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Dynamics of second order in time evolution equations with state-dependent delay

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ABSTRACT

We deal with a class of second order in time nonlinear evolution equations with state-dependent delay. This class covers several important PDE models arising in the theory of nonlinear plates. We first prove well-posedness in a certain space of functions which are C^1 in time. In contrast with the first order models with discrete state-dependent delay this result does not require any compatibility conditions. The solutions constructed generate a dynamical system in a C^1-type space over delay time interval. Our main result shows that this dynamical system possesses compact global and exponential attractors of finite fractal dimension. To obtain this result we adapt the recently developed method of quasi-stability estimates.

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1. Introduction

Our main goal is to study well-posedness and asymptotic dynamics of second order in time equations with delay of the form

$$\ddot{u}(t) + k\dot{u}(t) + Au(t) + F(u(t)) + M(u_t) = 0, \quad t > 0, \tag{1}$$

in some Hilbert space H. Here the dot over an element means time derivative, A is linear and $F(\cdot)$ is nonlinear operators, $M(u_t)$ represents (nonlinear) delay effect in the dynamics. All these objects will be specified later.

The main model we keep in mind is a nonlinear plate equation of the form

$$\partial_{tt}u(t,x) + k\partial_t u(t,x) + \Delta^2 u(t,x) + F(u(t,x)) + au(t-\tau[u(t)],x) = 0, \quad x \in \Omega,\ t > 0, \tag{2}$$

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in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with some boundary conditions on $\partial\Omega$. Here τ is a mapping defined on solutions with values in some interval $[0, h]$, k and a are constants. We assume that the plate is placed on some foundation; the term $au(t - \tau[u(t)], x)$ models effect of the Winkler type foundation (see [40,44]) with delay response. The nonlinear force F can be Kirchhoff, Berger, or von Karman type (see Section 6.1). Our abstract model covers also wave equation with state-dependent delay (see the discussion in Section 6.2).

We note that plate equations with *linear* delay terms were studied before mainly in Hilbert L_2-type spaces on delay interval (see, e.g., [2,3,10,11] and the references therein). However this L_2-type situation does not cover satisfactory the case of a state-dependent delay of the form described above. The point is that in this case the delay term in (2) is not even locally Lipschitz and thus difficulties related to uniqueness may arise. The desire to have Lipschitz property for this type of delay terms leads naturally to C-type spaces which are not even reflexive. This provides us with additional difficulties in contrast with the general theory well-developed for second order in time equations in the Hilbert space setting, see, e.g., [7] and also the literature cited there. In particular, in contrast with the non-delayed case (see [7–9]), in order to prove asymptotic smoothness of the flow (it is required for the existence of a global attractor) we are enforced to assume that the nonlinearity F is either subcritical (in the sense of [7]) or else the damping coefficient k in (1) is large enough. The main reason for this is that we are not able to apply Khanmamedov's or Ball's methods (see a discussion of both methods and more references in [9]). The point is that we cannot guarantee uniform in t weak continuity in the phase space of the corresponding functionals. Another reason is that the delay term destroys the gradient structure of the model in the case of potential nonlinearities F.

The studies of state-dependent delay models have a long history. As it is mentioned in [23] early discussion of differential equations with such a delay goes back to 1806 when Poisson studied a geometrical problem.[1] Since that time many problems, initially described by differential equations without delay or with constant delay, have been reformulated as equations with state-dependent delay. It seems rather natural because many models describing real world phenomena depend on the past states of the system. Moreover, it appears that in many problems the constancy of the time delay is just an extra assumption which makes the study easier. The waiver of this assumption is naturally lead to more realistic models and simultaneously makes analysis more difficult. The general theory of (ordinary) differential equations with state-dependent delay has been developed only recently (see, e.g., [25,31,45] and also the survey [23] and the references therein). This theory essentially differs from that of constant or time-dependent delays (see the references above and also Remark 2.1).

As for partial differential equations (PDEs) with delay their investigation requires the combination of both theories, methods and machineries (PDEs and delayed ODEs). The general theory of delayed PDEs was started with [19,43] at the abstract level and was developed in last decades mainly for parabolic type models with constant and time-dependent delays (see, e.g., the monographs [47] and the survey [39]). Abstract approaches for C-type [19,43] and L_p-type [26] phase spaces are available. We also mention a recent strong activity on dissipative PDEs with infinite delay which has the form of convolution in time with the proper kernel (see, e.g., [13,15,33] and the references therein).

Partial differential equations with state-dependent delay are essentially less investigated, see the discussion in the papers [12,34,35] devoted to the parabolic case. Some results (mainly, the existence and uniqueness) for the second order in time PDEs with delay are available. Most of them are based on a reformulation of the problem as a first order system and application of the theory of such systems (see, e.g., [19]). We also use this idea to get a local existence and uniqueness for problem (1). It is also worth mentioning the papers [20,24] which involve the theory of m-accretive (see [41], for instance) operators. However to the best of our knowledge, global and asymptotic dynamics of second order in time partial differential equations with state-dependent delay have not been studied before.

[1] We refer to [46] for a modern and detailed discussion of Poisson's example with state-dependent damping.

In our approach we employ the special structure of second order in time systems to get a globally well-posed initial value problem for mild solutions. As a phase space we choose some space of C^1-type functions. The solutions we deal with are also C^1 functions. To construct them we rewrite the second order in time equation (for unknown $u(t)$) as a first order system (for unknown vector $(u(t), \dot{u}(t))$) and look for continuous (mild) solutions to the system. However in contrast with approaches based on the general theory (see, e.g., [19] and also [45, Section 3] and [23, Section 2]) we take into account natural "displacement–velocity" compatibility from the very beginning at the level of the phase space. The solutions constructed have the desired Lipschitz (even C^1 in time) property for the first coordinate $u(t)$. In a sense it is an intermediate case between two standard classes of merely continuous (mild) and C^1 (classical) solutions $(u(t), v(t))$, $t \in [-h, T)$, $T > 0$ for a general first order in time system with delay:

$$\begin{cases} \dot{u}(t) = \mathcal{F}(u_t, v_t), \\ \dot{v}(t) = \mathcal{G}(u_t, v_t). \end{cases}$$

We emphasize that due to the structure of our problem we do not need any nonlinear compatibility type relations involving the right hand sides of equations which usually arise for general first order (even, finite-dimensional) systems when C^1 solutions are studied (see [45] and also the survey [23]). We also refer to Section 6.3 for a discussion of other features of our approach.

Our main result states that the dynamical system generated by (1) in the space W (see (3)) of C^1 functions on the delay time interval possesses a compact global attractor of finite fractal dimension. To achieve this result we involve the method of quasi-stability estimates suggested in [6] and developed in [7,8], see also the surveys in [9,5]. However owing to the structure of the phase space we cannot apply directly the results known for abstract quasi-stable systems and thus we are enforced to reconstruct and develop the corresponding argument in our state-dependent delay case.

The paper is organized as follows. In Section 2 we introduce our basic hypotheses and prove a well-posedness result. Further sections are devoted to long-time dynamics. We first prove that the system is dissipative (see Section 3). In Section 4 we show that the system satisfies some kind of quasi-stability estimate on an invariant bounded absorbing set. This allows us to establish the existence of compact finite-dimensional global and exponential attractors in Section 5. The concluding Section 6 illustrates main results by applications to plate and wave models.

2. Well-posedness and generation of a dynamical system

The main outcome of this section is the fact that problem (1) generates dynamical system in an appropriate linear phase space of C^1 functions.

In our study we assume that:

(A1) In (1), A is a positive operator with a discrete spectrum in a separable Hilbert space H with domain $D(A) \subset H$. Hence there exists an orthonormal basis $\{e_k\}$ of H such that

$$Ae_k = \mu_k e_k, \quad \text{with } 0 < \mu_1 \le \mu_2 \le \dots, \qquad \lim_{k\to\infty} \mu_k = \infty.$$

We can define the spaces $D(A^\alpha)$ for $\alpha > 0$ (see, e.g., [28]). For $h > 0$, we denote for short $C_\alpha = C([-h, 0]; D(A^\alpha))$ which is a Banach space with the following norm:

$$|v|_{C_\alpha} \equiv \sup\{\|A^\alpha v(\theta)\| : \theta \in [-h, 0]\}.$$

Here and below, $\|\cdot\|$ is the norm of H, and $(\cdot, \cdot)$ is the corresponding Hermitian product. We also write $C = C_0$.

(F1) The nonlinear (non-delayed) mapping $F : D(A^{\frac{1}{2}}) \to H$ is locally Lipschitz, i.e., for any $R > 0$ there is $L_{F,R} > 0$ such that for any u^1, u^2 with $\|A^{\frac{1}{2}} u^i\| \le R$, one has

$$\|F(u^1) - F(u^2)\| \le L_{F,R} \left\| A^{\frac{1}{2}}(u^1 - u^2) \right\|.$$

To describe the delay term M we need the following standard notations from the theory of delay differential equations. In (1) and below, if z is a continuous function from $\mathbb{R}$ into a metric space Y, then as in [21,47] $z_t \equiv z_t(\theta) \equiv z(t+\theta),\ \theta \in [-h, 0]$, denotes the element of $C([-h, 0]; Y)$, while $h > 0$ presents the (maximal) retardation time.

In our considerations an important role is played by the choice of a phase space (see Remark 2.1). We use the following one:

$$W \equiv C\left([-h, 0]; D\left(A^{\frac{1}{2}}\right)\right) \cap C^1([-h, 0]; H), \tag{3}$$

endowed with the norm $|\varphi|_W = |\varphi|_{C_{1/2}} + |\dot{\varphi}|_{C_0}$.

We accept the following (basic) hypothesis concerning the delay term.

(M1) The nonlinear delay term $M : W \mapsto H$ is locally Lipschitz in the sense that

$$\|M(\varphi^1) - M(\varphi^2)\| \le C_\varrho \left[|\varphi^1 - \varphi^2|_{C_{1/2}} + |\dot{\varphi}^1 - \dot{\varphi}^2|_{C_0}\right]$$

for every $\varphi^1,\ \varphi^2 \in W,\ |\varphi^j|_W \le \varrho,\ j = 1, 2.$

Remark 2.1. The main (benchmark) example[2] of a state-dependent delay term is

$$M(\varphi) = \varphi(-\tau(\varphi)), \quad \varphi \in C, \tag{4}$$

where τ maps C into some interval $[0, h]$. We notice that this (discrete time) delay term M is not locally Lipschitz in the classical space of continuous functions $C = C([-h, 0]; H)$, no matter how smooth the delay function $\tau : C \to [0, h]$ is. This may lead to the non-uniqueness of solutions (see a discussion in the survey [23] and the references therein). This makes the study of differential equations with state-dependent delays quite different from the one of equations with constant or time-dependent delays [16,21]. In such a situation the proof of the well-posedness of a system requires additional efforts. For instance, the main approach to C^1-solutions of general delay equations is the so-called "solution manifold method" [23,45] (see also [38] for a parabolic PDE case) which assumes some type of compatibility condition. It should be also noted that there is an alternative approach avoiding (nonlinear) compatibility hypotheses. However it is based on an additional hypotheses concerning the delay mechanism [35,37]. Thus it is important to deal with spaces in which we can guarantee a Lipschitz property for the mapping in (4). This is why to cover the case we are enforced to avoid the space C for the description of initial data. For the same reason we cannot also use the idea applied in [26] and also in the papers [2,3,10,11] which deal with L_2-type spaces over the time delay interval. In contrast, as we can see below the choice of a Banach space of the form (3) as a phase space allows us to guarantee local Lipschitz property for the term in (4). Moreover, this phase space takes into account the natural "displacement–velocity" relation from the very beginning.

