Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 259 (2015) 3–11

FUZZY sets and systems

www.elsevier.com/locate/fss

Superdecomposition integrals

Radko Mesiar [a,b], Jun Li [c,*], Endre Pap [d,e]

[a] *Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia*
[b] *UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic*
[c] *School of Sciences, Communication University of China, Beijing 100024, China*
[d] *Singidunum University, 11000 Belgrade, Serbia*
[e] *Óbuda University, H-1034 Budapest, Hungary*

Received 22 October 2013; received in revised form 9 May 2014; accepted 12 May 2014

Available online 14 May 2014

Abstract

This study introduces and discusses a new class of integrals based on superdecompositions of integrated functions, including an analysis of their relationship with decomposition integrals, which were introduced recently by Even and Lehrer. The proposed superdecomposition integrals have several properties that are similar or dual with respect to decomposition integrals, but they also have some significant differences. The convex integral is obtained by considering all possible superdecompositions with no constraints on the applied sets, which can be treated as the greatest convex homogeneous functional that is bounded from above by the measure we consider. The relationship with the universal integral of Klement et al. is also discussed. Finally, some possible generalizations are outlined.

Keywords: Choquet integral; Convex integral; Decomposition integral; Monotone measure; Superdecomposition integral

1. Introduction

Integrals play a key role in many theoretical and applied areas where the information contained in a measure (for example, weights of groups of criteria) and a function (for example, a score vector) is expressed using a single representative value. The origin of integrals is linked to measuring actual descriptions of the physical world, such as the length, area, and volume, which are sigma-additive, and real-valued functions. More general measures were considered only in the last century, especially their associations with human sciences. For example, the interactions within groups of people cannot be modeled directly using additive measures. Modified techniques were also introduced for constructing classical integrals such as the Riemann integral in 1854 and the Lebesgue integral in 1902. Various approaches developed for general integration include the Choquet integral [2] and the Sugeno integral [20]. For state-of-the-art accounts of generalized measures and integral theory (real-valued functions and monotone

* Corresponding author. Tel.: +86 10 65783583, fax: +86 10 65783583.
E-mail addresses: mesiar@math.sk (R. Mesiar), lijun@cuc.edu.cn (J. Li), pape@eunet.rs (E. Pap).

http://dx.doi.org/10.1016/j.fss.2014.05.003
0165-0114/

measures), we recommend a published handbook [16] and previous monographs [3,4,6,15,23] and [1,10,11,17,24,25]. Recently, Lehrer and Teper [8,9,22] introduced a concave integral, which can be treated as a solution to an optimization problem that maximizes the lower integral sums, i.e., it is based on a subdecomposition of a considered function. A common framework for Lehrer and Teper's concave integral and the Choquet integral was proposed by Even and Lehrer [5], who introduced the decomposition integrals. These decomposition integrals maximize the lower integral sums related to subdecompositions of the functions under consideration given some constraints on the sets being considered. Klement et al. [7] introduced a framework for functionals based on monotone measures, which should be referred to as universal integrals. Integrals that are both universal and decomposition integrals were characterized by Stupnaňová [19].

Inspired by the idea of decomposition integrals, we introduce and study a dual view of integration based on the upper integral sums, i.e., superdecompositions of the functions being considered. In the next section, we introduce decomposition and universal integrals, as well as some results related to these special integrals. Section 3 describes the proposed superdecomposition integrals, including several examples. In Section 4, we discuss the convex integral as a special superdecomposition integral related to convex functionals. Finally, some concluding remarks are provided, including possible further generalizations obtained by modifying the arithmetical operations applied.

2. Universal and decomposition integrals

2.1. Universal integral

Let $\mathcal{A}$ be a σ-algebra of subsets of set X. A set function $m : \mathcal{A} \to [0, \infty]$ is called a *monotone measure* whenever $m(\emptyset) = 0 < m(X)$, and for every $A, B \in \mathcal{A}$ such that $A \subseteq B$ we have $m(A) \le m(B)$. The following concepts are needed to define a universal integral.

Definition 1. (See [7].) Let $(X, \mathcal{A})$ be a measurable space.

