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A remark on multiobjective stochastic optimization via strongly convex functions

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Abstract Many economic and financial applications lead (from the mathematical point of view) to deterministic optimization problems depending on a probability measure. These problems can be static (one stage), dynamic with finite (multistage) or infinite horizon, single objective or multiobjective. We focus on one-stage case in multiobjective setting. Evidently, well known results from the deterministic optimization theory can be employed in the case when the "underlying" probability measure is completely known. The assumption of a complete knowledge of the probability measure is fulfilled very seldom. Consequently, we have mostly to analyze the mathematical models on the data base to obtain a stochastic estimate of the corresponding "theoretical" characteristics. However, the investigation of these estimates has been done mostly in one-objective case. In this paper we focus on the investigation of the relationship between "characteristics" obtained on the base of complete knowledge of the probability measure and estimates obtained on the (above mentioned) data base, mostly in the multiobjective case. Consequently we obtain also the relationship between analysis (based on the data) of the economic process characteristics and "real" economic process. To this end the results of the deterministic multiobjective optimization theory and the results obtained for stochastic one objective problems will be employed.

Keywords Stochastic multiobjective optimization problems · Efficient solution · Wasserstein metric and $\mathcal{L}_1$ norm · Stability and empirical estimates · Lipschitz property

Mathematics Subject Classification 90C15

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1 Introduction

To introduce a "rather general" multiobjective stochastic programming problem let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ s-dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F(z), z \in R^s)$, P_F and Z_F denote the distribution function, the probability measure and the support corresponding to ξ. Let, moreover, $g_i := g_i(x, z), i = 1, \ldots, l, l \geq 1$ be real-valued (say, continuous) functions defined on $R^n \times R^s$; $X_F \subset X \subset R^n$ be a nonempty set generally depending on F, and $X \subset R^n$ be a nonempty deterministic set. If the symbol E_F denotes the operator of mathematical expectation corresponding to F and if for every $x \in X$ there exist finite $\mathsf{E}_F g_i(x, \xi), i = 1, \ldots, l$, then a rather general "multiobjective" one-stage stochastic programming problem can be introduced in the form:

$$\text{Find} \quad \min \mathsf{E}_F g_i(x, \xi), \quad i = 1, \ldots, l \quad \text{subjectto} \quad x \in X_F. \tag{1}$$

The multiobjective problem (1) corresponds evidently to economic situation in which a "result" of an economic process is simultaneously influenced by a random factor ξ and a decision parameter x. It is reasonable to evaluate this process by a few (say $l, l \geq 1$) objective functions; moreover, the decision vector has to be determined without knowledge of the random element realization and it seems to be reasonable to determine "the decision" with respect to the mathematical expectation of the objectives.

Evidently, an "underlying" multiobjective problem with a random element can be in the form

$$\begin{aligned} &\text{Find} \quad \min \hat{g}_i(x, \xi), \quad i = 1, \ldots, l \\ &\text{subject to} \quad g_j^*(x, \xi) \leq 0, \quad j = 1, \ldots, l', \end{aligned}$$

where $\hat{g}_i(x\, z), i = 1, \ldots, l, \quad g_j^*(x, z), j = 1, \ldots, l'$ are functions defined on $R^n \times R^s$. Generally $g_i(x, z), i = 1, \ldots, l$ can depend on $\hat{g}_i(x, z),\ g_j^*(x, z), i = 1, \ldots, l, j = 1, \ldots, l'$. Of course, the approach defined by the last introduced problems with random elements is only one of possibilities. Evidently, there exists many others.

Other approaches how to treat the multiobjective problems with random elements are known from the literature [see e.g., in Ben Abdelaziz (2012), Caballero et al. (2001), Dupačová et al. (2002), Kuchta (2011) and Gutjahr and Pichler (2013)—where new approaches to stochastic multiobjective problems (including stochastic dominance) can be found]. A relationship between efficient points corresponding to the different approaches are discussed in Caballero et al. (2001). In this paper we study the approach that is defined by the multiobjective problem (1), generally with constraints set depending on F. Moreover, we restrict our consideration to the case when the functions $\mathsf{E}_F g_i(x, \xi), i = 1, \ldots, l$ are (mostly strongly) convex. The Markowitz model fulfills e.g., this condition [for the definition of the Markowitz model see e.g., Dupačová et al. (2002)]. But objective function in Problem (1) depends linearly on the probability measure and this condition is not fulfilled in the case of the Markowitz model. According to this fact, generalized stability results Kaňková (2013a) have to be employed to obtain practically the same results (to guarantee the same rate of the convergence it is necessary the support Z_F to be a bounded set.)

Applications to the multiobjective problems can be found also e.g., in Stancu-Minasian (1984) and Jablonský (1993). Many of them correspond (under some additional conditions) to two-stage stochastic problems with multiobjective inner problems. The assumptions, guaranteing convexity of the inner problems, can be found in the literature [see e.g., (Ahmed 2004, where convexity simultaneously with numerical tractability were studied; and Shapiro et al. 2009)]. Moreover, in Römisch and Schulz (1993a, b) and Schulz (1994) the assumptions are introduced under which the inner objective functions are strongly convex. Employing the results of Römisch and Schulz (1993b) we can see that even well known risk measure CVaR [for the definition see e.g., Shapiro et al. (2009)] is, under rather general conditions, the strongly convex function. Namely considering CVaR with the "underlying" linear problem and employing the results presented in Kall and Mayer (2005) we can see that CVaR can be reformulated in the form of a simple recourse problem. Consequently, employing the results of Römisch and Schulz (1993b) we can obtain for continuous P_F (with respect to the Lebesgue measure on R^s) that mostly CVaR is a strongly convex function.

The risk measures appear also often in a constraints set [see e.g., Dupačová and Kopa (2012)]. Replacing a variance by CVaR in Markowiz model we can obtain optimization problem with the strongly objective function (for more details about applications see the Sect. 5). At the end of this discussion we mention that a relationship between convex and strongly convex functions can be found e.g., in Merentes and Nikodem (2010).

It is possible only very seldom to find out simultaneously the solution with respect to all criteria in (1) and moreover, these problems depend on a probability measure P_F that usually has to be estimated on the data base. Consequently, in applications very often the "underlying" probability measure P_F has to be replaced by empirical one. Evidently, then the "solution" and an analysis of the problem have to be done with respect to an empirical problem:

$$\text{Find} \quad \min \mathsf{E}_{F^N} g_i(x, \xi), \quad i = 1, \ldots, l \quad \text{subject to} \quad x \in X_{F^N}, \qquad (2)$$

where F^N denotes an empirical distribution function determined by a random sample $\{\xi^i\}_{i=1}^N$ (not necessary independent) corresponding to the distribution function F.

Evidently, if $l = 1$ we obtain "classical" one objective stochastic programming problem. Solving the problem (2) we obtain empirical estimates of an optimal value and optimal solutions. Let us denote the optimal values of (1) and (2) (in the case $l = 1$) by $\varphi^1(F, X_F)$, $\varphi^1(F^N, X_{F^N})$ and the corresponding optimal solutions sets by $\mathcal{X}^1(F, X_F)$, $\mathcal{X}^1(F^N, X_{F^N})$. It follows from the stochastic programming literature that $\varphi^1(F^N, X_{F^N})$, $\mathcal{X}^1(F^N, X_{F^N})$ are under rather general assumptions "good" statistical estimates of $\varphi^1(F, X_F)$, $\mathcal{X}^1(F, X_F)$. It was shown that these estimates are consistent under rather general assumptions. The convergence rate and an asymptotic distribution has been studied as well. These results have been, first, obtained for the case $X_F = X$ and for "classical" thin tails distributions, see e.g., Wets (1974), Kaňková (1977, 1978, 1994), Dai et al. (2000), Dupačová and Wets (1984), Kaniovski et al. (1995), Kaňková and Houda (2006), Römisch and Wakolbinger (1987), Römisch (2003), Schulz (1996), Shapiro (1994) and Shapiro et al. (2009). Later, the results covering also "heavy" tails have arisen, see e.g., Kaňková (2010, 2012, 2013b) and Houda and Kaňková (2012).