Thus bearing in mind the discussion above we consider Eq. (1) with the following initial data

$$u_0 = u_0(\theta) \equiv u(\theta) = \varphi(\theta), \quad \text{for } \theta \in [-h, 0],\ \varphi \in W. \tag{5}$$

[2] A more general situation is described in hypothesis (M3) and Remark 3.1.

We can rewrite Eq. (1) as the following first order differential equation

$$\frac{d}{dt}U(t) + \mathcal{A}U(t) = \mathcal{N}(U_t), \quad t > 0, \tag{6}$$

in the space $Y = D(A^{1/2}) \times H$, where $U(t) = (u(t); \dot{u}(t))$. Here the operator $\mathcal{A}$ and the map $\mathcal{N}$ are defined by

$$\begin{aligned} &\mathcal{A}U = (-v; Au + kv), \quad \text{for } U = (u; v) \in D(\mathcal{A}) \equiv D(A) \times D(A^{1/2}) \\ &\mathcal{N}(\Phi) = -(0; F(\varphi(0)) + M(\varphi)) \quad \text{for } \Phi = (\varphi; \dot{\varphi}),\ \varphi \in W. \end{aligned} \tag{7}$$

One can show (see, e.g., [4]) that the operator $\mathcal{A}$ generates exponentially stable C_0-semigroup $e^{-\mathcal{A}t}$ in Y.

The representation in (6) motivates the following definition.

Definition 2.2. A *mild solution* of (1) and (5) on an interval $[0, T]$ is defined as a function

$$u \in C([-h, T]; D(A^{1/2})) \cap C^1([-h, T]; H),$$

such that $u(\theta) = \varphi(\theta),\ \theta \in [-h, 0]$ and $U(t) \equiv (u(t); \dot{u}(t))$[3] satisfies

$$U(t) = e^{-t\mathcal{A}}U(0) + \int_0^t e^{-(t-s)\mathcal{A}}\mathcal{N}(U_s)ds, \quad t \in [0, T]. \tag{8}$$

Similarly we can also define a mild solution on the semi-interval $[0, T)$.

We can easily prove the following local result.

Proposition 2.3. *Let* (A1), (F1) *and* (M1) *be valid. Then for any $\varphi \in W$ there exist $T_\varphi > 0$ and a unique mild solution $U(t) \equiv (u(t); \dot{u}(t))$ of* (1), (5) *on the semi-interval interval $[0, T_\varphi)$. Solution continuously depends on initial function $\varphi \in W$.*

Proof. The argument for the local existence, uniqueness and continuous dependence of a mild solution is standard (see, e.g., [43, Proposition 2.1 and Corollary 2.2], [19]) and uses the Banach fixed point theorem for a contraction mapping in the space $C([-h, T]; D(A^{1/2})) \cap C^1([-h, T]; H)$ with appropriately small T. We use here that F and M are locally Lipschitz by assumptions (F1) and (M1). □

To obtain a global well-posedness result we need additional hypotheses concerning F and M. As in the case of the second order models without delay (see [7,8]) we use the following set of assumptions concerning F.

(F2) The nonlinear mapping $F : D(A^{\frac{1}{2}}) \to H$ has the form

$$F(u) = \Pi'(u) + F^*(u),$$

where $\Pi'(u)$ denotes Fréchet derivative[4] of a C^1-functional $\Pi(u) : D(A^{\frac{1}{2}}) \to R$ and the mapping $F^* : D(A^{\frac{1}{2}}) \to H$ is globally Lipschitz, i.e.

$$\|F^*(u^1) - F^*(u^1)\|^2 \le c_0 \left\|A^{\frac{1}{2}}(u^1 - u^2)\right\|^2, \quad u^1, u^2 \in D\left(A^{\frac{1}{2}}\right). \tag{9}$$

Moreover, we assume that $\Pi(u) = \Pi_0(u) + \Pi_1(u)$, with $\Pi_0(u) \ge 0$, $\Pi_0(u)$ is bounded on bounded sets in $D(A^{\frac{1}{2}})$ and $\Pi_1(u)$ satisfies the property

$$\forall \eta > 0\ \exists C_\eta > 0 : \ |\Pi_1(u)| \le \eta \left(\left\|A^{\frac{1}{2}}u\right\|^2 + \Pi_0(u)\right) + C_\eta, \quad u \in D(A^{1/2}). \tag{10}$$

[3] Below $U(t)$ is also occasionally called a mild solution.

[4] This means that $\Pi'(u)$ is an element in $D(A^{\frac{1}{2}})'$ such that $|\Pi(u + v) - \Pi(u) - \langle \Pi'(u), v\rangle| = o(\|A^{1/2}v\|)$ for every $v \in D(A^{\frac{1}{2}})$.

As it is well-documented in [7,8] the second order models with nonlinearities satisfying (F2) arise in many applications (see also the discussion in Section 6).

We assume also

(M2) The nonlinear delay term $M: W \to H$ satisfies the linear growth condition:

$$\|M(\varphi)\| \le M_0 + M_1 \left\{ \max_{\theta\in[-h,0]} \|A^{1/2}\varphi(\theta)\| + \max_{\theta\in[-h,0]} \|\dot\varphi(\theta)\| \right\}, \quad \forall\, \varphi \in W, \tag{11}$$

for some $M_j \ge 0$, $j = 0, 1$.

The examples of F and M will be given in Section 6.

The main result of this section is the following assertion.

Theorem 2.4 (*Well-Posedness*)**.** *Let* (A1), (F1), (F2), (M1), *and* (M2) *be valid. Then for any $\varphi \in W$ there exists a unique global mild solution $U(t) \equiv (u(t); \dot u(t))$ of* (1), (5) *on the interval $[0, +\infty)$. Solutions satisfy an energy equality of the form*

$$\mathcal{E}(u(t), \dot u(t)) + k\int_0^t \|\dot u(s)\|^2 ds = \mathcal{E}(u(0), \dot u(0)) - \int_0^t (F^*(u(s)), \dot u(s))\, ds - \int_0^t (M(u_s), \dot u(s))\, ds. \tag{12}$$

Here we denote

$$\mathcal{E}(u, v) \equiv E(u, v) + \Pi_1(u), \qquad E(u, v) \equiv \frac{1}{2}\left(\|v\|^2 + \left\|A^{\frac{1}{2}} u\right\|^2 \right) + \Pi_0(u). \tag{13}$$

Moreover, for any $\varrho > 0$ and $T > 0$ there exists $C_{\varrho,T}$ such that

$$\|A^{1/2}(u^1(t) - u^2(t))\| + \|\dot u^1(t) - \dot u^2(t)\| \le C_{\varrho,T} |\varphi^1 - \varphi^2|_W, \quad t \in [0, T], \tag{14}$$

for any couple $u^1(t)$ and $u^2(t)$ of mild solutions with initial data φ^1 and φ^2 such that $|\varphi^j|_W \le \varrho$.

Proof. The local existence and uniqueness of mild solutions are given by Proposition 2.3. Let $U = (u; \dot u)$ be a mild solution of (1) and (5) on the (maximal) semi-interval $[-h, T_\varphi)$ and

$$f^u(t) \equiv F(u(t)) + M(u_t) \in C([0, T_\varphi); H).$$

It is clear that we can consider $(u(t); \dot u(t))$ as a mild solution of the linear *non-delayed* problem

$$\ddot v(t) + Av(t) + k\dot v(t) + f^u(t) = 0, \quad t \in [0, T_\varphi), \qquad (v(0); \dot v(0)) = (\varphi(0); \dot\varphi(0)) \in Y. \tag{15}$$

Here $Y = D(A^{1/2}) \times H$. Therefore (see, e.g., [4]) one can see that $u(t)$ satisfies the energy relation of the form

$$E_0(u(t), \dot u(t)) + k\int_0^t \|\dot u(s)\|^2 ds = E_0(u(0), \dot u(0)) - \int_0^t (f^u(s), \dot u(s))\, ds, \quad t < T_\varphi, \tag{16}$$

where $E_0(u, v) = \frac{1}{2}\left(\|A^{1/2}u\|^2 + \|v\|^2\right)$. Using the structure of f^u after some calculations (firstly performed on smooth functions) we can show that

$$\int_0^t (f^u(s), \dot u(s))\, ds = \Pi(u(t)) - \Pi(u(0)) + \int_0^t (F^*(u(s)) + M(u_s), \dot u(s))\, ds.$$

Therefore (16) yields (12) for every $t < T_\varphi$.

By (9) we have that $\|F^*(u)\| \le \sqrt{c_0}\|A^{1/2}u\| + \|F^*(0)\|$. Therefore using (12) and (11) we obtain that

$$\mathcal{E}(u(t),\dot u(t)) + \frac{k}{2}\int_0^t \|\dot u(s)\|^2 ds \le \mathcal{E}(u(0),\dot u(0)) + c_1\int_0^t \left(1 + \left\|A^{\frac12}u(s)\right\|^2\right) ds$$
$$+ c_2\int_0^t \left[\max_{\theta\in[-h,0]}\left\|A^{\frac12}u(s+\theta)\right\|^2 + \max_{\theta\in[-h,0]}\|\dot u(s+\theta)\|^2\right] ds. \tag{17}$$

One can see that

$$\max_{\theta\in[-h,0]}\left\|A^{\frac12}u(s+\theta)\right\|^2 + \max_{\theta\in[-h,0]}\|\dot u(s+\theta)\|^2 \le |\varphi|_W^2 + 2\max_{\sigma\in[0,s]} E(u(\sigma),\dot u(\sigma)) \tag{18}$$

for every $s\in[0,T_\varphi)$. It follows from (10) that there exists a constant $c>0$ such that

$$\frac12 E(u,v) - c \le \mathcal{E}(u,v) \le 2E(u,v) + c, \quad u\in D\left(A^{\frac12}\right),\ v\in H. \tag{19}$$

Therefore we use (19) and (18) to continue (see (17)) as follows

$$\max_{\sigma\in[0,t]} E(u(\sigma),\dot u(\sigma)) \le c\left(1 + t + E(u(0),\dot u(0)) + t\cdot|\varphi|_W^2 + \int_0^t \max_{\sigma\in[0,s]} E(u(\sigma),\dot u(\sigma))\,ds\right).$$

The application of Gronwall's lemma (to the function $p(t)\equiv \max_{\sigma\in[0,t]} E(u(\sigma),\dot u(\sigma))$) yields the following (a priori) estimate

$$\max_{\sigma\in[0,t]} E(u(\sigma),\dot u(\sigma)) \le C\left(1 + E(u(0),\dot u(0)) + |\varphi|_W^2\right)\cdot e^{at}, \quad a>0,\ t<T_\varphi,$$

which allows us in the standard way to extend the solution on the semi-axis $\mathbb{R}_+$.

