/2}u\|^2 + \Pi_0(u)\right]\\ &+ \left[-\frac{\mu}{r} + \mu\gamma + a_2M_F^2 r\right]\int_0^r \|\dot u(t-\xi)\|^2 d\xi + a_1(M_F).\end{aligned} \tag{42}$$

Therefore taking $\mu = 1/4$ and fixing $\gamma\le a_0c_1^{-1}$ we obtain that

$$\frac{d}{dt}\widetilde V(t) + \gamma\widetilde V(t) + \frac{1}{4}\|\dot u(t)\|^2 \le C,\ t\ge r, \tag{43}$$

provided $\gamma r + 4a_2 M_F^2 r^2 \le 1$. Thus under the condition $4a_2 m_F^2 r^2 < 1$ we can choose $\gamma \in (0, a_0 c_1^{-1}]$ and $M_F > m_F$ such that (43) holds. In particular we have that

$$\frac{d}{dt}\widetilde{V}(t) + \gamma \widetilde{V}(t) \le C,\ t \ge r,$$

which implies

$$\widetilde{V}(t) \le \widetilde{V}(r) e^{-\gamma(t-r)} + \frac{C}{\gamma}(1 - e^{-\gamma(t-r)}),\ t \ge r, \tag{44}$$

when $m_F r < \ell_0$. Using (41) and (17) we can conclude that $|\widetilde{V}(r)| \le C_B$ for all initial data from a bounded set B in $\mathcal{C}L$. Hence (see (1)) there exists R such that for every initial data from a bounded set B in $\mathcal{C}L$ we have that

$$\|A^{1/2}u(t)\| + \|A^{-1/2}\dot{u}(t)\| + \|\dot{u}(t) + Au(t)\| \le R \text{ for all } t \ge t_B.$$

Moreover, it follows from (43) that

$$\int_t^{t+1} \|\dot{u}(\tau)\|^2 d\tau \le C_R \text{ for all } t \ge t_B.$$

To get this one should multiply (43) by $e^{\gamma t}$, integrate over $[t, t+1]$ and multiply by $e^{-\gamma t}$. Then ultimate boundedness of $\widetilde{V}(t)$ (see (44)) and the relation $1 \le e^{\gamma(\tau - t)}$ for $\tau \ge t$ give the last estimate.

These relations imply (40) and allow us to complete the proof of Proposition 3. □

Remark 5. If the mapping F_0 has sublinear growth in H, i.e., there exists $\beta < 1$ such that

$$\|F_0(u)\| \le c_1 + c_2 \|u\|^\beta,\ u \in H,$$

then the linear growth parameter m_F given by (36) is zero. Thus in this case we have no restrictions concerning r in the statement of Proposition 3. In particular, this is true in the case of bounded mappings F_0. Moreover, in the latter case the argument can be simplified substantially (we can use a Lyapunov type function without delay terms). For more details in the bounded case we refer to [31, 33].

We use Proposition 3 to obtain the following assertion which means that the evolution semigroup S_t is (ultimately) compact.

Proposition 4 (Compact dissipativity)**.** *As in Proposition 3 we assume that $m_F r < \ell_0$. Then the evolution operator S_t possesses a compact absorbing set. More precisely, for every $0 < \beta \le 1$ and $\alpha < \min\{\beta, 1/2\}$ the set $D^R_{\alpha,\beta}$ given by (37) is absorbing for some R. This set $D^R_{\alpha,\beta}$ is compact in X provided $0 < \alpha < \beta < 1/2$.*

Proof. We first note that the compactness of $D^R_{\alpha,\beta}$ in $X \subset \mathcal{C}L$ for $0 < \alpha < \beta < 1/2$ follows from Arzelà-Ascoli theorem in Banach spaces (see, e.g., [34]).

Now we show that $D^R_{\alpha,\beta}$ is absorbing.