To analyze (for general l) the relationship between the characteristics obtained under the assumption of complete knowledge of P_F and them on the data base we employ the results for one-objective stochastic problems and the results of deterministic multiobjective theory. Our results cover also the case of the "underlying" distributions with heavy tails, that correspond just to many economic and financial situations [for more details see e.g., Klebanov (2003) or Meerschaert and Scheffler (2003)].

According to the above mentioned facts, the paper is organized as follows. First, we try to recall auxiliary assertions concerning deterministic multiobjective theory (Sect. 2.1). Section 2.2 will be devoted to the stability of one-stage stochastic programming problems. The stability results can be considered as the base for the investigation of empirical estimates (Sect. 2.3). Section 3 is devoted to an analysis of the multiobjective stochastic programming problems. Essential results can be found in Sect. 4. Applications (two special cases) and an analysis of a next possible and desirable investigation can be found in Sects. 5 and 6.

2 Some definition and auxiliary assertion

2.1 Deterministic multiobjective problems

To recall some results of the multiobjective deterministic optimization theory we consider a multiobjective deterministic optimization problem in the following form:

$$\text{Find} \quad \min f_i(x), \quad i = 1, \dots, l \quad \text{subject to} \quad x \in \mathcal{K}, \tag{3}$$

where $f_i(x), i = 1, \dots, l$ are real-valued functions defined on $R^n, \mathcal{K} \subset R^n$ is a nonempty set.

Definition 1 The vector x^* is an efficient solution of the problem (3) if and only if there exists no $x \in \mathcal{K}$ such that $f_i(x) \le f_i(x^*)$ for $i = 1, \dots, l$ and such that for at least one i_0 one has $f_{i_0}(x) < f_{i_0}(x^*)$.

Definition 2 The vector x^* is a properly efficient solution of the multiobjective optimization problem (3) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in \mathcal{K}$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \le M. \tag{4}$$

Proposition 1 (Geoffrion (1968)) *Let $\mathcal{K} \subset R^n$ be a nonempty convex set and let $f_i(x), i = 1, \dots, l$ be convex functions on $\mathcal{K}$. Then $x^0 \in \mathcal{K}$ is a properly efficient solution of the problem* (3) *if and only if x^0 is optimal in*

$$\min_{x \in \mathcal{K}} \sum_{i=1}^{l} \lambda_i f_i(x) \quad \text{for some} \quad \lambda_1, \ldots, \lambda_l > 0, \quad \sum_{i=1}^{l} \lambda_i = 1.$$

A relationship between efficient and properly efficient points is introduced e.g., in Ehrgott (2005) or in Geoffrion (1968). We summarize it in the following Remark.

Remark 1 Let $f(x) = (f_1(x), \ldots, f_l(x))\,, x \in \mathcal{K}$; $\mathcal{K}^{eff}$, $\mathcal{K}^{peff}$ be sets of efficient and properly efficient points of the problem (3). If $\mathcal{K}$ is a convex set, $f_i(x)$, $i = 1, \ldots, l$ are convex functions on $\mathcal{K}$, then

$$f\left(\mathcal{K}^{peff}\right) \subset f\left(\mathcal{K}^{eff}\right) \subset \bar{f}\left(\mathcal{K}^{peff}\right), \tag{5}$$

where $\bar{f}(\mathcal{K}^{peff})$ denotes the closure set of $f(\mathcal{K}^{peff})$.

Definition 3 Let $h(x)$ be a real-valued function defined on a nonempty convex se $\mathcal{K} \subset R^n$. $h(x)$ is a strongly convex function with a parameter $\rho > 0$ if

$$h\left(\lambda x^1 + (1-\lambda)\, x^2\right) \le \lambda h\left(x^1\right) + (1-\lambda)\, h\left(x^2\right) - \lambda\,(1-\lambda)\,\rho \|x^1 - x^2\|_2^2$$
$$\text{for every} \quad x^1, x^2 \in \mathcal{K}, \lambda \in \langle 0, 1\rangle.$$

($\|\cdot\|_2$ denotes the Euclidean norm in R^n.)

Proposition 2 (Kaňková (1993b)) *Let $\mathcal{K} \subset R^n$ be a nonempty, compact, convex set. Let, moreover, $h(x)$ be a strongly convex with a parameter $\rho > 0$, continuous, real-valued function defined on $\mathcal{K}$. If x^0 is defined by the relation* $x^0 = \arg\min_{x \in \mathcal{K}} h(x)$, *then*

$$\|x - x^0\|^2 \le \frac{2}{\rho} |h(x) - h(x^0)| \quad \text{for every} \quad x \in \mathcal{K}.$$

It has been proven in Schulz (1994) [see also Houda (2009)] that strong convexity condition is sufficient for the quadratic growth condition [for the definition of general growth conditions see e.g., Römisch (2003) or Houda (2009)].

Proposition 2 expresses the relationship between optimal value and optimal solution in the case of strongly convex objectives. The situation is more complicated in the general case, for more details see an analysis in Conclusion.

2.2 Stability in one-objective stochastic programming problems

To recall suitable (for us) assertions from single criterion stochastic optimization theory we start with the problem:

$$\text{Find} \quad \varphi^1\,(F, X_F) = \min \mathsf{E}_F g_1\,(x, \xi) \quad \text{subject to} \quad x \in X_F, \tag{6}$$

where $g_1(x, z)$ is a real-valued function defined on $R^n \times R^s$.

First, we recall some stability results corresponding to the problem (6). To this end let F and G be two s-dimensional distribution functions for which the Problem (6) is well defined. According to the triangular inequality we obtain

$$\left|\varphi^1(F, X_F) - \varphi^1(G, X_G)\right| \le \left|\varphi^1(F, X_F) - \varphi^1(G, X_F)\right| + \left|\varphi^1(G, X_F) - \varphi^1(G, X_G)\right|. \tag{7}$$

Consequently, it is easy to see that we can study separately stability of the problem (6) with respect to a perturbation in the objective function and in the constraints set. We restrict mostly our consideration to the case $X_F = X$. To this end let $\mathcal{P}(R^s)$ denote the set of all (Borel) probability measure on R^s. We shall try to introduce functions m (defined on R^+), d_s (may be metric, defined on $\mathcal{P}(R^s) \times \mathcal{P}(R^s)$) such that

$$\left|\varphi^1(F, X) - \varphi^1(G, X)\right| \le m\left(d_s\left(P_F, P_G\right)\right).$$

We introduce (for $i = 1, 2$) a system of the assumptions A1.(i):

A1. (i)
either $g_1(x, z)$ is a uniformly continuous function on $X \times (Z_F \cup Z_G)$ or X is a convex set and there exists $\varepsilon > 0$ such that $g_1(x, z)$ is a convex bounded function on $X(\varepsilon)$ ($X(\varepsilon)$ denotes the ε-neighborhood of X),
$g_1(x, z)$ is a Lipschitz function of $z \in R^s$ with the Lipschitz constant L_i (corresponding to the $\mathcal{L}_i$ norm) not depending on x.

($i = 1$ corresponds to $\mathcal{L}_1$ norm $\|\cdot\|_1$ and $i = 2$ to the Euclidean norm $\|\cdot\|_2$ in R^n.)

Furthermore, we define $\mathcal{M}_1^i(R^s) \subset \mathcal{P}(R^s)$, and $d_{W_1^i}(F, G) = d_{W_1^i}(P_F, P_G), i = 1, 2$ by

$$\begin{aligned} \mathcal{M}_1^i(R^s) &= \left\{ \nu \in \mathcal{P}(R^s), \int_{R^s} \|z\|_i \nu(dz) < \infty \right\} \\ d_{W_1^i}(F, G) &= \left[\inf \left\{ \int_{R^s \times R^s} \|z - \bar{z}\|_i \kappa(dz \times d\bar{z}) : \kappa \in \mathcal{D}(P_F, P_G) \right\} \right], \\ & \qquad P_F, P_G \in \mathcal{M}_1^i(R^s), \end{aligned} \tag{8}$$

where $\mathcal{D}(P_F, P_G)$ denotes the set of those measures on $\mathcal{P}(R^s \times R^s)$ whose marginal measures are P_F, P_G. $d_{W_1^i}(F, G) = d_{W_1^i}(P_F, P_G), i = 1, 2$ can be considered as metrics ("distances") in the space of the probability measures on R^s; $d_{W_1^2}(P_F, P_G)$ is based on the Euclidean norm, $d_{W_1^1}(P_F, P_G)$ is based on the $\mathcal{L}_1$ norm. They are known as the Wasserstein metrics [for more details about the Wasserstein metrics in the space of the probability measures see e.g., Rachev (1991) and Valander (1973)].