To prove (14) we use the fact that the difference $u(t) = u^1(t) - u^2(t)$ solves the problem in (15) with

$$f^u(t) = F(u^1(t)) + M(u_t^1) - F(u^2(t)) - M(u_t^2).$$

Lipschitz properties (F1), (M1) and standard estimates [43, Corollary 2.2]. This completes the proof of Theorem 2.4. □

Using Theorem 2.4 we can define an **evolution operator** $S_t : W \to W$ for all $t\ge 0$ by the formula $S_t\varphi = u_t$, where $u(t)$ is the mild solution of (1), (5), satisfying $u_0=\varphi$. This operator satisfies the semigroup property and generates a dynamical system $(S_t;W)$ with the phase space W defined in (3) (for the definition and more on dynamical systems see, e.g., [1,4,42]).

Remark 2.5. We can equivalently define the dynamical system on the linear space of vector-functions $\widetilde{W} \equiv \{\Phi = (\varphi;\dot\varphi) \mid \varphi\in W\} \subset C([-h,0];D(A^{\frac12})\times H)$. In this notations evolution operator reads $\widetilde{S}_t\Phi \equiv U_t$ and we have $W \ni \varphi \overset{G}{\longmapsto} (\varphi;\dot\varphi)\in\widetilde{W}$ satisfying $GS_t = \widetilde{S}_t G$. In fact we already have used this observation in Definition 2.2 and Proposition 2.3.

We conclude this section with a discussion of the existence of smooth solutions to problem (1) and (5). In the following assertion we show that under additional hypotheses mild solutions become strong.

Corollary 2.6 (*Smoothness*)**.** *Let the hypotheses of Theorem 2.4 be in force with assumption* (M1) *in the following (stronger) form*

$$\|M(\varphi^1) - M(\varphi^2)\| \le C_\varrho|\varphi^1 - \varphi^2|_{C_0} \tag{20}$$

for every $\varphi^1,\ \varphi^2\in W, |\varphi^j|_W\le\varrho,\ j=1,2$. *If the initial function* $\varphi(\theta)$ *possesses the property*

$$\varphi(0)\in D(A), \qquad \dot\varphi(0)\in D(A^{1/2}), \tag{21}$$

then the solution $u(t)$ satisfies the relations

$$u(t) \in L_\infty(0,T;D(A)), \qquad \dot{u}(t) \in L_\infty(0,T;D(A^{1/2})), \qquad \ddot{u}(t) \in L_\infty(0,T;H) \tag{22}$$

for every $T > 0$. If in addition $F(u)$ is Fréchet differentiable and $\|F'(u)v\| \le C_r\|A^{1/2}v\|$ for every $u \in D(A)$ with $\|Au\| \le r$, then we have

$$u(t) \in C(\mathbb{R}_+;D(A)), \qquad \dot{u}(t) \in C(\mathbb{R}_+;D(A^{1/2})), \qquad \ddot{u}(t) \in C(\mathbb{R}_+;H). \tag{23}$$

Proof. Let $u(t)$ be a solution. By Theorem 2.4 we have that

$$\max_{[-h,T]} \left(\|A^{1/2}u(t)\|^2 + \|\dot{u}(t)\|^2\right) \le R_T$$

for some R_T. Now we note that under condition (20) the function $t \mapsto f(t) \equiv M(u_t)$ is Lipschitz on any interval $[0,T]$ with values in H. Indeed, by (20) we have that

$$\|M(u_{t_1}) - M(u_{t_2})\| \le C_{R_T} \max_{[-h,0]} \left\| \int_{t_2+\theta}^{t_1+\theta} \dot{u}(\xi)d\xi \right\| \le C_{R_T} R_T |t_1 - t_2|.$$

Thus the derivative $\dot{f}(t)$ (in the sense of distributions) is bounded in H. This allows us to apply Theorem 2.3.8 [8, p. 63] (see also [41, Chapter 4]) to obtain the conclusion in (22).

Property (23) follows from [8, Proposition 2.4.37]. □

Remark 2.7. The property in (20) means that M is Lipschitz on subsets in $C = C([-h,0];H)$ which are bounded in W. Following [31, Definition 1.1, p. 106] we call this property as "locally almost Lipschitz" on C. It is also remarkable that in order to obtain strong solutions we need to assume an additional smoothness of initial data in the right end point of the interval $[-h,0]$ only (see (21)). A similar effect was observed earlier in [36,38] in the context of parabolic PDEs with discrete state-dependent delay.

We also note that under conditions of Corollary 2.6 with differentiable F we have that solutions are C^2 on the semi-axis $\mathbb{R}_+$ with values in H, and in C^1 on the extended semi-axis $[-h,+\infty)$. Assuming the smoothness of the initial data φ and some compatibility conditions we can show that the solutions are C^2-smooth on $[-h,+\infty)$. More precisely, if we assume that

$$\varphi \in W_{sm} = C^2([-h,0];H) \cap C^1([-h,0];D(A^{1/2})) \cap C([-h,0];D(A)), \tag{24}$$

then the solution u possesses the property in (23) with $[-h,+\infty)$ instead of $\mathbb{R}_+$ if and only if this smoothness property holds at time 0. The later property is obviously valid if and only if we have the following compatibility condition

$$\ddot{\varphi}(0) + k\dot{\varphi}(0) + A\varphi(0) + F(\varphi(0)) + M(\varphi) = 0. \tag{25}$$

Moreover, one can see that the set

$$\mathcal{L} = \{\varphi \in W_{sm} : \varphi \text{ satisfies (25)}\} \subset W \tag{26}$$

is forward invariant with respect to the flow S_t, i.e., $S_t\mathcal{L} \subset \mathcal{L}$ for all $t > 0$. Thus the dynamics is defined in a smother space. The set $\mathcal{L}$ is an analog to the solution manifold used in [45] for the ODE case and in [38] for the parabolic PDE case as a well-posedness class.

3. Asymptotic properties: dissipativity

Now we start to study the long-time dynamics of the system (S_t, W) generated by mild solutions to problem (1). For this we need to impose additional hypotheses. In analogy with [7] and [8, Chapter 8]

concerning the nonlinear (non-delayed) term F we assume

(F3) The nonlinear term $F : D(A^{\frac{1}{2}}) \to H$ (see (F2) above for notations) satisfies
(a) there are constants $\eta \in [0,1), c_4,\ c_5 > 0$ such that

$$-(u, F(u)) \le \eta \left\| A^{\frac{1}{2}} u \right\|^2 - c_4 \Pi_0(u) + c_5, \quad u \in D\left(A^{\frac{1}{2}}\right); \tag{27}$$

(b) for every $\widetilde{\eta} > 0$ there exists $C_{\widetilde{\eta}} > 0$ such that

$$\|u\|^2 \le C_{\widetilde{\eta}} + \widetilde{\eta}\left(\left\| A^{\frac{1}{2}} u \right\|^2 + \Pi_0(u)\right), \quad u \in D\left(A^{\frac{1}{2}}\right); \tag{28}$$

(c) the non-conservative term F^* satisfies the subcritical linear growth condition, i.e., there exist $\hat{\delta} > 0, c_6, c_7 \ge 0$ such that

$$\|F^*(u)\|^2 \le c_6 + c_7 \left\| A^{\frac{1}{2}-\hat{\delta}} u \right\|^2 \quad \text{for any } u \in D\left(A^{\frac{1}{2}}\right). \tag{29}$$

More examples of F will be given in Section 6.

As for the delay term, we concentrate on the case of *discrete state-dependent delay* and impose the following hypothesis.

(M3) The nonlinear delay term $M : W \mapsto H$ has the form $M(u_t) = G(u(t - \tau(u_t)))$, where τ maps W into the interval $[0, h]$ and G is a globally Lipschitz mapping from H into itself.

Remark 3.1. (1) Since the term $M(u_t)$ satisfying (M3) can be written in the form

$$M(u_t) = G(u(t - \tau(u_t))) \equiv G\left(u(t) - \int_{t-\tau(u_t)}^{t} \dot{u}(s)\, ds\right) \tag{30}$$

for $u_t \in W$, we have that

$$\|M(u_t)\| \le \|G(0)\| + L_G \left[\|u(t)\| + \int_{t-h}^{t} \|\dot{u}(s)\|\, ds\right],$$

where L_G is the Lipschitz constant of the mapping G. This yields that

$$\|M(u_t)\|^2 \le g_0 + g_1 \|u(t)\|^2 + g_2(h) \int_{t-h}^{t} \|\dot{u}(s)\|^2\, ds \tag{31}$$

with $g_0 = 4\|G(0)\|^2,\ g_1 = 4L_G^2$ and $g_2(h) = 2L_G^2 h$. Thus (M3) implies (M2). To guarantee (M1) we need to assume that τ is locally Lipschitz on W:

$$|\tau(\varphi^1) - \tau(\varphi^2)| \le C_\varrho \left[|\varphi^1 - \varphi^2|_{C_{1/2}} + |\dot{\varphi}^1 - \dot{\varphi}^2|_{C_0}\right]$$

for every $\varphi^1,\ \varphi^2 \in W,\ |\varphi^j|_W \le \varrho,\ j = 1, 2$. Indeed, from (30) we have that

$$\begin{aligned}
\|M(u_s^1) - M(u_s^2)\| &\le L_G \|u^1(s - \tau(u_s^1)) - u^1(s - \tau(u_s^2))\| + L_G \|u^1(s - \tau(u_s^2)) - u^2(s - \tau(u_s^2))\| \\
&\le \varrho L_G |\tau(u_s^1) - \tau(u_s^2)| + L_G \max_{\theta \in [-h,0]} \|u^1(s+\theta) - u^2(s+\theta)\| \\
&\le (1 + \varrho C_\varrho) L_G |u_s^1 - u_s^2|_W
\end{aligned}$$

for all $u_s^1, u_s^2 \in W, |u_s^j|_W \le \varrho,\ j = 1, 2$.