(i) $\mathcal{F}^{(X,\mathcal{A})}$ is the set of all $\mathcal{A}$-measurable functions $f : X \to [0, \infty]$.
(ii) For each number $a \in]0, \infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ is the set of all monotone measures that satisfy $m(X) = a$, and we take

$$\mathcal{M}^{(X,\mathcal{A})} = \bigcup_{a \in]0,\infty]} \mathcal{M}_a^{(X,\mathcal{A})}.$$

An equivalence relation between pairs of measures and functions was introduced in [7].

Definition 2. Two pairs, $(m_1, f_1) \in \mathcal{M}^{(X_1,\mathcal{A}_1)} \times \mathcal{F}^{(X_1,\mathcal{A}_1)}$ and $(m_2, f_2) \in \mathcal{M}^{(X_2,\mathcal{A}_2)} \times \mathcal{F}^{(X_2,\mathcal{A}_2)}$, which satisfy

$$m_1\big(\{f_1 \ge t\}\big) = m_2\big(\{f_2 \ge t\}\big) \text{ for all } t \in]0, \infty],$$

are called integral equivalent, which are represented as

$$(m_1, f_1) \sim (m_2, f_2).$$

Integral equivalence can be viewed as a generalization of the stochastic equivalence of random variables. Thus, two random variables (possibly defined on two different probability spaces) are integral equivalent, i.e., $(X, P_1) \sim (Y, P_2)$ if and only if they have coincident distribution functions, $F_X = F_Y$.

The notion of pseudo-multiplication is required to describe the universal integral.

Definition 3. (See [15,21].) A function $\otimes : [0, \infty]^2 \to [0, \infty]$ is called a pseudo-multiplication if it satisfies the following properties:

(i) It is non-decreasing in each component, i.e., for all $a_1, a_2, b_1, b_2 \in [0, \infty]$ with $a_1 \le a_2$ and $b_1 \le b_2$, we have $a_1 \otimes b_1 \le a_2 \otimes b_2$;
(ii) 0 is an annihilator of $\otimes$, i.e., for all $a \in [0, \infty]$, we have $a \otimes 0 = 0 \otimes a = 0$;

(iii) It has a neutral element that differs from 0, i.e., there exists an $e \in]0, \infty]$ such that for all $a \in [0, \infty]$, we have $a \otimes e = e \otimes a = a$.

Let $\mathcal{S}$ be the class of all measurable spaces and take

$$\mathcal{D}_{[0,\infty]} = \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}.$$

The Choquet, Sugeno, and Shilkret integrals are particular cases of the following integral, which was given in [7].

Definition 4. A function $\mathbf{I}: \mathcal{D}_{[0,\infty]} \to [0, \infty]$ is called a universal integral if the following axioms hold:

(I1) For any measurable space $(X, \mathcal{A})$ from $\mathcal{S}$, the restriction of the function $\mathbf{I}$ to $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ is non-decreasing in each coordinate;

(I2) There exists a pseudo-multiplication $\otimes: [0, \infty]^2 \to [0, \infty]$ such that for all pairs $(m, c \cdot \mathbf{1}_A) \in \mathcal{D}_{[0,\infty]}$ (where $\mathbf{1}_A$ is the characteristic function of the set A)

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A);$$

(I3) For all integral equivalent pairs $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$, we have

$$\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2).$$

For any fixed pseudo-multiplication $\otimes$, and for any two universal integrals $\mathbf{I}_1$ and $\mathbf{I}_2$ based on $\otimes$, the convex combination $\mathbf{I} = a \cdot \mathbf{I}_1 + (1 - a) \cdot \mathbf{I}_2$, where a is from $[0, 1]$, is also a universal integral based on $\otimes$. Moreover, for any fixed pseudo-multiplication $\otimes$, we can specify two distinct universal integrals based on $\otimes$ (for more details see Proposition 3.1 in [7]).