The following Proposition is only a little generalized assertion presented in Kaňková and Houda (2002a).

Proposition 3 *Let X be a nonempty compact set, $P_F \in \mathcal{M}_1^2(R^s)$. If*

1. *the assumption A.1* (2) *is fulfilled,*
2. *$P_G \in \mathcal{M}_1^2(R^s)$ is arbitrary,*

then

$$\left|\varphi^1(F, X) - \varphi^1(G, X)\right| \leq L_2 d_{W_1^2}(P_F, P_G). \tag{9}$$

The idea to employ the Wasserstein metric to the stability investigation belongs to Römisch and Schulz [see e.g., Römisch and Schulz (1993b)]. However to estimate the value $d_{W_1^2}(F, G)$ for the dimensions greater 2 is (from the numerical point of view) practically impossible.

Valander in Valander (1973) has proven for $s = 1$ that

$$d_{W_1^1}(F, G) = \int_{-\infty}^{\infty} |F(z) - G(z)|\, dz.$$

Consequently, replacing in the case of general s $d_{W_1^2}(F, G)$ by $d_{W_1^1}(F, G)$, the following assertion could be proven.

Proposition 4 (Kaňková and Houda (2006)) *Let $P_F, P_G \in \mathcal{M}_1^1(R^s)$, X be a nonempty set. If A.1* (1) *is fulfilled, then*

$$|\mathsf{E}_F g_1(x, \xi) - \mathsf{E}_G g_1(x, \xi)| \leq L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, dz_i \quad \text{for every} \quad x \in X.$$

If, moreover, X is a compact set, then also

$$\left|\varphi^1(F, X) - \varphi^1(G, X)\right| \leq L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, dz_i.$$

Proposition 4 reduces (from the mathematical point of view) s-dimensional case to one-dimensional. The idea to reduce an s-dimensional case to one dimensional is credited to Pflug (2001) [see also Šmíd (2009)]. In this paper we try to employ just the stability results introduced by Proposition 4 because they are suitable in the case of the distributions with "heavy" tails.

Remark 2 1. The right hand side of the both inequalities in Proposition 4 are equal to the value

$$L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, dz_i,$$

independently on the stochastic dependence between the components of the vector $\xi = (\xi_1, \ldots, \xi_s)$. Consequently, an information about the above mentioned dependence is neglected and so including it the lower "upper bound" may be obtained.

2. We have employed the Wasserstein metrics to obtain the stability results for one-stage stochastic programming problems (1) with $X_F = X$. Another metrics have been employed in this purpose, we can recall the Kolmogorov metric [see e.g., Kaňková (1993a)], Bounded Lipschitz metric [see e.g., Römisch and Wakolbinger (1987)], Fortet Mourrier metric [see. e.g., Römisch (2003)] or more general semi-metrics [see Römisch (2003)].

2.3 Empirical estimates in one-stage stochastic programming problems

Replacing G by an empirical estimate F^N of F we can employ Proposition 4 to investigate empirical estimates of Problem (6) (with $X_F = X$). Evidently, according to Proposition 4 it is reasonable to investigate the behaviour of $\int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i, \, i = 1, \ldots, s$. To this end, we define the following assumptions:

A.2 $\{\xi^i\}_{i=1}^{\infty}$ is independent random sequence corresponding to F; F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$,

A.3 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w.r.t. the Lebesgue measure on R^1,

and recall the following assertions:

Proposition 5 (Shorack and Wellner (1986)) *Let $s = 1$ and $P_F \in \mathcal{M}_1^1(R^1)$. Let, moreover A.2 be fulfilled. Then*

$$P\left\{\omega : \int_{-\infty}^{\infty} |F(z) - F^N(z)| dz \longrightarrow_{N \longrightarrow \infty} 0\right\} = 1.$$

Proposition 6 (Kaňková (2010)) *Let $s = 1$, $t > 0$ and Assumptions A.2, A.3 be fulfilled. If there exists $\beta > 0$, $R := R(N) > 0$ defined on $\mathcal{N}$ such that $R(N) \rightarrow_{N \rightarrow \infty} \infty$ and, moreover,*

$$N^{\beta} \int_{-\infty}^{-R(N)} F(z) dz \rightarrow_{N \rightarrow \infty} 0, \qquad N^{\beta} \int_{R(N)}^{\infty} [1 - F(z)] dz \rightarrow_{N \rightarrow \infty} 0,$$

$$2NF(-R(N)) \rightarrow_{N \rightarrow \infty} 0, \qquad 2N[1 - F(R(N))] \rightarrow_{N \rightarrow \infty} 0,$$

$$\left(\frac{12N^{\beta} R(N)}{t} + 1\right) \exp\left\{-2N \left(\frac{t}{12R(N)N^{\beta}}\right)^2\right\} \rightarrow_{N \longrightarrow \infty} 0, \tag{10}$$

then

$$P\left\{\omega : N^{\beta}\int\limits_{-\infty}^{\infty}\left|F(z)-F^{N}(z)\right|dz>t\right\}\rightarrow_{N\rightarrow\infty}0. \tag{11}$$

($\mathcal{N}$ denotes the set of natural numbers.)

Evidently, it follows from Propositions 4 and 6 that the convergence rate of $|\varphi^1(F^N, X)-\varphi^1(F, X)|$ depends on the convergence rate of

$$\int\limits_{-\infty}^{\infty}\left|F_i(z_i)-F_i^N(z_i)\right|dz_i,\quad i=1,\ldots,s. \tag{12}$$

Since it has been proven in Dvoretski et al. (1956) (under the assumptions of Proposition 6) that

$$P\left\{\omega : N^{1/2}\sup_{z}\left|F^N(z)-F(z)\right|>t\right\}\leq c\exp\left(-2t^2\right),$$

where c is universal constant,

it follows from Kaňková and Houda (2006) that

$$\left(\frac{12N^{\beta}R(N)}{t}+1\right)\exp\left\{-2N\left(\frac{t}{12R(N)N^{\beta}}\right)^2\right\}\rightarrow_{N\longrightarrow\infty}0$$

and, consequently, it follows from the relations (10), (11) that the convergence rate of (12) depends on the tails of the distribution function F.

The following auxiliary assertions have been successively proven.

Proposition 7 (Kaňková (2010)) *Let $s=1,\ t>0,\ \beta\in(0,\ \frac{1}{2})$ and the assumptions A.2, A.3 be fulfilled. If there exist constants $C_1,\ C_2$ and $T>0$ such that*

$$\bar{f}(z)\leq C_1\exp\{-C_2|z|\}\quad\text{for}\quad z\in(-\infty,-T)\cup(T,\infty),$$

then

$$P\left\{\omega : N^{\beta}\int\limits_{-\infty}^{\infty}\left|F(z)-F^{N}(z)\right|>t\right\}\longrightarrow_{N\longrightarrow\infty}0.$$

($\bar{f}$ denotes the probability density corresponding to F.)

Proposition 7 deals with the case of the distribution functions with "thin" tails. If we admit heavy tails we can introduce a weaker assertions.