(2) Instead of the structure presented in (M3) we can take a delay term of the form

$$M(u_t) = \sum_{k=1}^{N} G_k(u(t - \tau_k(u_t))),$$

or even consider an integral version of this sum. Moreover, instead of (M3) we can postulate the property in (31) with the constants g_0, g_1 independent of h and $g_2(h) \to 0$ as $h \to 0$. In particular, we can include into consideration velocity terms with a (distributed) state-dependent delay of the form

$$\int_{-h}^{0} r(\theta, u_t)\dot{u}(t+\theta)\, d\theta,$$

where $r : [-h, 0] \times W \mapsto H$ is measurable in the first variable and globally Lipschitz with respect to the second variable and satisfies appropriate properties. However, for the sake of transparency we do not pursue these generalizations.

Our first step in the study of qualitative behavior of the system (S_t, W) is the following (ultimate) dissipativity property.

Proposition 3.2. *Let assumptions* (A1), (F1), (F2), (F3), (M1) *and* (M3) *be valid. Then for any k_0 there exists $h_0 = h(k_0) > 0$ such that for every $(k, h) \in [k_0, +\infty) \times (0, h_0]$ the system (S_t, W) is dissipative, i.e., there exists $R > 0$ such that for every $\varrho > 0$ we can find $t_\varrho > 0$ such that*

$$|S_t\varphi|_W \le R \quad \text{for all } \varphi \in W, \qquad |\varphi|_W \le \varrho, \quad t \ge t_\varrho.$$

Moreover for every fixed $k_0 > 0$ the dissipativity radius R is independent of $k \ge k_0$ and the delay time $h \in (0, h_0]$. Thus the dynamical system (S_t, W) is dissipative (uniformly for $k \ge k_0$ and $h \le h_0$).

Remark 3.3. (1) The dissipativity property can be written in the form

$$\|\dot{u}(t)\|^2 + \left\|A^{\frac{1}{2}}u(t)\right\|^2 \le R^2 \quad \text{for all } t \ge t_\varrho,$$

provided the initial function $\varphi \in W$ possesses the property $|\varphi|_W \le \varrho$. We can also show in the standard way (see, e.g., [4] or [42]) that there exists a bounded *forward invariant* absorbing set $\mathcal{B}$ in W which belongs to the ball $\{\varphi \in W : |\varphi|_W \le R\}$ with the radius R independent of $k \in [k_0, +\infty)$ and $h \in [0, h_0]$.

(2) As we see in the proof below the restriction on the delay time h has the form $h \le \beta k_0$ for some $\beta > 0$. Thus increasing the low bound k_0 for the damping interval we can increase the corresponding admissible interval for h. This fact is compatible with observation that large time lag may destabilize the system. For instance, it is known from [14] that for the delayed 1D ODE

$$\ddot{u}(t) + k\dot{u}(t) + au(t) + u(t-\tau) = 0$$

with $a > 1$ and $2a > k^2$ there exist $0 < \tau_* < \tau^*$ such that the zero solution is stable for all $\tau < \tau_*$ and unstable when $\tau > \tau^*$. This example also demonstrates the role of the large damping. Indeed, if $k^2 > 2a > 2$, then (see [14]) the zero solution is stable for *all* $\tau \ge 0$. Thus large time delay requires large damping coefficient to stabilize the system.

Proof. We use the Lyapunov method to get the result. The presence of the delay term M requires some modifications of the standard functional V usually used for second order systems (see, e.g., the proof of Theorem 3.10 [7, pp. 43–46]).

We use the following functional

$$\widetilde{V}(t) \equiv \mathcal{E}(u(t), \dot{u}(t)) + \gamma(u(t), \dot{u}(t)) + \frac{\mu}{h}\int_0^h \left\{\int_{t-s}^{t} \|\dot{u}(\xi)\|^2 d\xi\right\} ds.$$

Here $\mathcal{E}$ is defined in (13) and the positive parameters γ and μ will be chosen later.

The main idea behind inclusion of an additional delay term in $\widetilde{V}$ is to find a compensator for $M(u_t)$. The compensator is determined by the structure of the mapping M (see (30) and (31)). This idea was already

applied in [8, p. 480] and [10] in the study of a flow-plate interaction model which contains a linear constant delay term with the critical spatial regularity. The corresponding compensator has a different form in the latter case and thus it is model-dependent.

One can see from (10) that there is $0 < \gamma_0 < 1$ such that

$$\frac{1}{2}E(u(t),\dot{u}(t)) - c \le \widetilde{V}(t) \le 2E(u(t),\dot{u}(t)) + \mu\int_0^h \|\dot{u}(t-\xi)\|^2 d\xi + c \tag{32}$$

for every $0 < \gamma \le \gamma_0$, where c does not depend on k.

Let us consider the time derivative of $\widetilde{V}$ along a solution. One can easily check that

$$\frac{d}{dt}(u(t),\dot{u}(t)) = \|\dot{u}(t)\|^2 - k(u(t),\dot{u}(t)) - \left\|A^{\frac{1}{2}}u(t)\right\|^2 - (u,F(u)) - (u,M(u_t)). \tag{33}$$

Combining (33) with the energy relation in (12) and using the estimate $k(u,\dot{u}) \le k^2\|\dot{u}\|^2 + \frac{1}{4}\|u\|^2$ we get

$$\begin{aligned}\frac{d}{dt}\widetilde{V}(t) \le{}& -(k-\gamma(1+k^2)-\mu)\|\dot{u}(t)\|^2 - (F^*(u(t)) + M(u_t),\dot{u}(t))\\ &-\gamma\left(-\frac{1}{4}\|u(t)\|^2 + \left\|A^{\frac{1}{2}}u(t)\right\|^2 + (u,F(u)) + (u,M(u_t))\right) - \frac{\mu}{h}\int_0^h \|\dot{u}(t-\xi)\|^2 d\xi.\end{aligned}$$

Using (29) we get

$$|(F^*(u(t)),\dot{u}(t))| \le \frac{1}{8}k\|\dot{u}(t)\|^2 + \frac{2}{k}\|F^*(u(t))\|^2 \le \frac{1}{8}k\|\dot{u}(t)\|^2 + \frac{2c_6}{k} + \frac{2c_7}{k}\|A^{1/2-\delta}u(t)\|^2.$$

Hence using the inequality $|(M(u_t),\dot{u}(t))| \le \frac{1}{8}k\|\dot{u}(t)\|^2 + \frac{2}{k}\|M(u_t)\|^2$ and also estimate (31) we obtain that

$$-(F^*(u(t)) + M(u_t),\dot{u}(t)) \le \frac{1}{4}k\|\dot{u}(t)\|^2 + \frac{c_0}{k}\left[1 + \|A^{1/2-\delta}u(t)\|^2 + \|u(t)\|^2\right] + \frac{2g_2(h)}{k}\int_0^h \|\dot{u}(t-\xi)\|^2 d\xi,$$

where $c_0 = 2\max\{c_7; c_6 + g_0, g_1\} > 0$ does not depend on k.

In a similar way (see (31)) we also have that

$$\frac{1}{4}\|u(t)\|^2 + |(u(t),M(u_t))| \le g_2(h)\int_0^h \|\dot{u}(t-\xi)\|^2 d\xi + C(g_0,g_1)(1+\|u(t)\|^2).$$

The relations in (27) and (28) with small enough $\widetilde{\eta} > 0$ yield

$$C(g_0,g_1)(1+\|u\|^2) - \left\|A^{\frac{1}{2}}u\right\|^2 - (u,F(u)) \le -3a_0E(u,\dot{u}) + \|\dot{u}\|^2 + a_1$$

for some $a_i > 0$. Thus it follows from the relations above that

$$\begin{aligned}\frac{d}{dt}\widetilde{V}(t) \le{}& -\left(\frac{3}{4}k - \gamma(2+k^2) - \mu\right)\|\dot{u}(t)\|^2 + \frac{c_0}{k}\left[1 + \|A^{1/2-\delta}u(t)\|^2 + \|u(t)\|^2\right]\\ &+\gamma\left(-3a_0E(u(t),\dot{u}(t)) + a_1\right) + \left[-\frac{\mu}{h} + \left(\frac{2}{k}+\gamma\right)g_2(h)\right]\int_0^h \|\dot{u}(t-\xi)\|^2 d\xi.\end{aligned}$$

As in [7, p. 45] using (28) we can conclude

$$\frac{c_0}{k}\left[\|A^{1/2-\delta}u(t)\|^2 + \|u(t)\|^2\right] \le \gamma a_0E(u(t),\dot{u}(t)) + \frac{1}{k}b\Big(\frac{1}{\gamma k}\Big),$$

where $b(s)$ is a non-decreasing function. Thus using (32) we arrive at the relation

$$\begin{aligned}\frac{d}{dt}\widetilde{V}(t) + \gamma a_0\widetilde{V}(t) \le{}& -\left(\frac{3}{4}k - \gamma(2+k^2) - \mu\right)\|\dot{u}(t)\|^2 + \gamma\left[\tilde{a} + \frac{1}{\gamma k}\tilde{b}\Big(\frac{1}{\gamma k}\Big)\right],\\ &+\left[-\frac{\mu}{h} + \mu\gamma a_0 + \left(\frac{2}{k}+\gamma\right)g_2(h)\right]\int_0^h \|\dot{u}(t-\xi)\|^2 d\xi.\end{aligned} \tag{34}$$

Take $\mu = \frac{k}{4}$ and $\gamma = \frac{\sigma k}{4+2k^2}$, where $0 < \sigma < 1$ is chosen such that $\gamma \le \gamma_0$ for all $k > 0$ (the bound γ_0 arises in (32)). Assume also that h is such that

$$-\frac{k}{4h} + \frac{\gamma k}{4} a_0 + \left(\frac{2}{k} + \gamma\right) g_2(h) \le 0. \tag{35}$$

Then (34) implies that

$$\frac{d}{dt}\widetilde{V}(t) + \gamma a_0 \widetilde{V}(t) \le \gamma \left[\tilde{a} + \frac{1}{\gamma k}\tilde{b}\Big(\frac{1}{\gamma k}\Big)\right]. \tag{36}$$

One can see there is $\sigma_0 = \sigma_0(k_0)$ such that $\sigma_0 \le \gamma k \le \sigma/2$ for all $k \ge k_0$. Therefore from (36) we obtain that

$$\widetilde{V}(t) \le \widetilde{V}(0)e^{-\gamma a_0 t} + \frac{1}{a_0}(1 - e^{-\gamma a_0 t}) \left[\tilde{a} + \frac{1}{\sigma_0}\tilde{b}\Big(\frac{1}{\sigma_0}\Big)\right], \tag{37}$$

provided

$$-\frac{k_0}{4h} + \frac{1}{8}a_0 + g_2(h)\left(\frac{2}{k_0} + \frac{1}{2}\right) \le 0. \tag{38}$$

Here we used (35) and properties $\gamma k < \frac{1}{2}$, $\gamma < \frac{1}{2}$ which follow from the choice of γ. One can see that there exists $\beta > 0$ such that (38) holds when $h \le \beta k_0$. Under this condition relation (37) implies the desired (uniform in k) dissipativity property[5] and completes the proof of Proposition 3.2. □

4. Asymptotic properties: quasi-stability

In this section we show that the system (S_t, W) generated by the delay equation in (1) possesses some asymptotic compactness property which is called "quasi-stability" (see e.g., [8,9]) and means that any two trajectories of the system are convergent modulo compact term. As it was already seen at the level of non-delayed systems (see, e.g., [7–9] and the references therein) this property usually leads to several important conclusions concerning global long-time dynamics of the system.

