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Can the bivariate Hurst exponent be higher than an average of the separate Hurst exponents?

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HIGHLIGHTS

- The bivariate Hurst exponent H_{xy} is studied in both time and frequency domain.
- The cases of $H_{xy} = \frac{1}{2}(H_x + H_y)$ and $H_{xy} < \frac{1}{2}(H_x + H_y)$ are shown to be feasible.
- The case of $H_{xy} > \frac{1}{2}(H_x + H_y)$ is shown to be infeasible.
- Further discussion of implications is provided together with the finite size effect.

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ABSTRACT

In this note, we investigate possible relationships between the bivariate Hurst exponent H_{xy} and an average of the separate Hurst exponents $\frac{1}{2}(H_x+H_y)$. We show that two cases are well theoretically founded. These are the cases when $H_{xy} = \frac{1}{2}(H_x + H_y)$ and $H_{xy} < \frac{1}{2}(H_x + H_y)$. However, we show that the case of $H_{xy} > \frac{1}{2}(H_x + H_y)$ is not possible regardless of stationarity issues. Further discussion of the implications is provided as well together with a note on the finite sample effect.

1. Introduction

Generalization of power-law correlations (long-term memory, long-range dependence) into a bivariate setting has brought a wide range of possibilities for studying connections between various series. These power-law cross-correlations have become popular especially in econophysics with applications to numerous financial series [1–6]. Formally, the bivariate long-term memory translates into a power-law decay of the cross-correlation function $\rho_{xy}(k)$ with lag k so that $\rho_{xy}(k) \propto k^{2H_{xy}-2}$ for $k \to +\infty$. The cross-correlation function is thus hyperbolically decaying (for negative lags, for positive lags, or for both) in the same manner as the auto-correlation function in the univariate case. Alternatively, the bivariate long-range dependence can be defined in the frequency domain via a divergence of spectrum close to the origin. Specifically, the cross-spectrum $f_{xy}(\omega)$ with frequency ω has a form of $|f_{xy}(\omega)| \propto \omega^{1-2H_{xy}}$ for $\omega \to 0+$. In the remainder of the text, we assume that these properties hold both for the power-law cross-correlations as well as the univariate power-law correlations. The bivariate Hurst exponent H_{xy} measures a strength of such power-law cross-correlations[1] [7,8].

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[1] Alternatively, parameters α, α_{xy} λ_{xy} or d_{12} are used as measures of power-law cross-correlations in the literature.

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The ideas of long-range cross-correlations have been reflected in an introduction of various estimators of the bivariate Hurst exponent. These are usually bivariate generalizations of the univariate estimators—detrended cross-correlation analysis (DCCA or DXA) [9–11], height cross-correlation analysis (HXA) [12] and detrending moving-average cross-correlation analysis (DMCA) [13,14]. In addition, new correlation coefficients have been proposed based on the ideas of the bivariate estimators. Most notably, Zebende [15,16] introduces the DCCA-based correlation coefficient and Kristoufek [17] adds the DMCA-based correlation coefficient. These two play an important role in our further discussion. Several tests of the power-law cross-correlations have been introduced as well [18,19].

In the applied literature, the main focus is usually put on the bivariate Hurst exponent H_{xy} and its comparison to the Hurst exponents of the separate processes, H_x and H_y. Numerically, it has been shown that various theoretical processes imply $H_{xy} = \frac{1}{2}(H_x + H_y)$ [8,18,20,21]. Several processes having $H_{xy} < \frac{1}{2}(H_x + H_y)$ have been proposed as well [8,20]. However, various studies report that the bivariate Hurst exponent is higher than the average of the separate processes, i.e. $H_{xy} > \frac{1}{2}(H_x + H_y)$ [3,6,22–24]. An unanswered question remains—are all these three possibilities feasible? More specifically, it is only not obvious whether the last option is feasible as the former two have been shown to exist analytically. In this short paper, we answer the posed question. The next section provides the needed instruments. The last section brings some novel insights into the topic with a discussion of implications.

2. Methodology

For studying the relationship between the bivariate Hurst exponent H_{xy} and the separate Hurst exponents H_x and H_y, we recall several concepts from both time and frequency domains. We present the spectrum coherence (frequency domain) and the DCCA and DMCA correlation coefficients (time domain). Both concepts are essential here.

The squared spectrum coherency is defined for two stationary series $\{x_t\}$ and $\{y_t\}$ with existing spectra $f_{xy}(\omega), f_x(\omega)$ and $f_y(\omega)$ at frequency $0 \leq \omega \leq \pi$. Squared spectrum coherency $K_{xy}^2(\omega)$ is defined as

$$K_{xy}^2(\omega) = \frac{|f_{xy}(\omega)|^2}{f_x(\omega)f_y(\omega)} \tag{1}$$

for a given frequency ω. The squared coherence can be understood as a squared correlation between processes $\{x_t\}$ and $\{y_t\}$ at frequency ω. Note that it holds that $0 \leq K_{xy}^2(\omega) \leq 1$ for all ω [25].