Theorem 1. *Let $\otimes: [0, \infty]^2 \to [0, \infty]$ be a pseudo-multiplication on $[0, \infty]$. Then, the smallest universal integral $\mathbf{I}_\otimes$ and the greatest universal integral $\mathbf{I}^\otimes$ based on $\otimes$ are given by*

$$\mathbf{I}_\otimes(m, f) = \sup\bigl\{t \otimes m\bigl(\{f \ge t\}\bigr) \bigm| t \in]0, \infty]\bigr\},$$
$$\mathbf{I}^\otimes(m, f) = \operatorname{essup}_m f \otimes \sup\bigl\{m\bigl(\{f \ge t\}\bigr) \bigm| t \in]0, \infty]\bigr\},$$

where $\operatorname{essup}_m f = \sup\{t \in [0, \infty] \mid m(\{f \ge t\}) > 0\}$.

Note that for the Sugeno integral $\mathbf{Su} = \mathbf{I}_{Min}$ and for the Shilkret integral $\mathbf{Sh} = \mathbf{I}_{Prod}$, where the pseudo-multiplications Min and $Prod$ are given by $Min(a, b) = \min(a, b)$ and $Prod(a, b) = a \cdot b$.

2.2. Decomposition integral

From Even and Lehrer [5], we recall some results related to decomposition integrals. Their construction copies the idea of lower integral sums and it is based on a system $\mathcal{H}$ of finite set systems from $\mathcal{A} \setminus \{\emptyset\}$ (called collections in [5], see also Section 3),

$$I_{\mathcal{H}}(m, f) = \sup\left\{\sum_{i\in I} a_i m(A_i) \,\middle|\, (A_i)_{i\in I} \in \mathcal{H}, \sum_{i\in I} a_i \mathbf{1}_{A_i} \le f\right\}, \tag{1}$$

where all constants $a_i, i \in I$, are non-negative. Depending on $\mathcal{H}$, several classical integrals can be constructed. For example, for a fixed measurable space $(X, \mathcal{A})$, if

$$\mathcal{H}_1 = \bigl\{\{A\} \bigm| A \in \mathcal{A}\bigr\},$$

then $I_{\mathcal{H}_1}$ is a Shilkret integral [18], whereas if

$$\mathcal{H}_2 = \{\mathcal{B} \mid \mathcal{B} \text{ is a finite chain in } \mathcal{A}\},$$

then $I_{\mathcal{H}_2}$ is a Choquet integral [2]. Recall that $\mathcal{B}$ is a finite chain in $\mathcal{A}$ if and only if there is an integer k and $\mathcal{B} = \{A_1, \dots, A_k\} \subset \mathcal{A}$ that satisfies $A_1 \subset A_2 \subset \dots \subset A_k$. Considering

$$\mathcal{H}_3 = \{\mathcal{B} \mid \mathcal{B} \text{ is a finite subset of } \mathcal{A}\},$$

$I_{\mathcal{H}_3}$ is the concave integral introduced by Lehrer [8]. Note that $I_{\mathcal{H}_3}(m, \cdot)$ is the smallest concave, positively homogeneous functional F that satisfies $F(\mathbf{1}_A) \geq m(A)$ for all $A \in \mathcal{A}$. For further details, we recommend [5,14,19]. Some other generalizations based on modifications of the classical arithmetical operations $+$ and $\cdot$ can be found in [13].

Each decomposition integral is positively homogeneous, $I_{\mathcal{H}}(m, cf) = cI_{\mathcal{H}}(m, f)$ for any $c \in]0, \infty[$. Moreover, considering a singleton space $X = \{x\}$, there is a unique decomposition integral I and $I(m, c \cdot \mathbf{1}_A) = c \cdot m(A)$. Therefore, decomposition integrals that are also universal integrals are necessarily related to the standard product as the underlying pseudo-multiplication. This fact reflects the general property that integrals based on standard multiplication can be linked to universal integrals only when the corresponding pseudo-multiplication $\otimes$ is the standard multiplication on $[0, \infty]$. For typical examples, we recall the Shilkret [18] and Choquet [2] integrals, which belong to both types of integrals. The study of integrals that are simultaneously decomposition and universal integrals was initiated previously in [5]. A complete solution of this problem was presented in [14], which led to a hierarchical family of integrals (by interpolating from the Shilkret integral to the Choquet integral) that are described in the next theorem.