Using the mild form of the problem and also the bound in (10) one can show that

$$\|A^{1-\delta}u(t)\| + \|A^{-\delta}\dot{u}(t)\| \le C_{R*}(\delta) \text{ for all } t \ge t_B, \tag{45}$$

for every $\delta > 0$, where $u(t)$ is a solution possessing property (40).

Now we consider the difference $u(t_1) - u(t_2)$ with $t_1 > t_2$. Namely, using the mild form we obtain

$$\|A^{1-\beta}(u(t_1) - u(t_2))\| \leq \|A^{1-\beta}(e^{-A(t_1-t_2)} - 1)u(t_2)\| + \int_{t_2}^{t_1} \|A^{1-\beta}e^{-A(t_1-\tau)}\| \cdot (\|F(u_\tau)\| + \|G(u(\tau)) - h\|)\, d\tau.$$

Since (see [21, Theorem 1.4.3, p.26] for related facts)

$$\|A^{-\alpha}(1 - e^{-At})\| \leq t^\alpha \text{ and } \|A^\alpha e^{-At}\| \leq \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha}$$

for all $t > 0$ and $0 \leq \alpha \leq 1$, we obtain

$$\|A^{1-\beta}(u(t_1) - u(t_2))\| \leq |t_1 - t_2|^\alpha \|A^{1-\beta+\alpha}u(t_2)\| + c_\beta \int_{t_2}^{t_1} \frac{1}{|t_1-\tau|^{1-\beta}} \left[C_{R_*} + c|u_\tau|_C\right] d\tau.$$

for $t_1 > t_2 \geq t_B$. Thus for every $0 < \alpha < \beta \leq 1$ we have

$$\|A^{1-\beta}(u(t_1) - u(t_2))\| \leq C_{R_*}|t_1 - t_2|^\alpha \text{ for all } t_i \geq t_B, |t_1 - t_2| \leq 1. \tag{46}$$

Similarly to (27) using (46) with $\beta = 1$ and $\alpha = 1/2$ we have that

$$\|F(u_{t_1}) - F(u_{t_2})\| \leq L_F \left| \int_{t_1-\eta(u_{t_1})}^{t_2-\eta(u_{t_2})} \|\dot{u}(\xi)\| d\xi \right| \leq C_{R_*} \left[|t_1 - t_2| + |u_{t_1} - u_{t_2}|_C^2\right]^{1/2} \leq C_{R_*} |t_1 - t_2|^{1/2}$$

for every $t_1, t_2 \geq t_B \geq r$. Thus from (1) and (46) we obtain

$$\|A^{-\beta}(\dot{u}(t_1) - \dot{u}(t_2))\| \leq C_{R_*}|t_1 - t_2|^\alpha \text{ for all } t_i \geq t_B,\ |t_1 - t_2| \leq 1,$$

for every $0 < \alpha < 1/2$. This and also Proposition 3 imply that the set $D^R_{\alpha,\beta}$ given by (37) is absorbing for some R provided $0 < \beta \leq 1$ and $\alpha < \min\{\beta, 1/2\}$. □

Proposition 4 allows us to apply the result from [28] (see Theorem 5.4 in the Appendix below) to guarantee the existence of a compact connected global attractor.

Remark 6. Using Proposition 4 we can show that there exists a *forward invariant* compact absorbing set which belongs to $D^R_{\alpha,\beta}$ for an appropriate choice of the parameters. More precisely, for every $0 < \alpha < \beta < 1/2$ we can find $R(\alpha, \beta)$ such that $D^R_{\alpha,\beta}$ is a compact absorbing set in X and $\mathcal{CL}$. Thus there exists $T = T(\alpha, \beta)$ such that $S_t D^R_{\alpha,\beta} \subset D^R_{\alpha,\beta}$ for all $t \geq T$. Hence $\cup_{t\geq T} S_t D^R_{\alpha,\beta}$ is a forward invariant absorbing set lying in $D^R_{\alpha,\beta}$. Since $D^R_{\alpha,\beta}$ is compact, the set

$$D_0^{\alpha,\beta} = \text{Closure}\left(\bigcup_{t\geq T} S_t D^R_{\alpha,\beta}\right) \tag{47}$$