Proposition 8 (Houda and Kaňková (2012)) *Let* $s = 1, t > 0, r > 0$, *the assumptions A.2, A.3 be fulfilled. Let, moreover,* ξ *be a random variable such that* $\mathsf{E}_F|\xi|^r < \infty$. *If constants* $\beta, \gamma > 0$ *fulfil the inequalities* $0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1 - r)\gamma < 0$, *then*

$$P\left\{\omega : N^{\beta} \int_{-\infty}^{\infty} \left|F(z) - F^N(z)\right| dz > t\right\} \longrightarrow_{N\longrightarrow\infty} 0.$$

Analyzing Proposition 8 we can obtain for $\beta := \beta(r)$ introduced by it that

$$\beta(r) \longrightarrow_{r\longrightarrow\infty} 1/2, \quad \beta(r) \longrightarrow_{r\longrightarrow 2^+} 0.$$

Proposition 8 covers some cases of heavy tails distributions. Unfortunately, we cannot obtain by this Proposition any results for the case when there exist only $\mathsf{E}_F|\xi|^r$ for $r < 2$. However, this case corresponds to the stable distributions with the tail (shape) parameter $\nu < 2$ [for more details see e.g., Kozubowski et al. (2003) or Rachev and Mitting (2000)]. The shape parameter expresses how "heavy" tails of the distribution are. The case $\nu = 2$ corresponds to the normal distribution, when the second moment exists. To deal with the case when the finite moment exists only for $r < 2$, we recall the results of Barrio et al. (1999).

Proposition 9 (Barrio et al. (1999)) *Let* $s = 1, \ \{\xi^i\}_{i=1}^N, \ N = 1, 2, \ldots$ *be a sequence of independent random values corresponding to a heavy tailed distribution* F *with the shape parameter* $\nu \in (1, 2)$ *and let*

$$\sup_{t>0} t^{\nu} P\{\omega : |\xi| > t\} < \infty, \tag{13}$$

then

$$\lim_{\bar{M}\longrightarrow\infty} \sup_{N} P\left\{\omega : \frac{N}{N^{1/\nu}} \int_{-\infty}^{\infty} \left|F(z) - F^N(z)\right| > \bar{M}\right\} = 0. \tag{14}$$

[For some more details see Omelchenko (2012).]

Remark 3 The convergence rate of the value $|\varphi^1(F, X) - \varphi^1(F^N, X)|$ has been investigated in the stochastic programming literature many times in the case of "thin" tails [see e.g., Dai et al. (2000), Kaňková (1978),Shapiro (1994) and Shapiro et al. (2009)]. The large deviations technique has been employed in these papers mostly.

Till now we have dealt with the case $X_F = X$. Another special case is considered in Kaňková (2012). To recall it we assume:

A.4 There exist $\delta > 0, \ \vartheta > 0$ such that $\bar{f}_i(z_i) > \vartheta$ for $z_i \in Z_{F_i}, |z_i - k_{F_i}(\alpha_i)| < 2\delta$, where
$k_{F_i}(\alpha_i) = \sup\{z_i : P_{F_i}\{\omega : z_i \leq \xi_i(\omega)\} \geq \alpha_i\}, \alpha_i \in (0, 1), i = 1, \ldots, s$. ($\bar{f}_i(z_i), \ i = 1, \ldots, s$ denotes the probability density corresponding to $F_i(z_i)$).

If $\bar{g}_i := \bar{g}_i(x)$, $i = 1, \ldots, s$ are defined on R^s, $\alpha_i \in \langle 0, 1 \rangle$, $i = 1, 2, \ldots, s$ and if we set

$$X_F \; (:= X_F(\alpha)) = \bigcap_{i=1}^{s} \{x \in X : P\,[\omega : \bar{g}_i(x) \leq \xi_i] \geq \alpha_i\}\,,$$
$$\alpha_i \in (0, 1), \quad i = 1, \ldots, s, \quad \alpha = (\alpha_1, \ldots, \alpha_s)\,, \tag{15}$$

then under Assumption A.4 we can see that

$$X_F = \bigcap_{i=1}^{s} \left\{x \in X : \bar{g}_i(x) \leq k_{F_i}(\alpha_i)\right\}. \tag{16}$$

Consequently, setting

$$\bar{X}(v) = \bigcap_{i=1}^{s} \{x \in X : \bar{g}_i(x) \leq v_i\}\,, \quad v = (v_1, \ldots, v_s)\,, v \in Z_F \tag{17}$$

we obtain

$$X_F = \bar{X}\,(k_F(\alpha))\,, \quad \alpha = (\alpha_1, \ldots, \alpha_s)\,, \quad k_F(\alpha) = \left(k_{F_1}(\alpha_1), \ldots, k_{F_s}(\alpha_s)\right).$$

Proposition 10 (Kaňková (2012)) *Let X be a convex, compact and nonempty set, $\alpha_i \in (0, 1)$, $i = 1, \ldots, s$, $\alpha = (\alpha_1, \ldots, \alpha_s)$. If*

1. *$\hat{g}_0(x)$, $x \in R^n$ is a Lipschitz function on X with the Lipschitz constant L,*
2. *A.2, A.3, A.4 are fulfilled,*
3. *$\bar{X}(v)$ are nonempty sets for $v \in Z_F$,*
4. *there exists $\varepsilon > 0$ such that $\bar{g}_i(x)$, $i = 1, \ldots, s$ are convex continuous functions on $X(\varepsilon)$,*

then there exists a constant $C > 0$ such that

$$P\left\{\omega : |\inf_{\bar{X}(k_F(\alpha))} \hat{g}_0(x) - \inf_{\bar{X}\left(k_{F^N}(\alpha)\right)} \hat{g}_0(x)| > t\right\} \leq 2s \exp\left\{-2N(\vartheta t/LCs)^2\right\}$$
$$\text{for} N \in \mathcal{N} \quad \text{and} \quad t > 0 \quad \text{such that} \quad 0 < t/LCs < \delta.$$

Remark 4 Observe that the assertion of Proposition 10 does not depend on the tails distributions. Especially, it is also valid for the stable distributions with the tails parameter $\nu \in (1, 2)$.

3 Multiobjective stochastic programming problems–problem analysis

To analyze the stability (and consequently empirical estimates) in the case of the multiobjective stochastic problem (1) we define the sets $\mathcal{G}(F, X_F)$, $\bar{\mathcal{X}}(F, X_F)$, $\bar{\mathcal{G}}^F(F, X_F)$, $\bar{\mathcal{G}}^F(F, X_G)$, Λ and the function $\bar{g}(x, z, \lambda)$ by the relations:

$$\mathcal{G}(F, X_F) = \Big\{ y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi), \quad j = 1, \ldots, l \\ \text{for some} \quad x \in X_F; \, y = (y_1, \ldots, y_l) \Big\},$$

$$\bar{\mathcal{X}}(F, X_F) = \{ x \in X_F : x \quad \text{is a properly efficient point of the problem}(1) \},$$

$$\bar{\mathcal{G}}^F(F, X_F) = \Big\{ y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi), \quad j = 1, \ldots, l \\ \text{for some} \quad x \in \bar{\mathcal{X}}(F, X_F) \Big\},$$

$$\bar{\mathcal{G}}^F(G, X_G) = \Big\{ y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi), \quad j = 1, \ldots, l \\ \text{for some} \quad x \in \bar{\mathcal{X}}(G, X_G) \Big\},$$

$$\Lambda = \left\{ \lambda \in R^l : \lambda = (\lambda_1, \ldots, \lambda_l), \lambda_i > 0, i = 1, \ldots, l, \sum_{i=1}^{l} \lambda_i = 1 \right\}, \tag{18}$$

$$\bar{g}(x, z, \lambda) = \sum_{i=1}^{l} \lambda_i g_i(x, z), \quad x \in R^n, z \in R^s, \lambda \in \Lambda. \tag{19}$$

Evidently,

1. $\mathcal{G}(F, X_F)$ is the image of the set X_F corresponding to the vector function $\mathsf{E}_F g_1(x, \xi), \ldots, \mathsf{E}_F g_l(x, \xi)$ and the underlying distribution F,
2. $\bar{\mathcal{G}}^F(F, X_F)$ is the image of the set $\bar{\mathcal{X}}(F, X_F)$ corresponding to the vector function $\mathsf{E}_F g_1(x, \xi), \ldots, \mathsf{E}_F g_l(x, \xi)$ and the underlying distribution F.,