Quasi-stability requires additional hypotheses concerning the system. We assume

(M4) There exists $\delta > 0$ such that the delay term M satisfies subcritical local Lipschitz property i.e. for any $\varrho > 0$ there exists $L(\varrho) > 0$ such that for any $\varphi^i,\ i = 1, 2$ such that $\|\varphi^i\|_W \le \varrho$, one has

$$\|M(\varphi^1) - M(\varphi^2)\| \le L(\varrho) \max_{\theta\in[-h,0]} \|A^{1/2-\delta}(\varphi^1(\theta) - \varphi^2(\theta))\|. \tag{39}$$

As in Remark 3.1 one can see that (39) holds for M given by (30) if we assume that

$$|\tau(\varphi^1) - \tau(\varphi^2)| \le L_\tau(\varrho) \max_{\theta\in[-h,0]} \|A^{1/2-\delta}(\varphi^1(\theta) - \varphi^2(\theta))\|. \tag{40}$$

Below we also distinguish the cases of critical and subcritical (non-delayed) nonlinearities F. We introduce the following hypothesis.

(F4) We assume that the nonlinear (non-delayed) mapping $F : D(A^{\frac{1}{2}}) \to H$ satisfies one of the following conditions:

[5] In fact for this property we only need that $g_2(h) \to 0$ as $h \to 0$ in estimate (31).

(a) **either** it is *subcritical*, i.e., there is positive η such that for any $R > 0$ there exists $L_F(R) > 0$ such that

$$\|F(u^1) - F(u^2)\| \le L_F(R) \left\| A^{\frac{1}{2}-\eta}(u^1 - u^2) \right\|, \quad \forall\, u^1, u^2 \in D\left(A^{\frac{1}{2}}\right), \ \left\| A^{\frac{1}{2}} u^i \right\| \le R; \tag{41}$$

(b) **or else** it is *critical*, i.e., (41) holds with $\eta = 0$, and the damping parameter k is large enough.

Theorem 4.1 (*Quasi-Stability*)**.** *Let assumptions* (A1), (F1), (F2), (F4), (M1), (M2) *and* (M4) *be in force. Then there exist positive constants* $C_1(R), \widetilde{\lambda}$ *and* $C_2(R)$ *such that for any two solutions* $u^i(t)$ *with initial data* φ^i *and possessing the properties*

$$\|\dot{u}^i(t)\|^2 + \left\| A^{\frac{1}{2}} u^i(t) \right\|^2 \le R^2 \quad \textit{for all } t \ge -h, \ i = 1, 2, \tag{42}$$

the following quasi-stability estimate holds:

$$\begin{aligned} &\|\dot{u}^1(t) - \dot{u}^2(t)\|^2 + \left\| A^{\frac{1}{2}}(u^1(t) - u^2(t)) \right\|^2 \\ &\quad \le C_1(R) e^{-\widetilde{\lambda} t} |\varphi^1 - \varphi^2|_W^2 + C_2(R) \max_{\xi \in [0,t]} \|A^{1/2-\delta}(u^1(\xi) - u^2(\xi))\|^2 \end{aligned} \tag{43}$$

with some $\delta > 0$*. In the critical case* $k \ge k_0(R)$ *for some* $k_0(R) > 0$*.*

We emphasize that Theorem 4.1 does not assume (F3) and (M3) and deals only with a pair of uniformly bounded solutions. However, if the conditions in (F3) and (M3) are valid, then by Proposition 3.2 and Remark 3.3(1) there exists a bounded forward invariant absorbing set. Thus under the conditions of Proposition 3.2 we can apply Theorem 4.1 on this set. Namely, we have the following assertion.

Corollary 4.2. *Let conditions* (A1), (F1)–(F4) *and* (M3) *with* (40) *be in force. Let* $\mathcal{B}_0$ *be a forward invariant absorbing set for* (S_t, W) *such that* $\mathcal{B}_0 \subset \{\varphi \in W : |\varphi|_W \le R\}$*. Then there exist* $C_i(R) > 0$ *and* $\widetilde{\lambda} > 0$ *such that* (43) *holds for any pair of solutions* $u^1(t)$ *and* $u^2(t)$ *starting from* $\mathcal{B}_0$*.*

Remark 4.3. Taking in (43) maximum over the interval $[t - h, t]$ yields

$$|S_t\varphi^1 - S_t\varphi^2|_W \le C_1(R) h e^{\widetilde{\lambda} h} e^{-\widetilde{\lambda} t} |\varphi^1 - \varphi^2|_W + C_2(R) h \max_{s \in [0,t]} \mu_W(u_s^1 - u_s^2), \quad t \ge h \tag{44}$$

where $\mu_W(\varphi) \equiv \left\{ \max_{\theta \in [-h,0]} \|A^{\frac{1}{2}-\delta}\varphi(\theta)\| \right\}$ is a compact semi-norm[6] on W. The quasi-stability property in (44) has the structure which is different from the standard form (see, e.g., [7–9]) of quasi-stability inequalities for (non-delayed) second order in time equations. However as we will see below the consequences in our case are the same as in the case of standard quasi-stable systems. We also note that quasi-stability properties in different forms were important in many situations in the long-time dynamics studies (see, e.g., the discussion in [8, Remark 7.9.3]).

We split the proof of Theorem 4.1 in two cases and start with the simplest one.

Proof of Theorem 4.1 in the subcritical case

We rely on the mild solutions form (8) of the problem and follow the line of argument given in [8, pp. 479–480] with modifications necessary for the case of state-dependent delay force M. We note that

[6] We recall that a semi-norm $\widetilde{n}(x)$ on a Banach space X is said to be compact iff for any bounded set $B \subset X$ there exists a sequence $\{x^n\} \subset B$ such that $\widetilde{n}(x^m - x^k) \to 0$ as $m, k \to \infty$.

similar to [7, pp. 58–62] we can also use here the multipliers method. However for the completeness we demonstrate here the constant variation method. The multipliers method is presented below in the case of the critical force F.

Let us consider two solutions $U^1 = (u^1, \dot{u}^1)$ and $U^2 = (u^2, \dot{u}^2)$ of (1), (5) possessing (42). Using (8) and exponential stability of the semigroup $e^{-\mathcal{A}t}$ in the space $Y = D(A^{1/2}) \times H$ we have that

$$\|U^1(t) - U^2(t)\|_Y \le e^{-\widetilde{\lambda}t}\|U^1(0) - U^2(0)\|_Y + \int_0^t e^{-\widetilde{\lambda}(t-s)}\|\mathcal{N}(U_s^1) - \mathcal{N}(U_s^2)\|_Y\, ds, \quad t > 0, \tag{45}$$

with $\widetilde{\lambda} > 0$, where $\mathcal{N}$ is given by (7). Since

$$\|\mathcal{N}(U_s^1) - \mathcal{N}(U_s^2)\|_Y \le \|F(u^1(t)) - F(u^2(t))\| + \|M(u_t^1) - M(u_t^2)\|,$$

using properties (39) and (41) we obtain

$$\|\mathcal{N}(U_s^1) - \mathcal{N}(U_s^2)\|_Y \le C(R) \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}(u^1(s+\theta) - u^2(s+\theta))\right\|$$

for some $\delta > 0$. Thus (45) yields

$$\|U^1(t) - U^2(t)\|_Y \le e^{-\widetilde{\lambda}t}\|U^1(0) - U^2(0)\|_Y + C(R)I(t, u^1 - u^2), \quad t > 0, \tag{46}$$

where

$$I(t,z) = \int_0^t e^{-\widetilde{\lambda}(t-s)} \max_{\ell\in[-h,0]} \left\|A^{\frac{1}{2}-\delta} z(s+\ell)\right\| ds \quad \text{with } z(s) = u^1(s) - u^2(s).$$

Now we split $I(t,z)$ as $I(t,z) = I^1(t,z) + I^2(t,z)$, where

$$\begin{aligned} I^1(t,z) &\equiv \int_0^h e^{-\widetilde{\lambda}(t-s)} \max_{\ell\in[-h,0]} \left\|A^{\frac{1}{2}-\delta} z(s+\ell)\right\| ds \le C_{R,h}|z_0|_W \int_0^h e^{-\widetilde{\lambda}(t-s)}\, ds \\ &= C_{R,h}|z_0|_W \cdot e^{-\widetilde{\lambda}t}(e^{\widetilde{\lambda}h} - 1)\widetilde{\lambda}^{-1} \end{aligned}$$

and

$$\begin{aligned} I^2(t,z) &\equiv \int_h^t e^{-\widetilde{\lambda}(t-s)} \max_{\ell\in[-h,0]} \left\|A^{\frac{1}{2}-\delta} z(s+\ell)\right\| ds \\ &\le \int_0^t e^{-\widetilde{\lambda}(t-s)} \max_{\xi\in[0,t]} \left\|A^{\frac{1}{2}-\delta} z(\xi)\right\| ds = (1 - e^{-\widetilde{\lambda}t})\widetilde{\lambda}^{-1} \cdot \max_{\xi\in[0,t]} \left\|A^{\frac{1}{2}-\delta} z(\xi)\right\|. \end{aligned}$$

Thus (46) yields the desired estimate in (43) for the subcritical nonlinearity F.

Proof of Theorem 4.1 in the critical case with large damping

We follow the line of the arguments of [7, p. 85, Theorem 3.58].

Let u^1 and u^2 be solutions satisfying (42). Then $z = u^1 - u^2$ solves the equation

$$\ddot{z}(t) + Az(t) + k\dot{z}(t) = -F_{1,2}(t) - M_{1,2}(t) \tag{47}$$

with

$$F_{1,2}(t) \equiv F(u^1(t)) - F(u^2(t)); \qquad M_{1,2}(t) \equiv M(u_t^1) - M(u_t^2).$$

We multiply the last equation by $\dot{z}(t)$ and integrate over $[t,T]$:

$$E_z(T) - E_z(t) + k\int_t^T \|\dot{z}(s)\|^2\,ds = -\int_t^T (F_{1,2}(s),\dot{z}(s))\,ds - \int_t^T (M_{1,2}(s),\dot{z}(s))\,ds. \tag{48}$$

Here we denote $E_z(t) \equiv \frac{1}{2}(\|\dot{z}(t)\|^2 + \|A^{\frac{1}{2}}z(t)\|^2)$.