is a compact absorbing set as well. Due to the closedness of S_t and the compactness of $D_0^{\alpha,\beta}$ this set $D_0^{\alpha,\beta}$ is also forward invariant. Moreover, it follows from Remark 3 that the restriction of S_t on $D_0^{\alpha,\beta}$ is *continuous* in both t and initial data in the topology induced by $\mathcal{CL}$ (see (9)). Thus a dynamical system $(S_t, D_0^{\alpha,\beta})$ in the classical (see [1, 4, 19, 36]) sense arises. In particular, this allows us to suggest an argument for the existence of a global attractor which avoids Theorem 5.4 on closed evolutions. For this we need to apply the standard existence result (see [1] or [36]) to the system $(S_t, D_0^{\alpha,\beta})$ and then to show that the attractor does not depend on α

and β. However, the approach to the existence based on Theorem 5.4 is more natural because it embeds the case in a general framework. At any case we emphasize that the observation concerning continuity of S_t on $D_0^{\alpha,\beta}$ is important. Below we use this property in the proof of the existence of a fractal exponential attractor.

4.2. **Dimension and exponential attractor.** Our further arguments are based on the notion of *quasi-stability* which says that the semigroup is asymptotically contracted up to a homogeneous compact additive term. For the convenience we remind the corresponding abstract result in the Appendix. The quasi-stability method was developed earlier in [5, 6, 7, 8, 9] for *continuous* evolution models. By Remark 6 we can apply this method to the system $(S_t, D_0^{\alpha,\beta})$ with $D_0^{\alpha,\beta}$ given by (47).

Proposition 5 (Quasi-stability). *Let Assumptions 2.1 and 4.1 be in force. Assume that (38) and (39) are valid for some $\gamma, \delta \in (0, 1/2]$. Let $D_0 = D_0^{\alpha,\beta}$ with $0 < \alpha < \beta < \gamma$. Then*

$$\begin{aligned} |S_t\varphi^1 - S_t\varphi^2|_{CL} \le & C_R e^{-\lambda_1 t}\left[\|\varphi^1(0) - \varphi^2(0)\|_{1/2} + |\varphi^1 - \varphi^2|_C\right] \\ & + C_R \max_{s\in[0,t]} \|A^{1/2-\gamma}(u^1(s) - u^2(s))\|, \ t \ge r, \end{aligned} \tag{48}$$

for every $\varphi^i \in D_0$, where $u^i(t) = (S_t\varphi^i)(\theta)\big|_{\theta=0}$.

Proof. Using the mild form presentation for $u^i(t)$ and (39) we have that

$$\|A^{1/2}(u^1(t) - u^2(t))\| \le e^{-\lambda_1 t}\|A^{1/2}(u^1(0) - u^2(0))\|$$
$$+\int_0^t \|A^{1-\delta} e^{-A(t-\tau)}\| \cdot \Big(C\|A^{-1/2+\delta}\big[F(u^1_\tau) - F(u^2_\tau)\big]\| + C_R\|u^1(\tau) - u^2(\tau)\|_{1/2-\gamma}\Big)\, d\tau.$$

As in (27) we also have that

$$\begin{aligned} \|A^{-1/2+\delta}\big[F(u^2_t) - F(u^1_t)\big]\| &\le C\left|\int_{t-\eta(u^1_t)}^{t-\eta(u^2_t)} \|A^{-\beta}\dot u^2(\xi)\| d\xi\right| + C|u^2_t - u^1_t|_C \\ &\le C(R) \max_{\theta\in[-r,0]} \|u^2(t+\theta) - u^1(t+\theta)\| \end{aligned}$$

for every $t \ge 0$ with $\beta \in (0, \gamma]$. Therefore

$$\begin{aligned} \|A^{1/2}(u^1(t) - u^2(t))\| \le c_1 e^{-\lambda_1 t} & \Big[\|A^{1/2}(\varphi^1(0) - \varphi^2(0))\| + |\varphi^1 - \varphi^2|_C\Big] \\ & + c_2(R) \max_{s\in[0,t]} \|A^{1/2-\gamma}(u^1(s) - u^2(s))\|. \end{aligned}$$

