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Fuzzy Bi-cooperative games in multilinear extension form

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Abstract

In this paper, we introduce the notion of a fuzzy Bi-cooperative game in multilinear extension form. An LG value as a possible solution concept is obtained using standard fuzzy game theoretic axioms.
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1. Introduction

In this paper, we propose a Bi-cooperative game with fuzzy bi-coalitions in multilinear extension form. Owen's [12] multilinear extension of a game is a very important tool in game theory particularly for computing the Shapley like solutions for large games. Moreover, it serves as a tool to characterize many related concepts, a remarkable one being identification of its linkage with the Choquet integral [7]. Following Owen, we integrate the multilinear extension over a simplex to construct a new class of Bi-cooperative games with fuzzy bi-coalitions. Meng and Zhang [10] defined a fuzzy cooperative game in multilinear extension form and obtained the characterization of the corresponding Shapley value.

Theory of Cooperative games since its inception by von Neumann and Morgenstern [13] has been instrumental in building decision models where a group of people (players) indulge in a joint endeavur with the single motive to gain more than what they would generate individually. However, it is found to be insensitive to the situations where a second group of players opposes the formation of the former group and the rest of the players remain indifferent. This idea extends the notion of a coalition to a bi-coalition: a pair of mutually exclusive coalitions of which the former coalition comprises of the positive contributors and the later coalition comprises of the negative contributors. Bipolarity of this kind was initially modeled in ternary voting games by Felsenthal and Machover [4]. Bilbao et al. [2] proposed a more general framework and called the corresponding games the Bi-cooperative games. They defined

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the order related to the bipolar monotonicity among the bi-coalitions to make the corresponding class a distributive lattice. Hsiao and Raghavan [8] introduced the notion of multi-choice games where players have contributions to coalitions at finitely many distinct levels. Here a partition of the player set exists with each member being labeled from zero (no participation) to some fixed number (highest level of participation). In this sense the Bi-cooperative games are a particular class of (or isomorphic to) the multi-choice games with three levels of participations. However in [9], Labreuche and Grabisch observed that the class of Bi-cooperative games should differ from their multi-choice counterparts because of the bipolarity they adhere. They have shown that by considering the product order instead of the one implied by monotonicity adopted by Bilbao et al. [1], one can distinguish them from the multi-choice games. Moreover it is interesting to note that under this product order, the class of bi-coalitions becomes an inf-semilattice.

A solution to a Bi-cooperative game is a payoff vector that satisfies some pre-imposed rationality conditions. The ith component of the vector represents the payoff to the ith player after she chooses her role in the game. Bilbao et al. [1] obtained the Shapley value for the class of Bi-cooperative games. Labreuche and Grabisch [9] proposed an alternative solution concept which we call here the LG value. These two rules differ by the underlying lattice structures imposed on the set of bi-coalitions. The LG value provides a more natural justification to the notion of bipolarity that distinguishes it from the multi-choice games.

In crisp Bi-cooperative game the membership of players (rates of participation) is assessed in binary terms (i.e., 1 for participation in the game, and 0 for nonparticipation). By contrast, fuzzy set theory permits the gradual assessment of the memberships of players (rates of participation) in a game. When in a coalition, players are participating partially with some membership degrees in the interval [0, 1], we call it a fuzzy coalition. In the similar fashion, when players participate partially in a bi-coalition, we can call it a fuzzy bi-coalition. The notion of a Bi-cooperative game with fuzzy bi-coalitions and its intuitive justifications were proposed by Borkotokey and Sarmah [3] where they obtained a set of axioms for the characterizations of the LG value [9] under fuzzy environment. The class of fuzzy Bi-cooperative games in Choquet integral type was introduced and finally the corresponding LG value for this class was obtained. In the present paper, we obtain an LG value for the class of fuzzy Bi-cooperative games in multilinear extension form. Possible relationships with the existing models are explored.