One can check that there is constant $C_R > 0$ such that

$$|(F_{1,2}(t),\dot{z}(t))| \le \varepsilon \left\|A^{\frac{1}{2}}z(t)\right\|^2 + \frac{C_R}{\varepsilon}\|\dot{z}(t)\|^2, \quad \forall\,\varepsilon > 0.$$

Similarly, using assumption (M4), we have

$$|(M_{1,2}(t),\dot{z}(t))| \le \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}z(t+\theta)\right\|^2 + C_R\|\dot{z}(t)\|^2.$$

Hence, we get from (48)

$$\begin{aligned}\left|E_z(T) - E_z(t) + k\int_t^T \|\dot{z}(s)\|^2\,ds\right| &\le \varepsilon\int_t^T \left\|A^{\frac{1}{2}}z(s)\right\|^2 ds + \int_t^T \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}z(s+\theta)\right\|^2 ds \\ &\quad + C_R\left(1+\frac{1}{\varepsilon}\right)\int_t^T \|\dot{z}(s)\|^2\,ds \end{aligned} \tag{49}$$

for every $\varepsilon > 0$. Below we choose (assume that) k is big enough to satisfy (see the last term in (49))

$$C_R\left(1+\frac{1}{\varepsilon}\right) < \frac{k}{2}, \quad \text{for all } k \ge k_0. \tag{50}$$

This choice is made for the simplification of the estimates only (the final choice of k_0 to be done after the choice of ε). Now we multiply (47) by $z(t)$ and integrate over $[0,T]$, using integration by parts. This yields

$$\begin{aligned}&(\dot{z}(T),z(T)) - (\dot{z}(0),z(0)) - \int_0^T \|\dot{z}(s)\|^2\,ds + \int_0^T \left\|A^{\frac{1}{2}}z(s)\right\|^2 ds + k\int_0^T (\dot{z}(s),z(s))\,ds \\ &\le \frac{1}{2}\int_0^T \left\|A^{\frac{1}{2}}z(s)\right\|^2 ds + \widetilde{C_R}\int_0^T \|z(s)\|^2\,ds + \widetilde{C_R}\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}z(s+\theta)\right\|^2 ds.\end{aligned}$$

Hence, using the definition of E_z after (48) and the relation

$$k\int_0^T (\dot{z}(s),z(s))\,ds \le \frac{1}{2}\int_0^T \|\dot{z}(s)\|^2\,ds + \frac{k^2}{2}\int_0^T \|z(s)\|^2\,ds,$$

we obtain that

$$\begin{aligned}\frac{1}{2}\int_0^T \left\|A^{\frac{1}{2}}z(s)\right\|^2 ds &\le \frac{3}{2}\int_0^T \|\dot{z}(s)\|^2\,ds + C(E_z(0)+E_z(T)) \\ &\quad + \widetilde{C_R}(k)\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}z(s+\theta)\right\|^2 ds.\end{aligned} \tag{51}$$

From (49) with $t=0$ and using (50) we get

$$E_z(0) \le E_z(T) + \frac{3k}{2}\int_0^T \|\dot{z}(s)\|^2\,ds + \varepsilon\int_0^T \left\|A^{\frac{1}{2}}z(s)\right\|^2 ds + \int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}z(s+\theta)\right\|^2 ds. \tag{52}$$

It follows from (49) with help of integration over $[0,T]$ (we use (50) again) that

$$TE_z(T) \le \int_0^T E_z(s)\,ds + \varepsilon T\int_0^T \left\|A^{\frac{1}{2}}z(s)\right\|^2 ds + T\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}-\delta}z(s+\theta)\right\|^2 ds. \tag{53}$$

Another consequence of (49) for $t = 0$, using (50), is

$$\frac{k}{2}\int_0^T \|\dot z(s)\|^2\,ds \le E_z(0) + \varepsilon\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + \int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds. \tag{54}$$

Considering the sum of (54) and (51) and assuming that $k \ge 8$ we can get

$$k\int_0^T \|\dot z(s)\|^2\,ds + \int_0^T E_z(s)\,ds \le C(E_z(0)+E_z(T)) + 4\varepsilon\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + C^*_{R,k}\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds. \tag{55}$$

Now we add to the both sides of (55) the value $\frac12 T E_z(T)$ and use (53)

$$k\int_0^T \|\dot z(s)\|^2\,ds + \frac12\int_0^T E_z(s)\,ds + \frac12 T E_z(T) \le 4\varepsilon(1+T)\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + C(E_z(0)+E_z(T)) + C_{R,k}(1+T)\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds. \tag{56}$$

Now we evaluate $E_z(0) + E_z(T)$. Using (52) we have that

$$E_z(0)+E_z(T) \le 2E_z(T) + \frac{3k}{2}\int_0^T \|\dot z(s)\|^2\,ds + \varepsilon\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + \int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds.$$

Substituting this into (56) we get that

$$\frac12\int_0^T E_z(s)\,ds + \left(\frac12 T - 2C\right) E_z(T) \le c_0 k\int_0^T \|\dot z(s)\|^2\,ds + c_1\varepsilon\,(1+T)\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + \widetilde{C_R}(k)(1+T)\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds.$$

Assuming that

$$\frac12 T - 2C > 1, \tag{57}$$

we get

$$E_z(T) + \frac12\int_0^T E_z(s)\,ds \le C_1\varepsilon\,(1+T)\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + (1+T)\,\widetilde{C_R}(k)\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds + \widetilde{c_0} k\int_0^T \|\dot z(s)\|^2\,ds. \tag{58}$$

To estimate the last term in (58) we use (49) with $t = 0$ (remind (50)) to get

$$\frac{k}{2}\int_0^T \|\dot z(s)\|^2\,ds \le E_z(0) - E_z(T) + \varepsilon\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + \int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds.$$

So, we can rewrite (58) as

$$E_z(T) + \frac12\int_0^T E_z(s)\,ds \le 2c_0(E_z(0)-E_z(T)) + C_1\varepsilon\,(1+T)\int_0^T \left\|A^{\frac12} z(s)\right\|^2 ds + (1+T)\,\widetilde{C_R}(k)\int_0^T \max_{\theta\in[-h,0]} \left\|A^{\frac12-\delta} z(s+\theta)\right\|^2 ds. \tag{59}$$

Since $\|A^{\frac12} z(s)\|^2 \le 2E_z(s)$, the choice of small $\varepsilon > 0$ to satisfy

$$C_1\varepsilon\,(1+T) < \frac14 \tag{60}$$

simplifies (59) as follows

$$E_z(T) \le \widetilde{c_0}(E_z(0) - E_z(T)) + (1+T)\,\widetilde{C_R}(k) \int_0^T \max_{\theta\in[-h,0]} \left\| A^{\frac{1}{2}-\delta} z(s+\theta) \right\|^2 ds.$$

The last step is

$$E_z(T) \le \frac{\widetilde{c_0}}{1+\widetilde{c_0}} E_z(0) + \widetilde{C_R}(T,k) \int_0^T \max_{\theta\in[-h,0]} \left\| A^{\frac{1}{2}-\delta} z(s+\theta) \right\|^2 ds.$$

Since $\gamma \equiv \frac{\widetilde{c_0}}{1+\widetilde{c_0}} < 1$ this means that there is $w > 0$ such that

$$E_z(T) \le e^{-wT} E_z(0) + C_{R,T,k} \int_0^T \max_{\theta\in[-h,0]} \left\| A^{\frac{1}{2}-\delta} z(s+\theta) \right\|^2 ds. \tag{61}$$

We mention that the parameters were chosen in the following order. First we choose $T > h$ to satisfy (57), next we choose small $\varepsilon > 0$ to satisfy (60) and finally we choose k big enough to satisfy (50).

Now using the same step by step procedure ($mT \mapsto (m+1)T$) as in the Remark 3.30 [7] we can derive the conclusion in (43) from the relation in (61) written on the interval $[mT, (m+1)T]$. Thus the proof of Theorem 4.1 is complete.

5. Global and exponential attractor

In this section relying on Proposition 3.2 and Theorem 4.1 we establish the existence of a global attractor and study its properties. We recall (see, e.g., [1,4,42]) that a *global attractor* of the dynamical system (S_t, W) is defined as a bounded closed set $\mathfrak{A} \subset W$ which is invariant ($S_t\mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and uniformly attracts all other bounded sets:

$$\lim_{t\to\infty} \sup\{\mathrm{dist}_W(S_t y, \mathfrak{A}) : \ y \in B\} = 0 \quad \text{for any bounded set } B \text{ in } W.$$

We note (see, e.g., [42]) that the global attractor consists of bounded full trajectories. In the case of the delay system (S_t, W) a full trajectory can be described as a function u from $C(\mathbb{R}, D(A^{1/2})) \cap C^1(\mathbb{R}, H)$ possessing the property $S_t u_s = u_{t+s}$ for all $s \in \mathbb{R}, t \ge 0$.

The main consequence of dissipativity and quasi-stability given by Proposition 3.2 and Theorem 4.1 is the following theorem.

Theorem 5.1 (*Global Attractor*)**.** *Let assumptions* (A1) *and* (F1)–(F4) *be in force. Assume that the term* $M(u_t)$ *has form* (30) *with* $\tau : W \mapsto [0,h]$ *possessing property* (40)*. Then the dynamical system* (S_t, W) *generated by* (1) *possesses the compact global attractor* $\mathfrak{A}$ *of finite fractal dimension.*[7] *Moreover, for any full trajectory* $\{u(t) : t \in \mathbb{R}\}$ *such that* $u_t \in \mathfrak{A}$ *for all* $t \in \mathbb{R}$ *we have that*

$$\ddot{u} \in L_\infty(\mathbb{R}, H), \qquad \dot{u} \in L_\infty(\mathbb{R}, D(A^{1/2})) \qquad u \in L_\infty(\mathbb{R}, D(A)) \tag{62}$$

and

$$\|\ddot{u}(t)\| + \|A^{1/2}\dot{u}(t)\| + \|Au(t)\| \le R_*, \quad \forall\, t \in \mathbb{R}. \tag{63}$$

Under the hypotheses of Corollary 2.6 we also have that $\mathfrak{A}$ *is a bounded set in* W_{sm} *and lies in* $\mathcal{L}$*, where* W_{sm} *and* $\mathcal{L}$ *are given by* (24) *and* (26)*.*

[7] For the definition and some properties of the fractal dimension, see, e.g., [4] or [42].