Theorem 2. *A function $I : \mathcal{D}_{[0,\infty]} \to [0, \infty]$ is both a universal and decomposition integral (on any measurable space $(X, \mathcal{A})$) if and only if*

$$I \in \{I_{(1)}, I_{(2)}, \dots, I_{(n)}, \dots, I_{(\infty)}\},$$

which, for $n \in \mathbb{N}$, $I_{(n)}/\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ (integral I_n is restricted to an (m, f) pair, which are linked to a fixed measurable space $(X, \mathcal{A})$), is given by

$$I_{(n)} = \sup\left\{\sum_{i=1}^{n} a_i \cdot m(A_i) \;\middle|\; (A_i)_{i\in\{1,\dots,n\}} \subset \mathcal{A},\ \text{is a chain},\ a_1, \dots, a_n \geq 0,\ \text{and}\ \sum_{i=1}^{n} a_i \cdot \mathbf{1}_{A_i} \leq f\right\},$$

and $I_{(\infty)} = \sup\{I_{(n)} \mid n \in \mathbb{N}\}$.

Note that $I_{(1)}$ is simply the Shilkret integral and $I_{(\infty)}$ is the Choquet integral. Moreover, $I_{(1)} \leq I_{(2)} \leq \dots \leq I_{(\infty)}$.

3. Superdecomposition integrals

For a fixed measurable space $(X, \mathcal{A})$ denoted by $\mathbb{X}$, the set of all collection systems $\mathcal{H}$ where $\mathcal{C} \subseteq \mathcal{A} \setminus \{\varnothing\}$ is a *collection* whenever it is finite. Considering that $(X, \mathcal{A})$ fixed and to shorten the notation, we denote $\mathcal{M}$ as the set of all monotone measures on $(X, \mathcal{A})$ and $\mathcal{F}$ is the set of all bounded measurable functions $f : X \to [0, \infty[$.

Definition 5. Let $\mathcal{H} \in \mathbb{X}$ be fixed. Then, the mapping $I^{\mathcal{H}} : \mathcal{M} \times \mathcal{F} \to [0, \infty]$ given by

$$I^{\mathcal{H}}(m, f) = \inf\left\{\sum_{A\in\mathcal{C}} a_A \cdot m(A) \;\middle|\; \mathcal{C} \in \mathcal{H}, a_A \geq 0 \text{ for each } A \in \mathcal{C}, \sum_{A\in\mathcal{C}} a_A \mathbf{1}_A \geq f\right\}$$

is called a superdecomposition integral.

It is obvious that each superdecomposition integral $I^{\mathcal{H}}$ is positively homogeneous and increasing in each coordinate.

Example 1.

(i) Consider $\mathcal{H}_1 = \{\{A\} \mid A \in \mathcal{A}\}$. Then,

$$I^{\mathcal{H}_1}(m,f) = \inf\{a \cdot m(A) \mid A \in \mathcal{A}, a \cdot \mathbf{1}_A \geq f\}$$
$$= \sup\{f(x) \mid x \in X\} \cdot m(\{f > 0\}).$$

Note that if m has no non-trivial null-sets, i.e., $m(A) = 0$ only if $A = \varnothing$, then $I^{\mathcal{H}_1}$ coincides with the greatest universal integral related to the product as the underlying multiplication.

(ii) Consider that $\mathcal{H}_2 = \{\mathcal{B} \mid \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $I_{\mathcal{H}_2}$ is the Choquet integral [2] given by

$$I^{\mathcal{H}_2}(m,f) = \int_0^\infty m(\{f \geq t\})\,dt.$$

Remark 1.

(i) Evidently, if for $\mathcal{H}, \mathcal{G} \subseteq \mathbb{X}$ it holds that $\mathcal{H} \subseteq \mathcal{G}$, then $I^{\mathcal{H}} \geq I^{\mathcal{G}}$, i.e., for each $m \in \mathcal{M}$ and $f \in \mathcal{F}$ it holds that $I^{\mathcal{H}}(m,f) \geq I^{\mathcal{G}}(m,f)$. Thus, considering that $\mathcal{H}_3 = \{\mathcal{B} \mid \mathcal{B}$ is a finite subset of $\mathcal{A}\}$, it holds that $\mathcal{H}_3 \supseteq \mathcal{H}$ for each $\mathcal{H} \in \mathbb{X}$, thus $I^{\mathcal{H}_3} \leq I^{\mathcal{H}}$, i.e., $I^{\mathcal{H}_3}$ is the smallest superdecomposition integral.