The rest of the paper is organized as follows. Section 2, presents the notion of Bi-cooperative games and corresponding solution concepts in both crisp and fuzzy environments. In Section 3, we introduce the notion of a fuzzy Bi-cooperative game in multilinear extension form. An LG value for this class has been proposed and shown to satisfy the LG axioms given in [3]. Section 4 includes the concluding remarks.

2. Bi-cooperative games in crisp and fuzzy settings

In this section, we present the basic definitions and results of Bi-cooperative games with both crisp and fuzzy coalitions and also define the LG value as a suitable solution concept. To a large extent, this section is based on Bilbao et al. [1,2], Labreuche and Grabisch [9] and Borkotokey and Sarmah [3]. Throughout the paper $N = \{1, 2, 3, \dots, n\}$ denotes the players' set and $\mathcal{Q}(N) = \{(S, T) \mid S, T \in N \text{ and } S \cap T = \emptyset\}$, the set of all bi-coalitions of N. Further, we assume that the members of $(S, T) \in \mathcal{Q}(N)$ exhibit bipolarity through their contributions to S or T. By what is called abuse of notations we alternatively use i for the singleton set $\{i\}$. Denote by small letters the cardinalities of sets, e.g., s for S etc.

2.1. Bi-cooperative games with crisp bi-coalitions and the LG value

A Bi-cooperative game is a pair (N, v) of which N is the players' set and $v : \mathcal{Q}(N) \to \mathbb{R}$, a real valued function such that $v(\emptyset, \emptyset) = 0$. In what follows, we formally present the two important ordering relations defined on $\mathcal{Q}(N)$ and their possible implications, see [2,9]. The first relation $\sqsubseteq_1$ defined by Bilbao et al. [2] is implied by monotonicity, i.e., for $(S, T), (S', T') \in \mathcal{Q}(N)$, $(S, T) \sqsubseteq_1 (S', T')$ iff $S \subseteq S'$ and $T' \subseteq T$. This makes the two elements $(\emptyset, N)$ and $(N, \emptyset)$ as the bottom and top elements of $\mathcal{Q}(N)$ and $\mathcal{Q}(N)$ becomes a distributive lattice. A second relation: the product order $\sqsubseteq_2$ given in [9] is defined as follows. For $(S, T), (S', T') \in \mathcal{Q}(N)$, $(S, T) \sqsubseteq_2 (S', T')$ iff $S \subseteq S'$ and $T \subseteq T'$ so that $(\emptyset, \emptyset)$ becomes the bottom element and all $(S, N \setminus S)$, $S \subseteq N$, the maximal elements. Under this ordering relation, $\mathcal{Q}(N)$ is an inf-semilattice. Note that this relation is in cognition with the notion of Bi-cooperative games as it incorporates bipolarity, a crucial concept for such games. Therefore, in this paper, we shall adopt this ordering relation and denote it simply without a suffix i.e., by $\sqsubseteq$.

In [9], a value on $\mathcal{BG}^N$ is defined as a function $\Phi : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ which associates each Bi-cooperative game v a vector $(\Phi_1(v), \Phi_2(v), \ldots, \Phi_n(v))$ representing a payoff distribution to the players in the game. There are two more definitions of a value found in the literature (see [1,5,6]). However, the definition given in [9] is more natural due to its endorsements with bipolarity of a Bi-cooperative game. To this end, we define the class of monotone games as follows, see [9].

Definition 2.1. Let $v \in \mathcal{BG}^N$. A player i is called *left monotone* with respect to v if

$$\forall (S,T) \in \mathcal{Q}(N \setminus i), \quad v(S \cup i, T) \geq v(S,T).$$

A player i is *right monotone* with respect to v if

$$\forall (S,T) \in \mathcal{Q}(N \setminus i), \quad v(S, T \cup i) \leq v(S,T).$$

The Bi-cooperative game v is monotone if all players are left and right monotone with respect to v.