Using (1), (4) and (11) we also have that

$$\|A^{-1/2}(\dot u^1(t) - \dot u^2(t))\| \le C(R)\Big[\|A^{1/2}(u^1(t) - u^2(t))\| + |u^2_t - u^1_t|_C\Big]$$

Thus

$$\begin{aligned} \|A^{-1/2}(\dot u^1(t) - \dot u^2(t))\| \le c_1 e^{-\lambda_1 t} & \Big[\|A^{1/2}(\varphi^1(0) - \varphi^2(0))\| + |\varphi^1 - \varphi^2|_C\Big] \\ & + c_2(R) \max_{s\in[0,t]} \|A^{1/2-\gamma}(u^1(s) - u^2(s))\|, \ t \ge r. \end{aligned}$$

This completes the proof of Proposition 5. □

Completion of the proof of Theorem 4.2. Since the global attractor belongs to an exponential one (see [15]), it is sufficient to prove the existence of a fractal exponential attractor only. For this we apply Theorem 5.6 on the set D_0 (see Proposition 5) with an appropriate choice of operators and spaces. Indeed, let $T > 0$ be chosen such that $q \equiv C_R e^{-\lambda_1 T} < 1$ where C_R is the constant from (48). We define the Lipschitz mapping

$$K : D_0 \mapsto Z_{[0,T]} \equiv C^1([0,T]; D(A^{-1/2})) \cap C([0,T]; D(A^{1/2}))$$

by the rule $K\varphi = u(t), t \in [0,T]$, with u be the unique solution of (1) and (7) with initial function $\varphi \in D_0$. The seminorm $n_Z(u) \equiv \max_{s\in[0,T]} \|A^{1/2-\gamma}u(s)\|$ is compact on $Z_{[0,T]}$ due to the compact imbedding of $Z_{[0,T]}$ into $C([0,T]; D(A^\alpha))$ for every $\alpha < 1/2$ by the Arzelà-Ascoli theorem (see, e.g.,[34]). If we take

$$Y \equiv \left\{\varphi \in C^1([-r,0]; H_{-1/2}) \cap C([-r,0]; H) \;:\; \varphi(0) \in H_{1/2}\right\}$$

equipped with the norm (32) and suppose $V = S_T$, then the (discrete) quasi-stability inequality in (50) is valid on D_0. Hence we can apply Theorem 5.6 with $V = S_T$ on D_0 to show that there exists a finite-dimensional set $A_\theta \subset D_0$ such that (51) holds. Then as in the standard construction (see, e.g., [15] or [27]) we suppose

$$\mathfrak{A}_{exp} = \text{Closure}\left(\bigcup_{t\in[0,T]} S_t A_\theta\right).$$

Since $V = S_T$, it is easy to see that $\mathfrak{A}_{exp}$ is exponentially attracting, see (49) in the Appendix.

Since D_0 is included in the set $D^R_{\alpha,\beta}$ given by (37), we have that $t \mapsto S_t\varphi$ is α-Hölder on D_0:

$$|S_{t_1}\varphi - S_{t_2}\varphi|_Y \le C_{D_0}|t_1 - t_2|^\alpha, \;\; t_1, t_2 \in [0,T], \;\; \varphi \in D_0.$$

Therefore in the standard way (see, e.g., [15] or [27]) we can conclude that $\mathfrak{A}_{exp}$ has finite fractal dimension in Y.

This completes the proof of Theorem 4.2.

5. Appendix.

Here, for the convenience of the reader, we remind some results used in our work. For more details we refer to the cited sources.

First we collect some definitions and properties, connected to (closed) evolution semigroups. We start with the following notion which was introduced in [28].