(ii) The decomposition integral $I_{\mathcal{H}_1}$ (i.e., the Shilkret integral) is the smallest universal integral linked to the product Prod, $I_{\mathcal{H}_1} = I_{Prod}$. However, as shown by Example 1, $I_{\mathcal{H}^1}$ need not coincide with the greatest universal integral based on *Prod*. In general, $I_{\mathcal{H}^1} > I^{Prod}$, i.e., $I_{\mathcal{H}^1}$ is not a universal integral.

Example 2. For a fixed universe $X = \{1, \dots, n\}, n \geq 2, \mathcal{A} = 2^X$, denoted by $m^*, m_* : \mathcal{A} \to [0, \infty]$, the monotone measures given by

$$m^*(A) = \begin{cases} 0 & \text{if } A = \varnothing, \\ 1 & \text{else,} \end{cases}$$
$$m_*(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{else,} \end{cases}$$

respectively. Then, for every $f \in \mathcal{F}$, it holds

$$I^{\mathcal{H}_1}(m^*,f) = I_{\mathcal{H}_1}(m^*,f) = \max\{f(i) \mid i \in X\},$$
$$I^{\mathcal{H}_1}(m_*,f) = I_{\mathcal{H}_1}(m_*,f) = \min\{f(i) \mid i \in X\},$$
$$I_{\mathcal{H}_3}(m^*,f) = \sum_{i=1}^n f(i),$$
$$I^{\mathcal{H}_3}(m^*,f) = \max\{f(i) \mid i \in X\},$$
$$I_{\mathcal{H}_3}(m_*,f) = \min\{f(i) \mid i \in X\},$$
$$I^{\mathcal{H}_3}(m_*,f) = 0.$$

Therefore, $I_{\mathcal{H}_1}(m^*,f) = I^{\mathcal{H}_3}(m^*,f)$ for all $f \in \mathcal{F}$, and $I_{\mathcal{H}_1}(m_*,f) > I^{\mathcal{H}_3}(m_*,f)$ whenever $\min\{f(i) \mid i \in X\} > 0$. For the sum $m_* + m^*$, we obtain

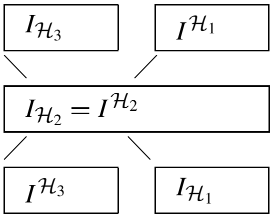
$$I_{\mathcal{H}_1}(m_* + m^*, f) = \max\{\max\{f(i) \mid i \in X\}, 2\min\{f(i) \mid i \in X\}\},$$
$$I^{\mathcal{H}_3}(m_* + m^*, f) = \max\{f(i) \mid i \in X\} + \min\{f(i) \mid i \in X\}.$$

Therefore, $I_{\mathcal{H}_1}(m^* + m_*, f) < I^{\mathcal{H}^3}(m^* + m_*, f)$ whenever

$$0 < \min\{f(i) \mid i \in X\} < \max\{f(i) \mid i \in X\}.$$

Thus, we can see that $I_{\mathcal{H}_1}$ and $I^{\mathcal{H}_3}$ are generally incomparable.

The following diagram represents the relationships between the introduced integrals.



4. Convex integral

The integrals $I^{\mathcal{H}}$ can be treated as counterparts of decomposition integrals $I_{\mathcal{H}}$. An interesting decomposition integral inspired by real-world problems is the concave integral $I_{\mathcal{H}_3}$ introduced by Lehrer [8], which can also be viewed as a special concave functional. In a similar manner, this section focuses on the superdecomposition integral $I^{\mathcal{H}_3}$, which can also be called a *convex integral*.