The monotonicity of a Bi-cooperative game v is, in fact, its increasingness when considering the ordering $\sqsubseteq_1$ (and thus it is not compatible with the second ordering $\sqsubseteq_2$). Denote by $\mathcal{BG}_M(N)$ the class of monotone games over $\mathcal{Q}(N)$.

Remark 2.2. The expression $v(S \cup i, T) - v(S,T)$ (respectively $v(S,T) - v(S, T \cup i)$) is called the marginal contribution of player i with respect to $(S,T) \in \mathcal{Q}(N \setminus i)$ when she is a positive contributor (respectively a negative contributor).

Prior to the definition of the LG value as a possible solution concept in a crisp Bi-cooperative game, we define the following:

Definition 2.3. Let $(S,T) \in \mathcal{Q}(N)$ and $v \in \mathcal{BG}^N$. Player $i \in N$ is a null player for v, if it satisfies

$$v(S \cup i, T) = v(S,T) = v(S, T \cup i) \tag{2.1}$$

for every $(S,T) \in \mathcal{Q}(N \setminus i)$.

We now define the LG value for the class $\mathcal{BG}^N$ as follows.

Definition 2.4. A function $\Phi : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ defines a value due to Labreuche and Grabisch (the LG value) if for every $(S,T) \in \mathcal{Q}(N)$ it satisfies the following axioms.

Axiom b1 (*Efficiency*). If $(N,v) \in \mathcal{BG}^N$, it holds that,

$$\sum_{i \in N} \Phi_i(N,v)(S,T) = v(S,T)$$

Axiom b2 (*Linearity*). For all $\alpha, \beta \in \mathbb{R}$ and $(N,b), (N,v) \in \mathcal{BG}^N$,

$$\Phi_i(N, \alpha b + \beta v)(S,T) = \alpha\, \Phi_i(N,b)(S,T) + \beta\, \Phi_i(N,v)(S,T).$$

Axiom b3 (*Null Player Axiom*). If player i is null for $(N,v) \in \mathcal{BG}^N$, then $\Phi_i(N,v)(S,T) = 0$.

Axiom b4 (*Intra-Coalition Symmetry*). If $(N,v) \in \mathcal{BG}^N$ and a permutation π is defined on N, such that $\pi S = S$ and $\pi T = T$, then it holds that, for all $i \in N$,

$$\Phi_{\pi i}\big(N, v \circ \pi^{-1}\big)(S,T) = \Phi_i(N,v)(S,T)$$

where $\pi v(\pi L, \pi M) = v(L,M)$ and $\pi L = \{\pi i : i \in L\}$ for every $(L,M) \in \mathcal{Q}(N)$.

Axiom b5 (*Inter-Coalition Symmetry*). Let $i \in S$ and $j \in T$, and (N, v_i), (N, v_j) be two Bi-cooperative games such that for all $(S', T') \in \mathcal{Q}((S \cup T) \setminus \{i, j\})$,

$$v_i(S' \cup i, T') - v_i(S', T') = v_j(S', T') - v_j(S', T' \cup j)$$
$$v_i(S' \cup i, T' \cup j) - v_i(S', T' \cup j) = v_j(S' \cup i, T') - v_j(S' \cup i, T' \cup j)$$

Then,

$$\Phi_i(N, v_i)(S, T) = -\Phi_j(N, v_j)(S, T). \tag{2.2}$$

Axiom b6 (*Monotonicity*). Given (N, v), $(N, v') \in \mathcal{BG}^N$ such that $\exists i \in N$ with

$$v'(S', T') = v(S', T') \tag{2.3}$$
$$v'(S' \cup i, T') \geq v(S' \cup i, T') \tag{2.4}$$
$$v'(S', T' \cup i) \geq v(S', T' \cup i) \tag{2.5}$$

for all $(S', T') \in \mathcal{Q}(N \setminus i)$, then $\Phi_i(N, v')(S, T) \geq \Phi_i(N, v)(S, T)$.