Definition 5.1 (Closed semigroup)**.** Let $\mathcal{X}$ be a complete metric space. A `closed semigroup` on $\mathcal{X}$ is a one-parameter family of (nonlinear) operators $S_t : \mathcal{X} \to \mathcal{X}$ ($t \in \mathbb{R}_+$) (or $t \in \mathbb{N}$) satisfying the conditions

(S.1) $S_0 = Id_{\mathcal{X}}$ - identical operator;
(S.2) $S_{t+\tau} = S_t S_\tau$ for all $t, \tau \in \mathbb{R}_+$;
(S.3) for every $t \in \mathbb{R}_+$ the relations $x_n \to x$ and $S_t x_n \to y$ imply that $S_t x = y$.

Assumptions (S.1) and (S.2) are the semigroup properties, while (S.3) says that S_t is a closed (nonlinear) map. We note the operator closedness is a well-known concept in the theory of linear (unbounded) operators. To our best knowledge in the context of evolution operators this notion was appeared in [1] as a (weak) closedness of an evolution (strongly continuous) semigroup (see also [4]).

The following notions are standard in the theory of infinite-dimensional evolution semigroups and dynamical systems (see, e.g., [1, 4, 19, 23, 36]).

Definition 5.2 (Dissipativity and compactness)**.** A semigroup S_t is `dissipative` if there is a bounded absorbing set $\mathcal{B}_{abs} \subset \mathcal{X}$. That means for any bounded set $B \subset \mathcal{X}$, there exists $t_0 = t_0(B)$ (the entering time) such that $S_t B \subset \mathcal{B}_{abs}$ for all $t \geq t_0$. A semigroup S_t is `compact` if there is a compact absorbing set.

Definition 5.3 (Global attractor)**.** A `global attractor` of an evolution semigroup S_t acting on a complete metric space $\mathcal{X}$ is defined as a bounded closed set $\mathfrak{A} \subset \mathcal{X}$ which is invariant ($S_t\mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and attracting.

We recall ([1, 36]) that a set $\mathcal{K} \subset \mathcal{X}$ is called `attracting` for S_t if, for any bounded set $B \subset \mathcal{X}$,

$$\lim_{t\to+\infty} d_{\mathcal{X}}\{S_t B, \mathcal{K}\} = 0,$$

where $d_{\mathcal{X}}\{A, B\} \equiv \sup_{x\in A} dist_{\mathcal{X}}(x, B)$ is the Hausdorff semi-distance between bounded sets $A, B \subset \mathcal{X}$.

The following assertion is a reformulation of Corollary 6 [28] which also takes into account the statement of [28, Theorem 2]).

Theorem 5.4 (Existence of a global attractor)**.** *Assume that $S_t : \mathcal{X} \to \mathcal{X}$ is a closed semigroup possessing a compact connected absorbing set $\mathcal{K}_{abs} \subset \mathcal{X}$. Then there exists a compact global attractor $\mathfrak{A}$ for S_t. This attractor is a connected set and $\mathfrak{A} = \omega(\mathcal{K}_{abs}) = \bigcap_{t\in\mathbb{R}_+} \overline{\bigcup_{\tau\geq t} S_\tau \mathcal{K}_{abs}}$.*

One of the desired qualitative properties of an attractor is its finite-dimensionality. We remind the following definition (see, e.g., [4, 36]).

Definition 5.5. Let $M \subset \mathcal{X}$ be a compact set. Then the `fractal (box-counting) dimension` $dim_f M$ of M is defined by

$$dim_f M = \limsup_{\varepsilon\to 0} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of the radius ε which cover the set M.

We also recall (see [15]) that a compact set $\mathfrak{A}_{\exp} \subset \mathcal{C}L$ is said to be `fractal exponential attractor` for S_t iff $\mathfrak{A}_{\exp}$ is a positively invariant set whose fractal dimension is finite and for every bounded set D there exist positive constants t_D, C_D and γ_D such that

$$\sup_{\varphi\in D} \operatorname{dist}_{\mathcal{C}L}(S_t\varphi, \mathfrak{A}_{\exp}) \leq C_D \cdot e^{-\gamma_D(t-t_D)}, \quad t \geq t_D. \tag{49}$$

For details concerning fractal exponential attractors in the case of continuous semigroups we refer to [15] and also to the recent survey [27]. We only mention that (i) a global attractor can be non-exponential and (ii) an exponential attractor is not unique and contains the global attractor.