Proposition 1. *The integral $I^{\mathcal{H}_3}(m, \cdot) : \mathcal{F} \to [0, \infty]$ is an increasing, convex, positively homogeneous functional.*

Proof. We only need to show the convexity of $I^{\mathcal{H}_3}$ because the remaining properties are satisfied by any superdecomposition integral $I^{\mathcal{H}}$. Due to its positive homogeneity, the convexity of $I^{\mathcal{H}_3}$ is equivalent to its subadditivity, i.e., we need to show that

$$I^{\mathcal{H}_3}(m, f + g) \leq I^{\mathcal{H}_3}(m, f) + I^{\mathcal{H}_3}(m, g) \tag{2}$$

for each $m \in \mathcal{M}$, $f, g \in \mathcal{F}$. For arbitrary superdecompositions $\sum_{i\in I} a_i \mathbf{1}_{A_i} \geq f$ and $\sum_{j\in J} b_j \mathbf{1}_{B_j} \geq g$ it holds

$$\sum_{i\in I} a_i \mathbf{1}_{A_i} + \sum_{j\in J} b_j \mathbf{1}_{B_j} \geq f + g.$$

Therefore,

$$\begin{aligned} I^{\mathcal{H}_3}(m, f) + I^{\mathcal{H}_3}(m, g) &= \inf\left\{\sum_{i\in I} a_i m(A_i) \,\middle|\, \sum_{i\in I} a_i \mathbf{1}_{A_i} \geq f\right\} + \inf\left\{\sum_{j\in IJ} b_j m(B_j) \,\middle|\, \sum_{j\in J} b_j \mathbf{1}_{B_j} \geq g\right\} \\ &= \inf\left\{\sum_{i\in I} a_i m(A_i) + \sum_{j\in J} b_j m(B_j) \,\middle|\, \sum_{i\in I} a_i \mathbf{1}_{A_i} \geq f \text{ and } \sum_{j\in J} b_j \mathbf{1}_{B_j} \geq g\right\} \\ &\geq \inf\left\{\sum_{k\in K} c_k m(C_k) \,\middle|\, \sum_{k\in K} c_k \mathbf{1}_{C_k} \geq f + g\right\}, \end{aligned}$$

thereby proving (2). □

Based on Proposition 1 and using similar arguments to those given in [9] regarding the concave integral, the next corollary holds true.

Corollary 1. *For a fixed monotone measure $m \in \mathcal{M}$, denote by $\mathcal{Q}_m$ the set of all convex increasing positively homogeneous functionals $H : \mathcal{F} \to [0, \infty]$ that satisfy $H(\mathbf{1}_A) \leq m(A)$ for all $A \in \mathcal{A}$. Then,*

$$I^{\mathcal{H}_3}(m, f) = \sup\{H(f) \mid H \in \mathcal{Q}_m\},$$

i.e., $I^{\mathcal{H}_3}$ is the maximal element of $\mathcal{Q}_m$.

Based on the convexity of $I^{\mathcal{H}_3}(m, \cdot)$, the superdecomposition integral $I^{\mathcal{H}_3}$ can also be called a convex integral. We illustrate the idea of a convex integral based on the next optimization problem.

Example 3. Suppose that we have to go to a market to buy four pieces of products of type a, three pieces of type b, and five pieces of type c. The price m depends on the groups of products we are buying. More precisely, suppose that we have $m(\{a,b,c\}) = 6$, this means that for 6 euro we can buy a group of three products that comprises one piece of each product. For the other groups A of the products considered, the corresponding price $m(A)$ will be given later. Our task is to minimize the price that we should pay to obtain the desired products (if by chance we buy more products than required, this has no affect on our strategy). The solution is exactly the integral value $I^{\mathcal{H}_3}(m, f)$, where $X = \{a,b,c\}$, $f(a) = 4$, $f(b) = 3$, $f(c) = 5$. Consider $m(X) = 6$, $m(A) = 2$ if $|A| = 1$, $m(A) = 3$ if $|A| = 2$, $m(\varnothing) = 0$. Then, we obtain $I^{\mathcal{H}_3}(m, f) = 18$, e.g., for the decomposition

$$f = 2 \cdot \mathbf{1}_X + 2 \cdot \mathbf{1}_{\{a,c\}} + \mathbf{1}_{\{b,c\}}.$$