The detrended cross-correlation coefficient $\rho_{DCCA}(s)$ for scale s [15] combines the detrended fluctuation analysis (DFA) [26–28] and the detrended cross-correlation analysis (DCCA) [9–11]. The DCCA-based coefficient for scale s is defined as

$$\rho_{DCCA}(s) = \frac{F_{DCCA}^2(s)}{F_{DFA,x}(s)F_{DFA,y}(s)}, \tag{2}$$

where $F_{DCCA}^2(s)$ is a detrended covariance between profiles of series $\{x_t\}$ and $\{y_t\}$ based on a window of size s, and $F_{DFA,x}^2$ and $F_{DFA,y}^2$ are detrended variances of profiles of the separate series, respectively, for a window size s. The detrending moving-average cross-correlation coefficient $\rho_{DMCA}(\lambda)$ for window size λ [17] connects the detrending moving average (DMA) procedure [29,30] and the detrending moving-average cross-correlation analysis (DMCA) [13,14]. The coefficient is defined as

$$\rho_{DMCA}(\lambda) = \frac{F_{DMCA}^2(\lambda)}{F_{x,DMA}(\lambda)F_{y,DMA}(\lambda)}, \tag{3}$$

where $F_{DMCA}^2(\lambda)$, $F_{DMA,x}^2(\lambda)$ and $F_{DMA,y}^2(\lambda)$ are a detrended covariance between profiles of the examined series and detrended variances of the separate series, respectively, with a moving average parameter λ. Both coefficients have been shown to range between $-1 \leq \rho_{DCCA}(s), \rho_{DMCA}(\lambda) \leq 1$ analytically for all scales s or window sizes λ [17,18].

3. Discussion

The squared spectrum coherency gives straightforward implications for the bivariate Hurst exponents. Rewriting the coherency using the definition of the power-law cross-correlations in the frequency domain, we obtain

$$K_{xy}^2(\omega) = \frac{|f_{xy}(\omega)|^2}{f_x(\omega)f_y(\omega)} \propto \frac{\omega^{2(1-2H_{xy})}}{\omega^{1-2H_x}\omega^{1-2H_y}} = \omega^{2(H_x+H_y-2H_{xy})}. \tag{4}$$

As the squared coherency lies between 0 and 1 for all frequencies, it does so for the long-range cross-correlations case of $\omega \to 0+$ as well. Therefore, this gives us two feasible and one infeasible possibilities:

- $H_{xy} = \frac{1}{2}(H_x + H_y) \Rightarrow 2(H_x + H_y - 2H_{xy}) = 0 \Rightarrow \lim_{\omega \to 0+} K_{xy}^2(\omega) \propto const.$
- $H_{xy} < \frac{1}{2}(H_x + H_y) \Rightarrow 2(H_x + H_y - 2H_{xy}) > 0 \Rightarrow \lim_{\omega \to 0+} K_{xy}^2(\omega) = 0.$
- $H_{xy} > \frac{1}{2}(H_x + H_y) \Rightarrow 2(H_x + H_y - 2H_{xy}) < 0 \Rightarrow \lim_{\omega \to 0+} K_{xy}^2(\omega) = +\infty \Rightarrow ↯.$

This implies that for stationary processes, we cannot have $H_{xy} > \frac{1}{2}(H_x + H_y)$ as it is in contradiction with the bounded squared spectrum coherency. This also translates into the non-stationary case with pseudo-spectra. To make the claim for the non-stationary case stronger, we show the contradiction in the time domain as well.

Both the DCCA and DMCA coefficients are fixed between −1 and 1 for all feasible scales but also for both stationary and non-stationary specifications of the underlying processes [17,18]. Similarly to the coherency case, we can rewrite the coefficients using the power-law correlations definition in the time domain. For this, we need to recall that for the long-range cross-correlated processes, we have $F^2_{DCCA}(s) \propto s^{2H_{xy}}$ for $s \to +\infty$ [9] and $F^2_{DMCA}(\lambda) \propto \lambda^{2H_{xy}}$ for $\lambda \to +\infty$ [13] so that the correlation coefficients can be rewritten as

$$\rho_{DCCA}(s) = \frac{F^2_{DCCA}(s)}{F_{DFA,x}(s)F_{DFA,y}(s)} \propto \frac{s^{2H_{xy}}}{s^{H_x+H_y}} = s^{2H_{xy}-(H_x+H_y)}$$
$$\rho_{DMCA}(s) = \frac{F^2_{DMCA}(\lambda)}{F_{DMA,x}(\lambda)F_{DMA,y}(\lambda)} \propto \frac{\lambda^{2H_{xy}}}{\lambda^{H_x+H_y}} = \lambda^{2H_{xy}-(H_x+H_y)}. \quad (5)$$

We then have the same implications as for the frequency domain argument—two feasible and one infeasible[2]:

- $H_{xy} = \frac{1}{2}(H_x + H_y) \Rightarrow 2H_xy - (H_x + H_y) = 0 \Rightarrow \lim_{s\to+\infty} \rho_{DCCA}(s) \propto const.$
- $H_{xy} < \frac{1}{2}(H_x + H_y) \Rightarrow 2H_xy - (H_x + H_y) < 0 \Rightarrow \lim_{s\to+\infty} \rho_{DCCA}(s) = 0.$
- $H_{xy} > \frac{1}{2}(H_x + H_y) \Rightarrow 2H_xy - (H_x + H_y) > 0 \Rightarrow \lim_{s\to+\infty} \rho_{DCCA}(s) = \pm\infty \Rightarrow \lightning.$

The implications are thus the same as in the frequency domain but here, they hold also for non-stationary series. Having $H_{xy} > \frac{1}{2}(H_x+H_y)$ is thus not feasible in the power-law cross-correlations setting. The consequences of the presented results and the logic of arguments are far reaching.