The following theorem ensures existence and uniqueness of the LG value.

Theorem 2.5. *There exists a unique value $\Phi(N, v)(S, T)$ on $\mathcal{BG}^N$ for $(S, T) \in \mathcal{Q}(N)$ that satisfies Axioms b1–b6 and is given by,*

$$\Phi_i(N, v)(S, T) = \sum_{K \subseteq (S \cup T) \setminus i} \frac{k!(s + t - k - 1)!}{(s + t)!}\left[V(K \cup i) - V(K)\right] \tag{2.6}$$

for all $i \in N$ where for $K \subseteq S \cup T$, $V(K) := v(S \cap K, T \cap K)$. Moreover, if $i \in N \setminus (S \cup T)$, $\Phi_i(N, v)(S, T) = 0$.

An important corollary to Theorem 2.5 given in [9] is as follows.

Result 2.6. *We have,*

$$\forall i \in N \setminus (S \cup T), \quad \Phi_i(v)(S, T) = 0 \tag{2.7}$$
$$\forall i \in S, \text{ with } i \text{ left monotone}, \quad \Phi_i(v)(S, T) \geq 0 \tag{2.8}$$
$$\forall i \in T, \text{ with } i \text{ right monotone}, \quad \Phi_i(v)(S, T) \leq 0 \tag{2.9}$$

2.2. Bi-cooperative games with fuzzy bi-coalitions and the Fuzzy LG value

Extending the notion of crisp bi-coalition to its fuzzy counterpart we define a fuzzy bi-coalition as follows:

Definition 2.7. Let $N = \{1, 2, \ldots, n\}$ be given. A fuzzy bi-coalition is an expression A on N given by

$$A = \left\{\langle i, \mu_A^N(i), \nu_A^N(i)\rangle \,\middle|\, i \in N, \min_{i \in N}\left(\mu_A^N(i), \nu_A^N(i)\right) = 0\right\}$$

where, $\mu_A^N : N \to [0, 1]$, $\nu_A^N : N \to [0, 1]$ represent respectively, the membership functions over N of the fuzzy sets of positive and negative contributors of A.

Remark that minimum condition in the above definition implies that the two options (positive and negative contributions) are mutually exclusive so that one cannot choose a little bit of both options simultaneously [3].

Thus, it follows from Definition 2.7 that a fuzzy bi-coalition A of N can be completely identified with the functions μ_A^N and ν_A^N. As N is fixed here, μ_A^N and ν_A^N can be simply written as μ_A and ν_A. Player i is a positive contributor in A if $\mu_A(i) > 0$ and a negative contributor if $\nu_A(i) > 0$. Let $\mathcal{F}_B(N)$ denote the set of all fuzzy bi-coalitions on N. Note that every crisp bi-coalition can be considered as a fuzzy bi-coalition with memberships either 0 or 1. Thus with an abuse of notations, we write $\mathcal{Q}(N) \subseteq \mathcal{F}_B(N)$.

For comparing the fuzzy bi-coalitions $A, B \in \mathcal{F}_B(N)$, the following operations and relations are adopted.

$$A \preceq B \quad \Leftrightarrow \quad \mu_A(i) \le \mu_B(i) \quad \text{and} \quad \nu_A(i) \le \nu_B(i) \quad \forall i \in N.$$
$$A = B \quad \Leftrightarrow \quad \mu_A(i) = \mu_B(i) \quad \text{and} \quad \nu_A(i) = \nu_B(i) \quad \forall i \in N.$$

For any $A \in \mathcal{F}_B(N)$, denote by $\mathcal{F}_B(A)$, the set of all fuzzy bi-coalitions B such that $B \preceq A$.