Indeed, by only considering singletons, we have a superdecomposition $f = 4 \cdot \mathbf{1}_{\{a\}} + 3 \cdot \mathbf{1}_{\{b\}} + 5 \cdot \mathbf{1}_{\{c\}}$ and the corresponding integral sum is $4 \cdot 2 + 3 \cdot 2 + 5 \cdot 2 = 24$. When considering only singletons or sets of cardinality 2, in several cases we are able to obtain the integral sum 19, such as in the case of the decomposition

$$f = 2 \cdot \mathbf{1}_X + 2 \cdot \mathbf{1}_{\{a,c\}} + \mathbf{1}_{\{b,c\}} + 2 \cdot \mathbf{1}_{\{c\}},$$

but never a smaller value. Next, if we also consider the full space X, the smallest integral sums when considering $3 \cdot \mathbf{1}_X$ is 23 when

$$f = 3 \cdot \mathbf{1}_X + \mathbf{1}_{\{a,c\}} + \mathbf{1}_{\{c\}}.$$

When $2 \cdot \mathbf{1}_X$ is considered in the decomposition of f and only singletons or two-point sets, we obtain the smallest possible output of integral sums 18, as shown above. Finally, considering $\mathbf{1}_X$ (and singletons or two-point sets), we can never obtain less than 19.

Remark 2. The dual problem with respect to that considered in Example 3 would concern a seller. Using the same numerical values, if $f(a) = 4$ means that a seller has only 4 products of type a, etc., and the measure m has the same meaning as in Example 3 (i.e., $m(A)$ is the price for the group A of products), the aim of the seller is to maximize his profit, i.e., to sell his products in groups that will yield the maximal total profit. Thus, this is simply the concave integral $I_{\mathcal{H}_3}$ and it is not difficult to check that it holds that $I_{\mathcal{H}_3}(m, f) = 24$, which is achieved if the goods are all sold as singletons (where the profit is $12 \cdot 2 = 24$), or if the seller sells 3 groups $\{a,b,c\}$ and the remaining goods as singletons (where the profit is $3 \cdot 6 + 3 \cdot 2 = 24$).

The concave integral $I_{\mathcal{H}_3}$ was shown to coincide with the Choquet integral $I_{\mathcal{H}_2}$ if and only if the underlying monotone measure $m \in \mathcal{M}$ is supermodular, i.e.,

$$m(A \cap B) + m(A \cup B) \geq m(A) + m(B), \quad A, B \in \mathcal{A},$$

for more details see [8,9].

We obtained a similar result for the convex integral. The arguments are similar to those in [8,9], thus we omit the proof of this result.

Proposition 2. *The convex integral coincides with the Choquet integral for all functions $f \in \mathcal{F}$ if and only if the underlying monotone measure m is submodular, i.e.,*

$$m(A \cup B) + m(A \cap B) \leq m(A) + m(B)$$

for all $A, B \in \mathcal{A}$.

Example 4. By extending the discussion presented in Example 3, we note that the monotone measure m is neither submodular nor supermodular. Moreover, the corresponding Choquet integral

$$I^{\mathcal{H}_2}(m, f) = 3 \cdot m(X) + m\big(\{a,c\}\big) + m\big(\{c\}\big) = 23 > 18 = I^{\mathcal{H}_3}(m, f).$$

In addition, note that the concave integral in this case is $I_{\mathcal{H}_3}(m, f) = 24$ (for example, this is obtained by only considering singletons). Obviously, $I_{\mathcal{H}_3}(m, f) > I_{\mathcal{H}_2}(m, f)$ in this case.

After considering the constraints from Example 3, $I^{\mathcal{H}_3}(m, f) = 18$ is the optimal solution for a buyer, whereas $I_{\mathcal{H}_3}(m, f) = 24$ is the best possible situation for a seller. The Choquet integral $I^{\mathcal{H}_2}(m, f) = I_{\mathcal{H}_2}(m, f) = 23$ is related to the same strategy for a buyer and a seller, i.e., first buying (or selling) the maximum number of full packages of products (in our case, 3 times the full package $X = \{a, b, c\}$), then restricting the choice to the remaining goods X' to be bought (or sold) by applying the same strategy (i.e., in our case, $X' = \{a, c\}$ and we have to buy/sell 1 package X'), and repeating this strategy until all of the goods are bought (or sold).