Proof. Since the system (S_t, W) is dissipative (see Proposition 3.2), for the existence of a compact global attractor we need to prove that (S_t, W) is asymptotically smooth.[8] For this we can use the Ceron–Lopes type criteria (see, e.g., [22] or [7]) which in fact states (see [7, p. 19, Corollary 2.7]) that the quasi-stability estimate in (44) implies that (S_t, W) is an asymptotically smooth dynamical system. Thus the existence of a compact global attractor is established.

To get the finite dimensionality of the attractor we apply the same idea as in [7,8] which is originated from the Málek–Nečas method of "short" trajectories (see [29] and also [30] where it was later developed as the method of l-trajectories). However we use a different choice of the space of "short" (or l-) trajectories which is motivated by the delay structure of the model and the choice of the phase space.

As in [7,8] we rely on the abstract result [7, Theorem 2.15, p. 23] on finite dimensionality of bounded closed sets in a Banach space which are invariant with respect to a Lipschitz mapping possessing some squeezing property. We consider the auxiliary space

$$W(-h,T) \equiv C\left([-h,T]; D\left(A^{\frac{1}{2}}\right)\right) \cap C^1([-h,T];H), \quad T>0,$$

endowed with the norm

$$|\varphi|_{W(-h,T)} = \max_{s\in[-h,T]} \|A^{1/2}\varphi(s)\| + \max_{s\in[-h,T]} \|\dot{\varphi}(s)\|.$$

We note that in the case $T=0$ we have $W(-h,0)=W$. Thus $W(-h,T)$ is the space of extensions with the same smoothness of functions from W on the interval $[-h,T]$.

Let $\mathcal{B}$ be a set in the phase space W. We denote by $\mathcal{B}_T$ the set of functions $u \in W(-h,T)$ which solve (1) with initial data $u_{t\in[-h,0]} = \psi \in \mathcal{B}$. We interpret $\mathcal{B}_T$ as a set of "pieces" of trajectories starting from $\mathcal{B}$. We also define the shift (along solutions to (1)) operator $\mathcal{R}_T : \mathcal{B}_T \mapsto W(-h,T)$ by the formula

$$(\mathcal{R}_T u)(t) = u(T+t), \quad t\in[-h,T], \tag{64}$$

where u is the solution to (1) with initial data from $\mathcal{B}$.

The following lemma states that the mapping $\mathcal{R}_T$ satisfies some contractive property modulo compact terms.

Lemma 5.2. *Let $\mathcal{B}$ be a forward invariant set for the dynamical system (S_t, W) such that $\mathcal{B} \in \{\phi : |\phi|_W \le R\}$ for some R. Let $T>h$. Then $\mathcal{B}_T$ is forward invariant with respect to the shift operator $\mathcal{R}_T$ and*

$$|\mathcal{R}_T\varphi^1 - \mathcal{R}_T\varphi^2|_{W(-h,T)} \le c_1(R)e^{-\widetilde{\lambda}(T-h)}|\varphi^1-\varphi^2|_{W(-h,T)} + c_2(R)\left[n(\varphi^1-\varphi^2) + n(\mathcal{R}_T\varphi^1 - \mathcal{R}_T\varphi^2)\right] \tag{65}$$

for every $\varphi^1, \varphi^2 \in \mathcal{B}_T$, where $n(\varphi) = \sup_{s\in[0,T]} \|A^{\frac{1}{2}-\delta}\varphi(s)\|$ is a compact seminorm (see the footnote in Remark 4.3 for the definition) on the space $W(-h,T)$.

Proof. The invariance of $\mathcal{B}_T$ is obvious due to the construction. The relation in (65) follows from Theorem 4.1. The compactness of the seminorm n is implied by the infinite dimensional version of Arzelà–Ascoli theorem, see the Appendix in [8], for instance. □

We choose $T>h$ such that $\eta_T = c_1(R)e^{-\widetilde{\lambda}(T-h)} < 1$ and take $\mathcal{B} = \mathfrak{A}$, where $\mathfrak{A}$ is the global attractor. It is clear that the set $\mathfrak{A}_T$ is strictly invariant. Therefore we can apply [7, Theorem 2.15, p. 23] to get the finite dimensionality of the set $\mathfrak{A}_T$ in $W(-h,T)$. The final step is to consider the restriction mapping

$$r_h : \{u(t),\ t\in[-h,T]\} \mapsto \{u(t),\ t\in[-h,0]\}$$

[8] According to [22] this means that for any bounded forward invariant set B in W there exists a compact set K in W which attracts uniformly S_tB as $t\to+\infty$.

which is obviously Lipschitz continuous from $W(-h,T)$ into W. Since $r_h\mathfrak{A}_T = \mathfrak{A}$ and Lipschitz mappings do not increase fractal dimension of a set, we conclude that

$$\dim_f^W \mathfrak{A} \le \dim_f^{W(-h,T)} \mathfrak{A}_T < \infty.$$

To prove the regularity properties in (62) and (63) we can use Theorem 4.1 and the same idea as in [7,8], see also [9]. Indeed, let $\gamma = \{u(t) : \ t \in \mathbb{R}\}$ be a full trajectory of the system, i.e., $(S_t u_s)(\theta) = u(t+s+\theta)$ for $\theta \in [-h,0]$. Assume that $u_t \in \mathfrak{A}$ for all $t \in \mathbb{R}$. Consider the difference of this trajectory and its small shift $\gamma_\varepsilon = \{u(t+\varepsilon) : \ t \in \mathbb{R}\}$ and apply the inequality in (43) with starting point at $s \in \mathbb{R}$:

$$\begin{aligned}&\|\dot u(t+\varepsilon) - \dot u(t)\|^2 + \left\|A^{\frac12}(u(t+\varepsilon) - u(t))\right\|^2 \\ &\quad \le C_1(R) e^{-\widetilde{\lambda}(t-s)} |u_{s+\varepsilon} - u_s|_W^2 + C_2(R) \max_{\xi\in[s,t]} \|A^{1/2-\delta}(u(\xi+\varepsilon) - u(\xi))\|^2.\end{aligned}$$

Since $u_s \in \mathfrak{A}$ for all $s \in \mathbb{R}$, in the limit $s \to -\infty$ we obtain that

$$\|\dot u(t+\varepsilon) - \dot u(t)\|^2 + \left\|A^{\frac12}(u(t+\varepsilon) - u(t))\right\|^2 \le C_2(R) \sup_{\xi\in[-\infty,t]} \|A^{1/2-\delta}(u(\xi+\varepsilon) - u(\xi))\|^2.$$

Now in the same way as in [7, pp. 102, 103] or in [8, pp. 386, 387] we can conclude that

$$\frac{1}{\varepsilon^2}\left[\|\dot u(t+\varepsilon) - \dot u(t)\|^2 + \left\|A^{\frac12}(u(t+\varepsilon) - u(t))\right\|^2\right]$$

is uniformly bounded in $\varepsilon \in (0,1]$. This implies (passing with the limit $\varepsilon \to 0$) that

$$\|\ddot u(t)\|^2 + \left\|A^{\frac12}\dot u(t)\right\|^2 \le C_R.$$

Now using Eq. (1) we conclude that $\|Au(t)\|^2 \le C_R$. This gives (62) and (63).

The final statement follows from Corollary 2.6 and Remark 2.7.

This completes the proof of Theorem 5.1. □

Now we present a result on the existence of exponential attractors. We recall the following definition.

Definition 5.3 (*Cf. [17]*). A compact set $\mathfrak{A}_{\exp} \subset W$ is said to be (generalized) exponential attractor for the dynamical system (S_t, W) iff $\mathfrak{A}_{\exp}$ is a positively invariant set whose fractal dimension is finite (in some extended space $\mathcal{W} \supset W$) and for every bounded set $D \subset W$ there exist positive constants $C_D = C(|D|_W)$ and γ such that

$$d_W\{S_t D \mid \mathfrak{A}_{\exp}\} \equiv \sup_{x\in D} \operatorname{dist}_W(S_t x, \mathfrak{A}_{\exp}) \le C_D \cdot e^{-\gamma t}, \quad t \ge 0. \tag{66}$$

This concept has been introduced in [17] in the case when $\mathcal{W}$ and W are the same. For details concerning exponential attractors we refer to [17] and also to recent survey [32]. We only mention that (i) a global attractor can be non-exponential and (ii) an exponential attractor is not unique and contains the global attractor.

Using the quasi-stability estimate and ideas presented in [7,8] we can construct exponential attractors for the system considered.

Theorem 5.4. *Let the hypotheses of Theorem 5.1 be in force. Then the dynamical system (S_t, W) possesses a (generalized) exponential attractor whose dimension is finite in the space*

$$\mathcal{W} \equiv C([-h,0]; D(A^{\frac12-\delta})) \cap C^1([-h,0]; H_{-\delta}), \quad \forall\, \delta > 0,$$

where $H_{-s}, s > 0$, denotes the closure of H with respect to the norm $\|A^{-s}\cdot\|$.

Proof. Let $\mathcal{B}$ be a forward invariant bounded absorbing set for (S_t, W) which exists due to Proposition 3.2 and Remark 3.3(1). Then we apply Lemma 5.2 to obtain (discrete) quasi-stability property for the shift mapping $\mathcal{R}_T$ defined in (64) on $\mathcal{B}_T$. We choose $T > h$ in (65) such that $\eta_T = c_1(R)e^{-\widetilde{\lambda}(T-h)} < 1$ and apply [7, Corollary 2.23] which gives us that the mapping $\mathcal{R}_T$ possesses an exponential attractor $\mathcal{A}_T$. Next, using (1) we can see that $\|\ddot{u}(t)\|_{-2} < C_R$ for all $t \in \mathbb{R}_+$. This allows us to show that $S_t\varphi$ is a Hölder continuous in t in the space $\mathcal{W}$, i.e.,

$$|S_{t_1}\varphi - S_{t_2}\varphi|_{\mathcal{W}} \le C_{\mathcal{B}}|t_1 - t_2|^{\gamma}, \quad t_1, t_2 \in \mathbb{R}_+,\ y \in \mathcal{B}, \tag{67}$$

for some positive $\gamma > 0$. Now we consider the restriction map r_h (see above) and the sets $r_h\mathcal{A}_T = \mathcal{A} \subset W, \mathfrak{A}_{\exp} \equiv \bigcup\{S_t\mathcal{A} : t \in [0,T]\} \subset W$. It is clear that A_{exp} is forward invariant. Since r_h is Lipschitz from $W(-h,T)$ into $W, \mathcal{A}$ is finite-dimensional. Therefore the property in (67) implies that $\mathfrak{A}_{\exp}$ has a finite fractal dimension in $\mathcal{W}$. As in [7, p. 123] we can see that $\mathfrak{A}_{\exp}$ is an exponentially attracting set for (S_t, W). This completes the proof of Theorem 5.4. □

We note that there is a technology which allows to prove the existence of an exponential attractor which has finite dimension in the pivot space W. This is the so-called method of transitivity of exponential attraction suggested in [18], see also the survey [32]. However it seems an application of this method in our situation requires additional hypotheses related with smoothness and subcriticality of the nonlinear terms. We do not pursue these issues here and postpone the corresponding analysis for future.