If for $\varepsilon > 0$ the symbol $X(\varepsilon)$ denotes the $\varepsilon-$ surroundings of the set X and if the assumption

B.1
- X is a nonempty, convex set and, moreover, there exists $\varepsilon > 0$ such that $g_i(x, z), i = 1, \ldots, s$ are convex bounded functions on $X(\varepsilon)$,
- $g_i(x, z), i = 1, \ldots, l$ are (for every $x \in X$) Lipschitz functions of $z \in R^s$ with the Lipschitz constant L_1 (corresponding to $\mathcal{L}_1$ norm) not depending on $x \in X$,

is fulfilled, then $\bar{g}(x, z, \lambda)$ is (for every $z \in Z_F, \lambda \in \Lambda$) a convex function on $X(\varepsilon)$ and, moreover, (for every $x \in X, \lambda \in \Lambda$) a Lipschitz function of $z \in Z_F$ with the Lipschitz constant L_1 not depending on $x \in X, \lambda \in \Lambda$. Consequently, applying the parametric optimization problem:

$$\text{Find} \quad \varphi^{\lambda}(F, X) = \inf \mathsf{E}_F \bar{g}(x, \xi, \lambda) \quad \text{subject to} \quad x \in X \quad \text{for} \quad \lambda \in \Lambda, \tag{20}$$

we can apply Proposition 4 to obtain.

Proposition 11 *Let $P_F, P_G \in \mathcal{M}_1^1(R^s)$, B.1 be fulfilled, X be a nonempty set, then*

$$|\mathsf{E}_F g_i(x, \xi) - \mathsf{E}_G g_i(x, \xi)| \le L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, dz_i \\ \text{for every} \quad x \in X, \; i = 1, \ldots, l,$$

$$|\mathsf{E}_F \bar{g}(x, \xi, \lambda) - \mathsf{E}_G \bar{g}(x, \xi, \lambda)| \le L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i$$

$$\text{for every} \quad x \in X, \lambda \in \Lambda.$$

If, moreover, X is a compact set, then also

$$\left|\varphi^{\lambda}(F, X) - \varphi^{\lambda}(G, X)\right| \le L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, dz_i \quad \text{for every} \quad \lambda \in \Lambda.$$

To present the next assertion, first, we recall the definition of the Hausdorff distance of two subsets of R^n, $n \ge 1$. To this end let $\mathcal{K}', \mathcal{K}'' \subset R^n$ be two nonempty sets. The Hausdorff distance of these sets $\Delta[\mathcal{K}', \mathcal{K}''] := \Delta_n[\mathcal{K}', \mathcal{K}'']$ is defined by

$$\begin{aligned} \Delta_n \left[\mathcal{K}', \mathcal{K}''\right] &= \max \left[\delta_n \left(\mathcal{K}', \mathcal{K}''\right), \delta_n \left(\mathcal{K}'', \mathcal{K}'\right)\right], \\ \delta_n \left(\mathcal{K}', \mathcal{K}''\right) &= \sup_{x' \in \mathcal{K}'} \inf_{x'' \in \mathcal{K}^{\cdot}} \left\|x' - x''\right\|_2, \end{aligned}$$

[for more details about the Hausdorff distance see e.g., Rockafellar and Wets (1983)].

Evidently, the following assertion follows from Proposition 11.

Proposition 12 *Let $P_F, P_G \in \mathcal{M}_1^1(R^s)$, X be a compact nonempty set. If B.1 is fulfilled, then*

$$\Delta_l \left[\mathcal{G}(F, X), \mathcal{G}(G, X)\right] \le l L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, dz_i.$$

Furthermore, if $g_i(x, z), i = 1, \ldots, l$ are for every $z \in Z_F \cup Z_G$ and some $\varepsilon > 0$ strongly convex functions on $X(\varepsilon)$ with a parameter $\rho > 0$, then also $\bar{g}(x, z, \lambda)$ is for every $z \in Z_F, \lambda \in \Lambda$ a strongly convex function on $X(\varepsilon)$ with the parameter ρ. Employing the triangular inequality, Propositions 2 and Proposition 4 we can obtain successively for $P_F, P_G \in \mathcal{M}_1^1(R^s)$ and compact set X that

$$\begin{aligned} \left\|x_F^{\lambda}(X) - x_G^{\lambda}(X)\right\|^2 &\le \frac{2}{\rho} \left|\mathsf{E}_F \bar{g}\left(x_F^{\lambda}(X), \xi, \lambda\right) - \mathsf{E}_F \bar{g}\left(x_G^{\lambda}(X), \xi, \lambda\right)\right| \\ &\le \frac{2}{\rho} \left[\left|\mathsf{E}_F \bar{g}\left(x_F^{\lambda}(X), \xi, \lambda\right) - \mathsf{E}_G \bar{g}\left(x_G^{\lambda}(X), \xi, \lambda\right)\right| \right. \\ &\quad \left. + \left|\mathsf{E}_G \bar{g}\left(x_G^{\lambda}(X) \xi, \lambda\right) - \mathsf{E}_F \bar{g}\left(x_G^{\lambda}(X), \xi, \lambda\right)\right|\right] \\ &\le \frac{2}{\rho} L_1 \left[\sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| \, dz_i \right. \\ &\quad \left. + \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| \, dz_i \right] \end{aligned}$$

$$\leq \frac{4}{\rho}\left[L_1 \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)|\, dz_i\right] \quad \text{for every} \quad \lambda \in \Lambda, \tag{21}$$

where $x_F^{\lambda}(X),\ x_G^{\lambda}(X)$ are solutions of problem (20) with the "underlying" distributions F and G.
Consequently

$$\left\| x^{\lambda}(F) - x^{\lambda}(G) \right\| \leq \left[\frac{4}{\rho} L_1 \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)|\, dz_i\right]^{\frac{1}{2}} \quad \text{for every} \quad \lambda \in \Lambda. \tag{22}$$

Now already we can formulate the next assertion.

Proposition 13 *Let $P_F, P_G \in \mathcal{M}_1^1(R^s)$, X be a compact set. If B.1 is fulfilled and, moreover, $g_i(x, z)$, $i = 1, \ldots, l$ are for every $z \in Z_F \cup Z_G$ strongly convex functions on X with a parameter $\rho > 0$, then*

$$\Delta_n \left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}(G, X)\right] \leq \left[L_1 \frac{4}{\rho} \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, dz_i\right]^{1/2}.$$

Proof The assertion of Proposition 13 follows immediately from the relation (22), Proposition 1 and the properties of the Hausdorff distance. □

Evidently, the next Corollary follows from Proposition 13, properties of the Lipschitz functions and the Hausdorff distance.

Corollary 1 *If the assumptions of Proposition 13 are fulfilled and, moreover, $\mathsf{E}_F g_i(x, z)$, $i = 1, \ldots, l$ are Lipschitz functions of $x \in X$, then there exists $L_2 > 0$ such that*

$$\Delta_l \left[\bar{\mathcal{G}}^F(F, X), \bar{\mathcal{G}}^F(G, X)\right] \leq \left[L_2 \frac{4}{\rho} \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, dz_i\right]^{1/2}.$$

Remark 5 Let X be compact nonempty set, $\varepsilon > 0$. If $\mathsf{E}_F g_i(x, \xi)$, $i = 1, \ldots, l$ are for every $z \in Z_F \cup Z_G$ convex continuous bounded functions on $X(\varepsilon)$, then they are also Lipschitz functions on X [for more details see Rockafellar (1970)].

4 Multiobjective stochastic programming problems: empirical estimates

Replacing (in the assertions of the previous section) G by F^N and employing the properties of the probability measure we can investigate the corresponding empirical estimates of the multiobjective problems (1) with $X_F = X$ to obtain the following results.