The intersection of two fuzzy bi-coalitions A and B is obtained using the minimum operator '$\wedge$' as follows.

$$A \cap B = \{\langle i, \mu_A(i) \wedge \mu_B(i), \nu_A(i) \wedge \nu_B(i)\rangle \mid i \in N\}. \tag{2.10}$$

This ensures the existence of the infimum of every pair of elements in $\mathcal{F}_B(N)$ and thus similar to its crisp counterpart $\mathcal{Q}(N)$, $\mathcal{F}_B(N)$ can be considered as an inf-semilattice. The union however can be defined only on a restricted sub-domain of $\mathcal{F}_B(N)$. Formally we have the following.

For $A, B \in \mathcal{F}_B(N)$ such that $\{\mu_A(i) \vee \mu_B(i)\} \wedge \{\nu_A(i) \vee \nu_B(i)\} = 0$, $\forall i \in N$, we define, $A \cup B$ as follows.

$$A \cup B = \{\langle i, \mu_A(i) \vee \mu_B(i), \nu_A(i) \vee \nu_B(i)\rangle : i \in N\} \tag{2.11}$$

The Support of a fuzzy bi-coalition A, denoted by Supp(A) is given by

$$\text{Supp}(A) = (\{i \in N \mid \mu_A(i) > 0\}, \{i \in N \mid \nu_A(i) > 0\}) \tag{2.12}$$

Definition 2.8. The null fuzzy bi-coalition $\emptyset_B$ is given by

$$\emptyset_B = \{\langle i, \mu_{\emptyset_B}(i), \nu_{\emptyset_B}(i)\rangle \mid i \in N\}$$

where $\mu_{\emptyset_B}(i) = 0$, and $\nu_{\emptyset_B}(i) = 0 \ \forall i \in N$.

Thus a Bi-cooperative game with fuzzy bi-coalitions can be defined as follows:

Definition 2.9. A Bi-cooperative game with fuzzy bi-coalitions is a function $w : \mathcal{F}_B(N) \to \mathbb{R}$ with $w(\emptyset_B) = 0$. We call the value $w(A)$, the worth of A due to the fuzzy or partial contributions by the members of N.

The worth $w(A)$ for every $A \in \mathcal{F}_B(N)$ is interpreted as the gain (whenever $w(A) > 0$) or loss (whenever $w(A) < 0$) that A can receive when the players can participate in it in either of the three distinct capacities: positive, negative or absentees. We call a "Bi-cooperative game with fuzzy bi-coalitions" a "fuzzy Bi-cooperative game" in short. Let $\mathcal{G}_{\mathcal{FB}}(N)$ denote the class of all fuzzy Bi-cooperative games. It follows that the class $\mathcal{BG}^N$, of crisp Bi-cooperative games is a subclass of the class $\mathcal{G}_{\mathcal{FB}}(N)$ of fuzzy Bi-cooperative games.

Definition 2.10. Let $w \in \mathcal{G}_{\mathcal{FB}}(N)$. Player $i \in N$ is called left monotone in fuzzy sense with respect to w if for every $A, B \in \mathcal{F}_B(N)$ such that $\mu_A(i) > \mu_B(i)$ with $\mu_A(j) = \mu_B(j)$ and $\nu_A(j) = \nu_B(j)$ for $i \neq j \in N$, we have $w(A) \ge w(B)$. Similarly, player i is right monotone in fuzzy sense with respect to w if for every $A, B \in \mathcal{F}_B(N)$ such that $\nu_A(i) > \nu_B(i)$ with $\mu_A(j) = \mu_B(j)$ and $\nu_A(j) = \nu_B(j)$ for $i \neq j \in N$, we have $w(A) \le w(B)$.

The game $w \in \mathcal{G}_{\mathcal{FB}}(N)$ is monotone in fuzzy sense if every player is both left and right monotone in fuzzy sense.

Note that Definition 2.10 provides a more general form of a monotone game in fuzzy sense than in [3]. Preparatory to the definition of the LG axioms for a fuzzy Bi-cooperative game, we describe the following.