In conclusion of this section we note that using quasi-stability property (43) we can also establish some other asymptotic properties of the system (S_t, W). For instance, in the same way as it is done in [7,8] we can suggest criteria which guarantee the existence of finite number of determining functionals. We also note that at this moment we do not know how to avoid the assumption of large damping in the case of critical nonlinearities F in Theorem 5.4.

6. Examples

In this section we discuss several possible applications of the results above.

6.1. Plate models

Our main applications are related to nonlinear plate models.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. In the space $H = L_2(\Omega)$ we consider the following problem

$$\partial_{tt}u(t,x) + k\partial_t u(t,x) + \Delta^2 u(t,x) + F(u(t,x)) + au(t - \tau[u_t], x) = 0, \quad x \in \Omega,\ t > 0, \tag{68a}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad u(\theta) = \varphi(\theta) \quad \text{for } \theta \in [-h, 0]. \tag{68b}$$

We assume that τ is a continuous mapping from $C(-h,0;H_0^2(\Omega)) \cap C^1(-h,0;L_2(\Omega))$ into the interval $[0,h]$. As it was already mentioned in Introduction the delay term in (68a) models the reaction of foundation.

The model in (68) can be written in the abstract form (1) with $A = \Delta^2$ defined on the domain $D(A) = H^4 \cap H_0^2(\Omega)$. Here and below $H^s(\Omega)$ is the Sobolev space of the order s and $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. In this case we have $D(A^s) = H_0^{4s}(\Omega)$ for $0 \le s \le 1/2,\ s \ne 1/8, 3/8$.

As the simplest example of delay terms satisfying all hypotheses in (M1)–(M4) we can consider

$$\tau[u_t] = g(Q[u_t]), \tag{69}$$

where g is a smooth mapping from $\mathbb{R}$ into $[0,h]$ and

$$Q[u_t] = \sum_{i=1}^{N} c_i u(t - \sigma_i, a_i).$$

Here $c_i \in \mathbb{R}, \sigma_i \in [0,h], a_i \in \Omega$ are arbitrary elements. We could also consider the term Q with the Stieltjes integral over delay interval $[-h,0]$ instead of the sum. Another possibility is to consider combination of averages like

$$Q[u_t] = \sum_{i=1}^{N} \int_\Omega u(t-\sigma_i, x)\xi_i(x)dx, \tag{70}$$

where $\sigma_i \in [0,h]$ and $\{\xi_i\}$ are arbitrary functions from $L_2(\Omega)$. We can also consider linear combinations of these Q's and also their powers and products. The corresponding calculations are simple and related to the fact that for every $s > 1/4$ the space $D(A^s)$ is an algebra belonging to $C(\overline{\Omega})$.

As for nonlinearities F satisfying requirements (F1)–(F4) they are the same as in [7,8]. Therefore delay perturbations of the models considered in these sources in the case of linear damping provides us with a series of examples. Here we only mention three of them.

Kirchhoff model: In this case $F(u) = f(u) - h(x)$, where $h \in L_2(\Omega)$, and

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \quad \text{satisfies } \liminf_{|s|\to\infty} f(s)s^{-1} = \infty. \tag{71}$$

This is a subcritical case (see assumption (F4), (41) with $\eta > 0$). The growth condition in (71) is needed to satisfy (28) in (F3).

The following two examples are critical (assumption (F4), (41) with $\eta = 0$).

Von Karman model: In this model (see, e.g., [8,27]) $F(u) = -[u, v(u) + F_0] - h(x)$, where $F_0 \in H^4(\Omega)$ and $h \in L_2(\Omega)$ are given functions,

$$[u,v] = \partial^2_{x_1} u \cdot \partial^2_{x_2} v + \partial^2_{x_2} u \cdot \partial^2_{x_1} v - 2 \cdot \partial_{x_1x_2} u \cdot \partial_{x_1x_2} v,$$

and the function $v(u)$ satisfies the equations:

$$\Delta^2 v(u) + [u,u] = 0 \quad \text{in } \Omega, \qquad \frac{\partial v(u)}{\partial n} = v(u) = 0 \quad \text{on } \partial\Omega.$$

For details concerning properties (F1)–(F4) we refer to [7, Chapter 6] and [8, Chapters 4,9].

Berger model: In this case $F(u) = \Pi'(u)$, where

$$\Pi(u) = \frac{\kappa}{4}\left[\int_\Omega |\nabla u|^2 dx\right]^2 - \frac{\mu}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega u(x)h(x)dx,$$

where $\kappa > 0$ and $\mu \in \mathbb{R}$ are parameters, $h \in L_2(\Omega)$. The analysis presented in [4, Chapter 4] and [7, Chapter 7] yields the assumptions in (F1)–(F4).

6.2. Wave model

Let $\Omega \subset \mathbb{R}^n, n = 2,3$, be a bounded domain with a sufficiently smooth boundary Γ. The exterior normal on Γ is denoted by ν. We consider the following wave equation

$$\partial_{tt}u - \Delta u + k\partial_t u + f(u) + u(t - \tau[u_t]) = 0 \quad \text{in } Q = [0,\infty) \times \Omega$$

subject to boundary condition either of Dirichlet type

$$u = 0 \quad \text{on } \Sigma \equiv [0,\infty) \times \Gamma, \tag{72}$$

or else of Robin type

$$\partial_\nu u + u = 0 \quad \text{on } \Sigma. \tag{73}$$

The initial conditions are given by $u(\theta) = \varphi(\theta)$, $\theta \in [-h, 0]$. In this case $H = L_2(\Omega)$ and A is $-\Delta$ with either the Dirichlet (72) or the Robin (73) boundary conditions. So $D(A^{1/2})$ is either $H_0^1(\Omega)$ or $H^1(\Omega)$ in this case.

We assume that k is a positive parameter and the function $f \in C^2(\mathbb{R})$ satisfies the following polynomial growth condition: there exists a positive constant $M > 0$ such that

$$|f''(s)| \le M(1 + |s|^{q-1}),$$

where $q \le 2$ when $n = 3$ and $q < \infty$ when $n = 2$. Moreover, we assume the same lower growth condition as (71). One can see that the hypotheses in (F1)–(F4) are satisfied (see [7, Chapter 5] for the detailed discussion). Moreover we have the subcritical case if $n = 2$ or $n = 3$ and $q < 2$. The case $n = 3$ and $q = 2$ is critical.

As for the delay term $u(t - \tau[u_t])$ we can assume that, as in the plate models above, $\tau[u_t]$ has the form (69) with $Q[u_t]$ given by (70). Moreover, instead of the averaging we can consider an arbitrary family of linear functionals on $H^{1-\delta}(\Omega)$ for some $\delta > 0$, i.e., we can take

$$Q[u_t] = \sum_{i=1}^{N} c_i l_i[u(t - \sigma_i)],$$

where $c_i \in \mathbb{R}, \sigma_i \in [0, h]$ and $l_i \in [H^{1-\delta}(\Omega)]'$ are arbitrary elements.

6.3. Ordinary differential equations

The results above can be also applied in the ODE case when $H = \mathbb{R}^n, A$ is a symmetric $n \times n$ matrix A and the nonlinear mappings $F : \mathbb{R}^n \to \mathbb{R}^n, M : C([-h, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ obey appropriate requirements. The space of initial states becomes $W = C^1([-h, 0]; \mathbb{R}^n)$ (cf. (3)) and hence possesses a linear structure.

Thus in contrast with the solution manifold method suggested in [45] (see also [23]) our approach does not assume any nonlinear compatibility conditions and provides us with a well-posedness result in a *linear* phase space. In addition, both approaches produce the same class of solutions after some time. To illustrate this effect we consider the same second order delay ODE as it was used in [45] as a motivating example:

$$\dot{u} = v, \qquad \dot{v} + kv = f(cs(u_t) - w), \quad t > 0, \tag{74a}$$

$$u(\theta) = \varphi^0(\theta), \qquad v(\theta) = \varphi^1(\theta) \quad \text{for } \theta \in [-h, 0]. \tag{74b}$$

Here k, c and w are positive reals, s is a state-dependent delay (implicitly defined in [45]), $f : \mathbb{R} \to \mathbb{R}$ is a smooth function (for more details see [45, pp. 61–64]). In the model u is a position of a moving object and v is its velocity. The result of [45] applied to this system says that if the initial data $(\varphi^0; \varphi^1)$ belong to $C^1([-h, 0]; \mathbb{R}^2)$ and satisfy the compatibility condition

$$\dot{\varphi}^0(0) = \varphi^1(0), \qquad \dot{\varphi}^1(0) + k\varphi^1(0) = f(cs(\varphi^0) - w), \tag{75}$$

then (74) generates (local) C^1-semiflow on the solution manifold

$$\mathcal{M} = \left\{(\varphi^0; \varphi^1) \in C^1([-h, 0]; \mathbb{R}^2) : \ (75) \text{ is satisfied}\right\}.$$

Application of our Theorem 2.4 to the same system (written as a second order equation with respect to u) says that if the initial data $(\varphi^0; \varphi^1)$ belong to $C^1([-h, 0]; \mathbb{R}) \times C([-h, 0]; \mathbb{R})$ and are compatible in the natural way (as a position and the velocity): $\dot{\varphi}^0(\theta) = \varphi^1(\theta)$ for all $\theta \in [-h, 0]$, then under the same conditions as in [45] we can avoid the (nonlinear) compatibility in (75) and construct a local semiflow in the space

$$\widetilde{W} = \left\{(\varphi^0; \varphi^1) : \ \dot{\varphi}^0(\theta) = \varphi^1(\theta) \text{ for all } \theta \in [-h, 0], \ \varphi^0 \in C^1([-h, 0]; \mathbb{R})\right\}.$$

Thus we obtain another well-posedness class for the model in (74). Moreover, by Corollary 2.6 the corresponding solution $(u(t); v(t))$ is C^1 for $t \geq 0$ and satisfies (75) for $t > h$. Hence after time $t > h$ solutions arrive at the same solution manifold $\mathcal{M}$ as in [45]. Similarly, starting at $\mathcal{M}$ after time $t > h$ we obviously arrive at $\widetilde{W}$ (see the first equation in (74a)). Thus both classes of initial functions $\widetilde{W}$ and $\mathcal{M}$ lead to exactly the same class of solutions for $t > h$.

As a bottom line we emphasize that in the case of the second order delay equations, the natural (linear) "position–velocity" compatibility provides us with an alternative point of view on dynamics and leads to a simpler well-posedness argument comparing to the method of a solution manifold presented in [45].

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