Theorem 1 *Let Assumptions B.1 and A.2 be fulfilled,* $P_F \in \mathcal{M}_1^1(R^s)$, X *be a compact nonempty set. Then*

1.

$$P\left\{\omega : \Delta_l\left[\mathcal{G}(F, X), \mathcal{G}(F^N, X)\right] \longrightarrow_{N\longrightarrow\infty} 0\right\} = 1.$$

If, moreover,

2. $g_i(x, z), i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex functions with a parameter* $\rho > 0$ *on* X, *then also*

$$P\left\{\omega : \Delta_n\left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}\left(F^N, X\right)\right] \longrightarrow_{N\longrightarrow\infty} 0\right\} = 1.$$

3. – $g_i(x, z), i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex functions with a parameter* $\rho > 0$ *on a convex set* X,
   – $\mathsf{E}_F g_i(x, \xi), i = 1, \ldots, l$ *are Lipschitz functions of* $x \in X$,

   then also

$$P\left\{\omega : \Delta_l\left[\mathcal{G}^F(F, X), \mathcal{G}^F\left(F^N, X\right)\right] \longrightarrow_{N\longrightarrow\infty} 0\right\} = 1.$$

Proof The assertion 1 of Theorem 1 follows from Propositions 5, and Proposition 12. The assertion 2 follows from Proposition 5, Proposition 13 and the properties of the Hausdorff distance.
The assertion 3 follows from Corollary 1, Proposition 5 and the properties of the Lipschitz functions. □

Remark 6 See that the stable probability measures with one dimensional marginals having tails parameters $\nu_i > 1, i = 1, \ldots, s$ belong to $\mathcal{M}_1^1(R^s)$. Consequently, the assertions of Theorem 1 are valid (under the corresponding assumptions) also for the stable distributions with the tail parameter $\nu_i \in (1, 2), i = 1, \ldots, s$.

Replacing G by F^N in Proposition 13 we can furthermore successively obtain for $\gamma > 0$ that

$$P\left\{\omega : N^\gamma \Delta_n\left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}(F^N, X)\right] \geq t\right\}$$
$$\leq P\left\{\omega : N^\gamma\left[L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} \left|F_i(z_i) - F_i^N(z_i)\right| dz_i\right]^{1/2} \geq t\right\},$$
$$P\left\{\omega : N^\gamma \Delta_n\left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}(F^N, X)\right] \geq t\right\}$$
$$\leq P\left\{\omega : \left[N^{2\gamma} L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} \left|F_i(z_i) - F_i^N(z_i)\right| dz_i\right]^{1/2} \geq t\right\},$$

$$P\left\{\omega : N^{\gamma}\Delta_n\left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}(F^N, X)\right] \geq t\right\}$$
$$\leq P\left\{\omega : \left[N^{2\gamma}\left(L_1\sum_{i=1}^{s}\int_{-\infty}^{+\infty}\left|F_i(z_i) - F_i^N(z_i)\right|dz_i\right)\right] \geq t^2\right\}. \tag{23}$$

Applying this system of inequalities, Proposition 7 and Proposition 11 we can obtain.

Theorem 2 *Let B.1, A.2, A.3 be fulfilled, X be a compact nonempty set,* $t > 0$. *If there exists constants* C_1, C_2 *and* $T > 0$ *such that*

$$\bar{f}_j(z) \leq C_1 \exp\left\{-C_2|z_j|\right\} \quad \text{for} \quad z_j \in (-\infty, -T) \cup (T, \infty), \quad j = 1, \ldots, s,$$

then

1. $P\left\{\omega : N^{\beta}\Delta_l\left[\mathcal{G}(F, X), \mathcal{G}\left(F^N, X\right)\right] > t\right\} \longrightarrow_{N\longrightarrow\infty} = 0 \quad \text{for} \quad \beta \in \left(0, \frac{1}{2}\right).$

   If moreover

2. $g_i(x, z_i)$, $i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex functions on convex set X with the parameter* $\rho > 0$, *then also*

$$P\left\{\omega : N^{\beta}\Delta_n\left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}(F^N, X)\right] > t\right\} \longrightarrow_{N\longrightarrow\infty} = 0 \quad \text{for} \quad \beta \in \left(0, \frac{1}{4}\right).$$

3. – $g_i(x, z)$, $i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex functions on a convex set X with a parameter* $\rho > 0$,
   – $\mathsf{E}_F g_i(x, \xi)$, $i = 1, \ldots, l$ *Lipschitz functions of* $x \in X$,

   then also

$$P\left\{\omega : N^{\beta}\Delta_l\left[\bar{\mathcal{G}}^F(F, X), \bar{\mathcal{G}}^F(F^N, X)\right] > t\right\} \longrightarrow_{N\longrightarrow\infty} = 0 \quad \text{for} \quad \beta \in \left(0, \frac{1}{4}\right).$$

Proof The first assertion of Theorem 2 follows from Proposition 7 and Proposition 12.

The assertion 2 follows Proposition 7, Proposition 13 and the relation (23).

The last assertion follows from the second assertion and the properties of Lipschitz functions. □

Theorem 3 *Let* $t > 0$, $r > 0$, *Assumptions B.1, A.2, A.3 be fulfilled. Let, moreover,* ξ *be a random vector with the components* ξ_i, $i = 1, \ldots, s$ *such that* $\mathsf{E}_F|\xi_i|^r < \infty$. *If constants* $\beta, \gamma > 0$ *fulfil the inequalities* $0 < \beta + \gamma < 1/2$, $\gamma > 1/r$, $\beta + (1 - r)\gamma < 0$, *then*

1. $P\left\{\omega : N^{\beta}\Delta_l\left[\mathcal{G}(F, X), \mathcal{G}(F^N, X)\right] > t\right\} \longrightarrow_{N\longrightarrow\infty} = 0.$

   If moreover

2. $g_i(x, z_i), i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex function on convex set* X *with the parameter* ρ*, then also*

$$P\left\{\omega : N^{\bar{\beta}} \Delta_n \left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}(F^N, X)\right] > t\right\} \longrightarrow_{N \longrightarrow \infty} = 0 \ \text{ for } \ \bar{\beta} = \frac{\beta}{2}.$$

3. – $g_i(x, z), i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex functions on a convex set* X *with a parameter* $\rho > 0$,
   – $\mathsf{E}_F g_i(x, \xi), i = 1, \ldots, l$ *are for every* $z \in Z_F$ *Lipschitz functions of* $x \in X$,
   then also

$$P\left\{\omega : N^{\bar{\beta}} \Delta_l \left[\bar{\mathcal{G}}^F(F, X), \bar{\mathcal{G}}^F(F^N, X)\right] > t\right\} \longrightarrow_{N \longrightarrow \infty} = 0 \ \text{ for } \ \bar{\beta} = \frac{\beta}{2}.$$

Proof The proof of Theorem 3 is the same as the proof of Theorem 2, only a Proposition 7 is necessary to replace by Proposition 8. □

Theorem 4 *Let Assumptions B.1, A.2 and A.3 be fulfilled,* $P_F \in \mathcal{M}_1^1(R^s)$, $\bar{M} > 0$, X *be a compact set. If one-dimensional components* ξ_i, $i = 1, \ldots, s$ *of the random vector* ξ *have the distribution functions* F_i *with the tails parameters* $\nu_i \in (1, 2)$ *fulfilling the relations*

$$\sup_{t>0} t^{\nu_i} P_F\{\omega : |\xi_i| > t\} < \infty, \quad i = 1, \ldots, s,$$

then

1.

$$\lim_{\bar{M} \longrightarrow \infty} \sup_N P\left\{\omega : \frac{N}{N^{1/\nu}} \Delta_l \left[\mathcal{G}(F, X), \mathcal{G}(F^N, X)\right] > \bar{M}\right\} = 0$$
$$\text{with} \quad \nu = \min(\nu_1, \ldots, \nu_s).$$

If moreover

2. $g_i(x, z), i = 1, \ldots, l$ *are strongly convex with a parameter* $\rho > 0$ *function on* X, *then also for* $\gamma = \frac{\nu - 1}{2\nu}$

$$\lim_{\bar{M} \longrightarrow \infty} \sup_N P\left\{\omega : N^{\gamma} \Delta_n \left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}\left(F^N, X\right)\right] > \bar{M}\right\} = 0$$
$$\text{with} \quad \nu = \min(\nu_1, \ldots, \nu_s).$$

3. – $g_i(x, z), i = 1, \ldots, l$ *are for every* $z \in Z_F$ *strongly convex functions on a convex set* X *with a parameter* $\rho > 0$,
   – $\mathsf{E}_F g_i(x, \xi), i = 1, \ldots, l$ *are Lipschitz functions of* $x \in X$
   then also for $\gamma = \frac{\nu - 1}{2\nu}$

$$\lim_{\bar{M} \longrightarrow \infty} \sup_N P\left\{\omega : N^{\gamma} \Delta_n \left[\bar{\mathcal{G}}^F(F, X)), \bar{\mathcal{G}}^F\left(F^N, X\right)\right] > \bar{M}\right\} = 0$$
$$\text{with} \quad \nu = \min(\nu_1, \ldots, \nu_s\}.$$

Proof Evidently, the first assertion follows from Proposition 9 and Proposition 12.