First, the bivariate Hurst exponent is not necessarily equal to the average of the separate Hurst exponents. Second, unless at least one of the series is long-range correlated with $H > 0.5$, the processes cannot be power-law cross-correlated with $H_{xy} > 0.5$. Long-term memory of one of the underlying processes is thus needed and necessary. The power-law cross-correlations thus do not emerge out of nowhere but these are rather a by-product of the persistent separate process(es). This is well in hand with analytical results about long-range cross-correlated processes [18,20,21]. Third, the case of $H_{xy} = \frac{1}{2}(H_x+H_y)$ is a natural limiting case for various processes with the non-zero squared coherency. These are not limited to the quite well studied and documented correlated ARFIMA processes or the mixtures of autoregressive and long-range dependent processes [8,18,20,21] but they encompass rich possibilities. Fourth, the case of $H_{xy} > \frac{1}{2}(H_x+H_y)$ cannot happen which means that results suggesting it does fall victims to inefficient estimators of the bivariate Hurst exponent or are due to the finite sample effect.[3] Interpretations based on such results are then misleading. Fifth, the case of $H_{xy} < \frac{1}{2}(H_x + H_y)$ is feasible and potentially interesting. Sela and Hurvich [8] refer to such processes as the anti-cointegration as the separate processes are long-range correlated but pairwise uncorrelated in a long-term horizon (at low frequencies). This is in evident opposition to the (fractional) cointegration for which it holds that $K^2_{xy}(\lambda) = 1$ as $\lambda \to 0+$. The authors propose to use $d_\rho = d_{12} - \frac{d_1+d_2}{2}$ where d_{12}, d_1 and d_2 are fractional integration parameters for the joint long-term memory and the separate long-term memories, respectively, as a measure of power-law coherency. As we mainly function with the Hurst exponent definitions, we can rewrite the measure as $H_\rho = H_{xy} - \frac{H_x+H_y}{2} = d_{12} + \frac{d_1+d_2}{2} = d_\rho$ so that these are equivalent. If it holds that $H_{xy} = \frac{1}{2}(H_x+H_y)$, we have $H_\rho = 0$, and for the anti-cointegration case, we have $H_\rho < 0$. The latter case is only sparsely

2 Only the implications for DCCA are shown as the ones for DMCA are the same.

3 As the reviewers have suggested, the finite sample effect can play an important role. We focus on the time domain implications here but similar outcomes can be shown for the frequency domain as well. Recall that $|\rho_{DCCA}(s)| \leq 1$ for all scales s (again the same logics can be applied to the DMCA coefficient). The proportionality in Eq. (5) can be written as

$$\rho_{DCCA}(s) = Ks^{2H_{xy}-(H_x+H_y)} \quad (6)$$

where K is a proportionality term. Without a loss of generality, we assume $K > 0$ and we thus focus on $0 < \rho_{DCCA}(s) \leq 1$ (for $K < 0$, we can perform a symmetric examination of the problem, and the problem is not interesting for $K = 0$). Taking logarithm of Eq. (6), we have

$$\log K + (2H_{xy} - H_x - H_y) \log s = \log \rho_{DCCA}(s) \leq 0 \quad (7)$$

which implies

$$(2H_{xy} - H_x - H_y) \leq -\frac{\log K}{\log s}. \quad (8)$$

In the limiting case of $s \to +\infty$, we simply have $\lim_{s\to+\infty} -\frac{\log K}{\log s} = 0$. However, for a finite sample case, we observe that even though the term $\log s$ goes to infinity, the divergence is quite slow. Therefore, the $\log K$ term can play a role in the finite sample analysis as K can be either $0 < K < 1$, or $K = 1$ or $K > 1$ (note that we still assume $K > 0$ here). Labelling the finite sample bias as ζ, we have $H_{xy} \leq \frac{H_x+H_y}{2} + \zeta$, where $\zeta > 0$, $\zeta = 0$ and $\zeta < 0$ for $0 < K < 1$, $K = 1$ and $K > 1$, respectively. We can thus have a case when $H_{xy} > \frac{H_x+H_y}{2}$ caused by a finite sample bias for $0 < K < 1$. Nevertheless, such possibility still remains unfeasible in the asymptotic case.

investigated in the literature [8,20] and it thus provides a relatively open field for further research, both theoretical and applied.

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