To prove the existence of exponential attractors we need the following assertion which was established in [5] and is a version of the result proved in [7] for metric spaces (some particular forms of Theorem 5.6 are also known from [6, 8]).

Theorem 5.6. *Let M be a bounded closed set in some Banach space Y and $V : M \to M$ be a continuous mapping. Assume there exist a Lipschitz mapping K from M into some Banach space Z and a compact seminorm $n_Z(x)$ on Z such that*

$$\|Vv^1 - Vv^2\| \leq q\|v^1 - v^2\| + n_Z(Kv^1 - Kv^2) \tag{50}$$

for any $v^1, v^2 \in M$*, where* $0 < q < 1$ *is a constant. Then for every* $\theta \in (q,1)$ *there exists a positively invariant compact set* $A_\theta \subset M$ *of finite fractal dimension satisfying*

$$\sup\left\{\operatorname{dist}(V^k u, A_\theta) \,:\, u \in M\right\} \le r\theta^k, \quad k = 1, 2, \dots, \tag{51}$$

for some constant $r > 0$*. Moreover,*

$$\dim_f A_\theta \le \ln m_Z\left(\frac{2L_K}{\theta - q}\right) \cdot \left[\ln \frac{1}{\theta}\right]^{-1},$$

where $L_K > 0$ *is the Lipschitz constant for* K*:*

$$\|Kv^1 - Kv^2\|_Z \le L_K \|v^1 - v^2\|, \; v^1, v^2 \in M,$$

and $m_Z(R)$ *is the maximal number of elements* z_i *in the ball* $\{z \in Z : \|z_i\|_Z \le R\}$ *possessing the property* $n_Z(z_i - z_j) > 1$ *when* $i \ne j$*.*

Acknowledgments. This work was supported in part by GA CR under project P103/12/2431. We also thank the anonymous referee for valuable comments and suggestions.

REFERENCES

[1] A. V. Babin and M. I. Vishik, *Attractors of Evolutionary Equations*, Amsterdam, North-Holland, 1992.

[2] R. Bellman and K. L. Cooke, *Differential-difference Equations*, in "Mathematics in Science and Engineering". Vol. 6. New York-London: Academic Press, XVI, 1963.

[3] N. F. Britton, Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model, *SIAM. J. Appl. Math.*, **50** (1990), 1663–1688.

[4] I. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, Acta, Kharkov, 1999, English translation, 2002; `http://www.emis.de/monographs/Chueshov`

[5] I. Chueshov, *Dynamics of Quasi-Stable Dissipative Systems*, Springer, 2015 (to be published).

[6] I. Chueshov and I. Lasiecka, Attractors for second-order evolution equations with a nonlinear damping, *J. of Dyn. and Diff. Equations*, **16** (2004), 469–512.

[7] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Mem. Amer. Math. Soc. 195 (2008), no. 912, viii+183 pp.

[8] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations. Well-posedness and Long-time Dynamics*, Springer Monographs in Mathematics. Springer, New York, 2010.

[9] I. Chueshov and I. Lasiecka, Well-posedness and long time behavior in nonlinear dissipative hyperbolic-like evolutions with critical exponents, in *Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations (HCDTE Lecture Notes, Part I)*, AIMS on Applied Mathematics Vol.6 (G. Alberti et al. eds.), AIMS, Springfield, 2013, 1–96.

[10] I. Chueshov, I. Lasiecka and J. T. Webster, Attractors for delayed, non-rotational von Karman plates with applications to flow-structure interactions without any damping, *Communications in Partial Differential Equations*, **39** (2014), 1965–1997.

[11] I. Chueshov, I. Lasiecka and J. T. Webster, Flow-plate interactions: well-posedness and long-time behavior, *Discrete Continuous Dynamical Systems Ser. S*, **7** (2014), 925–965.