Definition 2.11. If $A \in \mathcal{F}_B(N)$, and $w \in \mathcal{G}_{\mathcal{FB}}(N)$, the player $i \in N$ is said to be null for w in A if $w(B \cup I) = w(B)$ for all $B \in \mathcal{F}_B(A)$ with $\mu_B(i) = \nu_B(i) = 0$ and all $I \in \mathcal{F}_B(N)$ such that $\mu_I(j) = \nu_I(j) = 0$ when $j \neq i$, where $\cup$ is as defined in (2.11).

Definition 2.12. Let $A \in \mathcal{F}_B(N)$, for any permutation π on N, define the fuzzy bi-coalition πA by

$$\mu_{\pi A}(i) = \mu_A(\pi^{-1} i) \tag{2.13}$$
$$\nu_{\pi A}(i) = \nu_A(\pi^{-1} i) \tag{2.14}$$

Then πA is called a permutation of the fuzzy bi-coalition A.

Now we define the LG value for Bi-cooperative games with fuzzy bi-coalitions as follows.

Definition 2.13. A function $\Phi : \mathcal{G}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{F}_B(N)}$ is said to be an LG value on $\mathcal{G}_{\mathcal{FB}}(N)$ if it satisfies the following six axioms:

Axiom f1 (*Efficiency*). If $w \in \mathcal{G}_{\mathcal{FB}}(N)$ and $A \in \mathcal{F}_B(N)$, then

$$\sum_{i \in N} \Phi_i(w)(A) = w(A).$$

Axiom f2 (*Linearity*). For $\alpha, \beta \in \mathbb{R}$ and $w, w' \in \mathcal{G}_{\mathcal{FB}}(N)$ we must have

$$\Phi\big(\alpha w + \beta w'\big) = \alpha \Phi(w) + \beta \Phi\big(w'\big).$$

Axiom f3 (*Null Player Axiom*). If player $i \in N$ is a null player for $w \in \mathcal{G}_{\mathcal{FB}}(N)$, in $A \in \mathcal{F}_B(N)$, then, $\Phi_i(w)(A) = 0$.

Axiom f4 (*Intra-Coalition Symmetry*). For any $w \in \mathcal{G}_{\mathcal{FB}}(N)$, a fuzzy bi-coalition A, and a permutation π defined on N such that $\pi A = A$, it holds for all $i \in N$,

$$\Phi_i(w)(A) = \Phi_{\pi i}(\pi w)(A) \tag{2.15}$$

where $\pi w \in \mathcal{G}_{\mathcal{FB}}(N)$ is defined by $\pi w(\pi B) = w(B)$, with πB defined for every $B \in \mathcal{F}_B(N)$ as in Definition 2.12.

Axiom f5 (*Inter-Coalition Symmetry*). Given $A \in \mathcal{F}_B(N)$ and $i, j \in N$, if w_i and w_j are two Bi-cooperative games with fuzzy bi-coalitions such that for every $B \in \mathcal{F}_B(A)$ with $i, j \notin \mathrm{Supp}(B)$ (i.e. $\mu_B(i) = 0 = \mu_B(j)$ and $\nu_B(i) = 0 = \nu_B(j)$), and every pair of $I, J \in \mathcal{F}_B(N)$, such that $\mu_I(i) = \nu_J(j) > 0$ or $\mu_J(j) = \nu_I(i) > 0$ and $\mu_I(k) = \mu_J(k) = 0 = \nu_I(k) = \nu_J(k)\ \forall k \in N \setminus \{i, j\}$, it holds that

$$w_i(B \cup I) - w_i(B) = w_j(B) - w_j(B \cup J)$$
$$w_i(B \cup I \cup J) - w_i(B \cup J) = w_j(B \cup I) - w_j(B \cup I \cup J)$$

then, $\Phi_i(w_i)(A) = -\Phi_j(w_j)(A)$.