To prove the assertion 2 we can see that it follows from the relation (23) that for $\gamma = 1 - 1/\nu$

$$P\left\{\omega : N^{\gamma} \Delta_n \left[\bar{\mathcal{X}}(F, X), \bar{\mathcal{X}}\left(F^N, X\right)\right] \geq t\right\} \leq$$
$$P\left\{\omega : \left[N^{2\gamma}\left(L_1 \sum_{i=1}^{s} \int_{-\infty}^{+\infty} \left|F_i(z_i) - F_i^N(z_i)\right| dz_i\right)\right] \geq t^2\right\}.$$

Evidently now the assertion 2 follows from the last inequality, the properties of the probability measure and the first assertion.

The last assertion follows from the second assertion and the properties of the Lipschitz functions. □

In the next section let us give a sketch of two possible applications.

5 Applications: analysis of two special examples

We consider and analyze two very simple typical problems from practise.

1. First, we shall rather modify an example from Stancu-Minasian (1984) p. 287 belonging to a group of classical "production planning problems". A factory is producing two main groups of goods: group A having n_1 assortments and group B including n_2 assortments. A goal of a company is:
   - to minimize the total cost of production,
   - to minimize the next expense connected with ecological tax,
   - to minimize the cost of material,
   - to minimize expense connected with transport goods to the main stock.

   Furthermore, we recall the following conditions and notions:

- A capacity of the factory is the value K (deterministic), values $x_1,\ x_2$ correspond to a partition of the capacity to the groups A and B, $C_1 x_1 + C_2 x_2$ is a cost that the company has to pay for partition,
- demands on assortments are random values $A_i,\ i = 1, \ldots n_1$ in the group A and $B_j,\ j = 1, \ldots, n_2$ in the group B (of course with probability measures supports bounded sets),
- the company wishes to invest into production of group A the value $\bar{A} \sum_{i=1}^{n_1} A_i$ and into group B the value $\bar{B} \sum_{j=1}^{n_2} B_j$,
- it is necessary (maybe after the realization of $A_i, B_j, i = 1, \ldots, n_1, j = 1, \ldots, n_2$) to determine values $y_{A,i},\ y_{B,j}, i = 1, \ldots, n_1, j = 1, \ldots, n_2$ corresponding to the production of A_i, B_j,
- the cost of production $y_{A,i}, y_{B,j}, i = 1, \ldots, n_1, j = 1, \ldots, n_2$ is equal to $d_{A,i} y_{A,i}, d_{B,j} y_{B,j}$,
- the expense connected with ecological taxes are $\bar{c}_{A,i} y_{A,i},\ \bar{c}_{B,j}\ y_{B,i},\ i = 1, \ldots, n_1, j = 1, \ldots, n_2$,

– the costs of the material corresponding to $y_{A,i}$, $y_{B,j}$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$ are $c_{A,i}y_{A,i}$, $c_{B,j}y_{B,j}$,
– expense connected with transport $t_{A,i}y_{A,i}$, $t_{B,j}y_{B,j}$,

$$C_1, C_2, d_{A,i}, d_{B,j}, c_{A,i}, c_{B,j}, \bar{c}_{A,i}, \bar{c}_{B,j}, t_{A,i}, t_{B,j}, \bar{A}, \bar{B} \geq 0,$$

Evidently, the company situation leads simultaneously to problems:

$$\begin{aligned}
&\text{Find} \quad \min\left\{C_1x_1 + C_2x_2 + Q^1(x_1, x_2) : x_1 + x_2 \leq K,\ x_1,\ x_2 \geq 0\right\},\\
&Q^1(x_1, x_2) = \mathsf{E}_F \min\left\{\sum_{i=1}^{n_1} d_{A,i}y_{A,i} + \sum_{j=1}^{n_2} d_{B,j}y_{B,j} :\right.\\
&\sum_{i=1}^{n_1} d_{A,\,i}y_{A,\,i} + C_1x_1 = \bar{A}\sum_{1=1}^{n_1} A_i,\\
&\sum_{j=1}^{n_2} d_{B,\,j}y_{B,\,j} + C_2x_2 = \bar{B}\sum_{j=1}^{n_2} B_j,\\
&\left. A_i \leq y_{A,i}, \quad i = 1, \ldots, n_1,\ B_j \leq y_{B,j}, \quad j = 1, \ldots, n_2\right\},
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\text{Find} \quad \min\left\{Q^2(x_1, x_2) : x_1 + x_2 \leq K,\ x_1, x_2 \geq 0\right\},\\
&Q^2(x_1,\ x_2) = \mathsf{E}_F \min\left\{\sum_{i=1}^{n_1} \bar{c}_{A,i}y_{A,i} + \sum_{j=1}^{n_2} \bar{c}_{B,j}y_{B,j} :\right.\\
&\sum_{i=1}^{n_1} d_{A,\,i}y_{A,\,i} + C_1x_1 = \bar{A}\sum_{1=1}^{n_1} A_i,\\
&\sum_{j=1}^{n_2} d_{B,\,j}y_{B,\,j} + C_2x_2 = \bar{B}\sum_{j=1}^{n_2} B_j,\\
&\left. A_i \leq y_{A,i},,\ i = 1, \ldots, n_1,\ B_j \leq y_{B,j},\ j = 1, \ldots, n_2\right\},
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\text{Find} \quad \min\left\{Q^3(x_1,\ x_2) : x_1 + x_2 \leq K,\ x_1,\ x_2 \geq 0\right\},\\
&Q^3(x_1,\ x_2) = \mathsf{E}_F \min\left\{\sum_{i=1}^{n_1} c_{A,i}y_{A,i} + \sum_{j=1}^{n_2} c_{B,j}y_{B,j} :\right.\\
&\sum_{i=1}^{n_1} d_{A,\,i}y_{A,\,i} + C_1x_1 = \bar{A}\sum_{1=1}^{n_1} A_i,
\end{aligned}$$

$$\sum_{j=1}^{n_2} d_{B,\,j} y_{B,\,j} + C_2 x_2 = \bar{B} \sum_{j=1}^{n_2} B_j,$$

$$\left. A_i \le y_{A,i},\,,\; i = 1,\,\ldots,\,n_1,\;\; B_j \le y_{B,j},\;\; j = 1,\,\ldots,\,n_2 \right\},\tag{26}$$

$$\text{Find} \quad \min \left\{ Q^4(x_1,\, x_2) : x_1 + x_2 \le K,\; x_1,\, x_2 \ge 0 \right\},$$

$$Q^4(x_1,\, x_2) = \mathsf{E}_F \min \left\{ \sum_{i=1}^{n_1} t_{A,i} y_{A,i} + \sum_{j=1}^{n_2} t_{B,j} y_{B,j} : \right.$$

$$\sum_{i=1}^{n_1} d_{A,\,i} y_{A,\,i} + C_1 x_1 = \bar{A} \sum_{1=1}^{n_1} A_i,$$