[12] I. Chueshov and A. Rezounenko, Dynamics of second order in time evolution equations with state-dependent delay, *Nonlinear Analysis: Theory, Methods and Applications*, **123-124** (2015), 126–149.

[13] O. Diekmann, S. van Gils, S. Verduyn Lunel and H-O. Walther, *Delay Equations: Functional, Complex, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.

[14] R. D. Driver, A two-body problem of classical electrodynamics: the one-dimensional case, *Ann. Physics*, **21** (1963), 122–142.

[15] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Research in Appl. Math. 37, Masson, Paris, 1994.

[16] W. E. Fitzgibbon, Semilinear functional differential equations in Banach space, *J. Differential Equations*, **29** (1978), 1–14.

[17] S. Gourley, J. So and J. Wu, Non-locality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics, in *Contemporary Mathematics, Thematic Surveys* (D. V. Anosov AND A. Skubachevskii eds.), Kluwer, Plenum, Dordrecht, NewYork, 2003, 84–120; (see also *Journal of Mathematical Sciences*, **124** (2004), 5119–5153).

[18] J. K. Hale, *Theory of Functional Differential Equations*, Springer, Berlin- Heidelberg- New York, 1977.

[19] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[20] F. Hartung, T. Krisztin, H.-O. Walther and J. Wu, Functional differential equations with state-dependent delays: Theory and applications, In *Handbook of Differential Equations, Ordinary Differential Equations* (A. Canada, P. Drabek and A. Fonda eds.), vol. 3, Elsevier Science B. V., North Holland, 2006, pp. 435–545.

[21] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, New York, Springer, 1981.

[22] T. Krisztin and O. Arino, The 2-dimensional attractor of a differential equation with state-dependent delay, *J. Dynamics and Differential Equations*, **13** (2001), 453–522.

[23] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.

[24] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[25] J. L. Lions and E. Magenes, *Problèmes aux Limites Non Homogénes et Applications*, Dunon, Paris, 1968.

[26] J. Mallet-Paret, R. D. Nussbaum and P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, *Topol. Methods Nonlinear Anal.*, **3** (1994), 101–162.

[27] A. Miranville and S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains, In *Handbook of Differential Equations: Evolutionary Equations* (C. M. Dafermos and M. Pokorny eds.), vol. 4, Elsevier, Amsterdam, 2008, pp.103–200.

[28] V. Pata and S. Zelik, A result on the existence of global attractors for semigroups of closed operators, *Commun. Pure. Appl. Anal.*, **6** (2007), 481–486.

[29] A. V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, *Journal of Mathematical Analysis and Applications*, **326** (2007), 1031–1045.

[30] A. V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, *Nonlinear Analysis: Theory, Methods and Applications*, **70** (2009), 3978–3986.

[31] A. V. Rezounenko, Non-linear partial differential equations with discrete state-dependent delays in a metric space, *Nonlinear Analysis: Theory, Methods and Applications*, **73** (2010), 1707–1714.

[32] A. V. Rezounenko, A condition on delay for differential equations with discrete state-dependent delay, *Journal of Mathematical Analysis and Applications*, **385** (2012), 506–516.

[33] A. V. Rezounenko and P. Zagalak, Non-local PDEs with discrete state-dependent delays: well-posedness in a metric space, *Discrete and Continuous Dynamical Systems - Series A*, **33** (2013), 819–835.

[34] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Annali di Mat. Pura ed Appl.*, **146** (1987), 65–96.

[35] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, Mathematical Surveys and Monographs: vol. 49, 1997.

[36] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, Berlin-Heidelberg-New York, 1988.

[37] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Transactions of AMS*, **200** (1974), 395–418.

[38] H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, *Journal of Differential Equations*, **195** (2003), 46–65.

[39] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

Received October 2014; revised February 2015.

E-mail address: chueshov@karazin.ua
E-mail address: rezounenko@yahoo.com