5. Conclusion

In this study, we introduced and discussed superdecomposition integrals, which are counterparts of the decomposition integrals introduced by Even and Lehrer [5], although they are not related by duality. For example, while $I_{\mathcal{H}_1}$ is the smallest universal integral based on the standard product, $I^{\mathcal{H}_1}$ is generally a stronger functional than the greatest universal integral based on the product [7]. Moreover, the concave integral introduced by Lehrer and Teper [8,9] acts on all non-negative measurable functions, but the boundedness of integrands is essential in the case of convex integrals.

In further research, we could modify the standard arithmetic operations $+$ and $\cdot$ into pseudo-operations, i.e., pseudo-addition $\oplus$ and pseudo-product $\odot$, which may be compared with [12,13]. Note that we may obtain unusual integrals in these cases, such as $\oplus = \max$ and $\otimes = \min$. In the case of decomposition integrals, such as $\mathcal{H}_1, \mathcal{H}_2$, or $\mathcal{H}_3$, the Sugeno integral is always obtained [12,13]. However, we obtain three different integrals based on the generalization of superdecomposition integrals. A deeper study of this topic is an interesting area for future research.

Acknowledgements

We thank the reviewers for their very useful comments. This research was supported by the grants: APVV 0073-10, GAČR P402/11/0378, MNPRS 174009, and by National Natural Science Foundation of China (Grant No. 11371332), and by the project “Mathematical models of intelligent systems and their applications,” which was supported by the Provincial Secretariat for Science and Technological Development of Vojvodina.

References

[1] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals, in: E. Pap (Ed.), Handbook of Measure Theory, vol. II, Elsevier, 2002, pp. 1329–1379.
[2] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[3] D. Denneberg, Non-additive Measure and Integral, Kluwer Academic Publisher, 1994.
[4] M. Grabisch, T. Murofushi, M. Sugeno, Fuzzy Measures and Integrals, Stud. Fuzziness Soft Comput., vol. 40, Physica-Verlag, Heidelberg, 2000.
[5] Y. Even, E. Lehrer, Decomposition-integral, unifying Choquet and Concave integrals, Econom. Theory 56 (2014) 33–58.
[6] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[7] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (1) (2010) 178–187.
[8] E. Lehrer, A new integral for capacities, Econom. Theory 39 (2009) 157–176.
[9] E. Lehrer, R. Teper, The concave integral over large spaces, Fuzzy Sets Syst. 159 (2008) 2130–2144.
[10] J. Li, R. Mesiar, P. Struk, Pseudo-optimal measures, Inf. Sci. 180 (20) (2010) 4015–4021.
[11] R. Mesiar, Choquet-like integrals, J. Math. Anal. Appl. 194 (1995) 477–488.
[12] R. Mesiar, J. Li, E. Pap, Pseudo-concave integrals, in: NLMUA’2011, in: Adv. Intell. Syst. Comput., vol. 100, Springer-Verlag, Berlin Heidelberg, 2011, pp. 43–49.
[13] R. Mesiar, J. Li, E. Pap, Discrete pseudo-integrals, Int. J. Approx. Reason. 54 (2013) 357–364.
[14] R. Mesiar, A. Stupňanová, Decomposition integrals, Int. J. Approx. Reason. 54 (2013) 1252–1259.
[15] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht–Boston–London, 1995.
[16] E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002.
[17] E. Pap, Pseudo-additive measures and their applications, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002, pp. 1403–1465.
[18] N. Shilkret, Maxitive measure and integration, Indag. Math. 33 (1971) 109–116.
[19] A. Stupňanová, A note on decomposition integrals, in: Advances in Computational Intelligence, in: Commun. Comput. Inf. Sci., vol. 300, 2012, pp. 542–548.
[20] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. thesis, Tokyo Institute of Technology, 1974.

[21] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, J. Math. Anal. Appl. 122 (1987) 197–222.
[22] R. Teper, On the continuity of the concave integral, Fuzzy Sets Syst. 160 (2009) 1318–1326.
[23] Z. Wang, G.J. Klir, Generalized Measure Theory, Springer, Boston, 2009.
[24] Q. Yang, The pan-integral on fuzzy measure space, J. Fuzzy Math. 3 (1985) 107–114 (in Chinese).
[25] Q. Zhang, R. Mesiar, J. Li, P. Struk, Generalized Lebesgue integral, Int. J. Approx. Reason. 52 (2011) 427–443.