$$\sum_{j=1}^{n_2} d_{B,\,j} y_{B,\,j} + C_2 x_2 = \bar{B} \sum_{j=1}^{n_2} B_j,$$

$$\left. A_i \le y_{A,i}, i = 1, \ldots, n_1,\;\; B_j \le y_{B,j}, j = 1, \ldots, n_2 \right\},\tag{27}$$

$$\xi = \left(A_1, \ldots, A_{n_1}, B_1, \ldots, B_{n_2} \right).$$

Evidently, we have obtained multiobjective stochastic optimization problem. This type of the problems has been introduced in Cho (1995), where the stability has been investigated. If at least one of the objective functions is strongly convex, then all results of this paper are valid. If assumptions of strongly convexity is not fulfilled, then problem belongs only to multiobjective convex group of optimization and a little weaker results are valid. However, it is possible "to believe" that the strong convexity can be fulfilled for standard continuous (with respect to the Lebesgue measure) probability distribution. Moreover, very often (in the above investigated model) is reasonable to add the next problem:

$$\text{Find} \quad \min \left\{ C_1 x_1 + C_2 x_2 + Q^5\,(x_1, x_2) : x_1 + x_2 \le K,\; x_1, x_2 \ge 0 \right\},$$

$$Q^5(x_1, x_2) = \mathsf{E}_F \left\{ \min\{q^+ y^+ + q^- y^- : \right.$$

$$\left. C_1 x_1 + C_2 x_2 + y^+ - y^- = \bar{A} \sum_{i=1}^{n_1} A_i + \bar{B} \sum_{j+1}^{n_2} B_j, \quad y^+,\; y_- \ge 0 \right\},\tag{28}$$

where $q^+,\; q^-$ are constants (generally they can also be random values).

It follows from the paper (Römisch and Schulz 1993a) that under relatively general conditions the last problem is strongly convex. Consequently, if we add this problem to the list of objectives we can see that results of this paper are valid.

The next example fulfils the conditions of strongly convexity.

2. Let us consider an "underlying" problem of classical portfolio selection:

$$\text{Find} \quad \max \sum_{k=1}^{n} \xi_k x_k \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \leq 1, \quad x_k \geq 0, k =, \ldots, n, s = n, \tag{29}$$

with x_k a fraction of the unit wealth invested in the asset k, ξ_k (random values with finite second moments) return of the asset. Evidently the above mentioned problem is (under an assumption of knowledge $\xi_k, k = 1, \ldots, n$) a very simple problem of linear programming. However, mostly the decision $x_k, k = 1, \ldots n$ has to be determined before the realization of $\xi_k, k = 1, \ldots, n$. Of course it is possible to maximize the average of profit. It means to solve the problem:

$$\text{Find} \quad \max \sum_{k=1}^{n} \mathsf{E}_F \xi_k x_k \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \leq 1, \quad x_k \geq 0, \ k = 1, \ \ldots, \ n. \tag{30}$$

This approach is not usually reasonable because it does not include a risk (the realization of the vector $(\xi_1, \ldots, \ \xi_n)$ can be relatively "far" from the vector of its mathematical expectation) consequently profit can be "great" negative value. Evidently, it is reasonable to consider as second criterion risk. Markowitz [see, e.g., Dupačová et al. (2002)] first connected these two criteria in one and introduced the following problem:

$$\text{Find} \quad \max \left\{ \sum_{k=1}^{n} \mu_k x_k - K \sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,\,j} x_j \right\}$$

$$\text{s.t.} \quad \sum_{k=1}^{n} x_k \leq 1, \quad x_k \geq 0, \quad k = 1, \ \ldots, \ n, \quad K > 0 \quad \text{constant}, \tag{31}$$

where $\mu_k = \mathsf{E}_F \xi_k, \quad c_{k,\,j} = \mathsf{E}_F \left(\xi_k - \mu_k\right) \left(\xi_j - \mu_j\right), \quad k, \ j = 1, \ \ldots, \ n.$

Evidently $\sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,\,j} x_j$ can be considered as a risk measure and according to the properties of the mathematical statistics we can see that

$$- \left\{ \sum_{k=1}^{n} \mu_k x_k - K \sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,\,j} x_j \right\}$$

is (under general conditions) a strongly convex function. $\sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,\,j} x_j$ is a symmetric function with respect to $\mu_1, \ \ldots, \ \mu_n$. This property can be sometimes suitable but it has not to be ideal everywhere. Consequently, it can be reasonable to replace $\sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k\,j} x_j$ by another risk measures. In financial mathematics are well known risk measures VaR and CVaR. We recall them (for our case) by

$$Z(x) = \sum_{k=1}^{n} \xi_k x_k, \quad \alpha \in (0,\ 1), \quad x = (x_1,\ \ldots,\ x_n),$$
$$VaR\alpha(Z(x)) = \inf \{t : P\,[Z(x) \le t] \ge \alpha\},$$
$$CVaR_\alpha(Z(x)) = \min_{v \in R} \left[v + \frac{1}{1-\alpha} y \right], \quad y = \mathsf{E}_F \left(\sum_{i=1}^{n} \xi_i x_i - v \right)^+$$

It is known [see, e.g., Shapiro et al. (2009)] that VaR does not fulfill conditions of "good" risk measure, however $CVaR$ these conditions fulfils. Moreover, employing the results presented in Kall and Mayer (2005) we can see that CVaR, with the"underlying" linear problem, can be reformulated in the form of simple recourse problem. Consequently employing the results of Römisch and Schulz (1993a) we can obtain for a continuous P_F (with respect to the Lebesgue measure on R^s) that CVaR is (under general additional assumptions) a strongly convex function.

Consequently, considering the problems (30), (31) we have two problems with strongly convex objective functions; from which one is useful for an investor preferring symmetrical risk measure and the other that protects before a great loss. Evidently, it can be often very useful to consider simultaneously both these objective functions and to consider the problem as multiobjective problem.

6 Conclusion: discussion

The paper deals with the multiobjective stochastic programming problems. In particular the aim of the paper is to show that the multiobjective deterministic objective theory and the results of one objective stochastic programming problems can be employed to investigate a relationship between characteristics (of multiobjective problems) obtained under complete knowledge of the probability measure and them determined on the data base. To this end, we have restricted our investigation to (properly) efficient points and their functions values [for the definition see (18)]. According to (5) it is easy to see that a restriction of the investigation to properly efficient (instead efficient) points is not essential.

To obtain the above mentioned results we have mostly supposed that the objective functions $g_i(x, z), i = 1, \ldots, l$ are strongly convex on convex set X with the same parameter $\rho > 0$.

This assumption is rather strong, but it guarantees one point solution set in the simple objective case and consequently also [for every $\lambda \in \Lambda$ for the problem (20)]. Evidently, it is suitable for many quadratic cases.

A little weaker results can be considered when at least one functions from the set $\{g_i(x, z), i = 1, \ldots, l\}$ is strongly convex and the other are only convex, evidently, linear functions do not fulfill the assumption of strongly convexity however if h is a linear function $h(x) = a_1x_1 + \cdots + a_nx_n, x = (x_1, \ldots, x_n) \in R^n, a_i \in R^1, i = 1, \ldots, n$ and if we denote

$$\mathcal{A}(c) = \{x_c \in X : h(x_c) = c\},$$

then (under some additional assumptions)

$$\underline{L}|c - c'| \quad \le \rho'[\mathcal{A}(c),\ \mathcal{A}(c')] \le \quad \bar{L}\,|c - c'|, \quad c,\ c' \in R^1,$$

where $\underline{L}$, $\bar{L} > 0$, and $\rho'[\mathcal{A}(c),\ \mathcal{A}(c']$ is the corresponding metric. Evidently, then we can obtain by another technique some similar results (maybe also for some more general types of objectives).

Furthermore, we have restricted ourselves also to the case when $X_F = X$. Considering X_F fulfilling the relation (15), and fulfilling Proposition 10, the relation (7) and taking the proof idea of Theorem 23 in Kaňková (2012) we can obtain the results for more general type of X_F.

The investigation in all above mentioned directions is beyond the scope of this paper.

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