Axiom f6 (*Monotonicity*). Let w and w' be two Bi-cooperative games with fuzzy bi-coalitions and $A \in \mathcal{F}_B(N)$. Let further that there exists an $i \in N$ such that for every $I \in \mathcal{F}_B(N)$ with $\mu_I(i) > 0$ or $\nu_I(i) > 0$, $\mu_I(j) = \nu_I(j) = 0$, $\forall j \neq i$, and for all $B \in \mathcal{F}_B(A)$ such that $\mu_B(i) = \nu_B(i) = 0$, it holds that

$$w'(B) = w(B)$$
$$w'(B \cup I) \geq w(B \cup I)$$

Then, $\Phi_i(w')(A) \geq \Phi_i(w)(A)$.

Note that if Φ satisfies Axioms f1–f6 then its restriction to the class of crisp Bi-cooperative games namely, $\Phi|_{\mathcal{BG}^N}$ satisfies Axioms f1–f6. As a matter of fact the Axioms f1–f6 are intuitive extensions of their crisp analogues. Furthermore, the above definition can apply to any class of Bi-cooperative games with fuzzy bi-coalitions. The reader may look into [3] for a detailed discussion on the solution concept that has been developed through the characterization process. In the next section, we propose the notion of fuzzy Bi-cooperative games in multilinear extension form and discuss their properties.

3. Fuzzy Bi-cooperative games in multilinear extension form

Take $\mathrm{Supp}\, A = (L, M) \in \mathcal{Q}(N)$. Given $v \in \mathcal{BG}^N$ and $A \in \mathcal{F}_B(N)$, define the fuzzy Bi-cooperative game $w_M : \mathcal{F}_B(N) \to \mathbb{R}$ in multilinear extension form as follows.

$$w_M(A)=\frac{1}{2}\sum_{(S,T)\subseteq \mathrm{Supp}\,A}\left\{\prod_{i\in S}\mu_A(i)\prod_{j\in L\setminus S}\big(1-\mu_A(j)\big)+\prod_{i\in T}\nu_A(i)\prod_{j\in M\setminus T}\big(1-\nu_A(j)\big)\right\}v(S,T) \tag{3.1}$$

Denote by $\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$ the class of fuzzy Bi-cooperative games in multilinear extension form. In what follows, we define an LG value for $\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$. Prior to this we have the following. Given $K\subseteq N$, define the fuzzy bi-coalition A_K as follows.

$$\mu_{A_K}(j)=\mu_A(j)\wedge\chi_K(j),\quad \forall j\in N \tag{3.2}$$

$$\nu_{A_K}(j)=\nu_A(j)\wedge\chi_K(j),\quad \forall j\in N \tag{3.3}$$

where χ_K is the characteristic (membership) function of K given by

$$\chi_K(j)=\begin{cases}1 & \text{if } j\in K\\ 0 & \text{otherwise}\end{cases} \tag{3.4}$$

Then we have,

$$\begin{aligned}\mathrm{Supp}\,A_K&=\big(\{j\in N: \mu_{A_K}(j)>0\},\{j\in N: \nu_{A_K}(j)>0\}\big)\\&=(L\cap K, M\cap K)\end{aligned} \tag{3.5}$$

Similarly, for $I\in\mathcal{F}_B(N)$ such that,

$$\mu_I(i)=\mu_A(i),\qquad \nu_I(i)=\nu_A(i) \tag{3.6}$$

$$\mu_I(j)=\nu_I(j)=0,\quad \text{if } i\neq j \tag{3.7}$$

we have $\mathrm{Supp}(A_K\cup I)=(L\cap(K\cup i), M\cap(K\cup i))$ which follows from the facts that

$$\mu_{(A_K\cup I)}(j)=\mu_A(j)\wedge\chi_{K\cup\{i\}}(j) \tag{3.8}$$

$$\nu_{(A_K\cup I)}(j)=\nu_A(j)\wedge\chi_{K\cup\{i\}}(j) \tag{3.9}$$

Set $Q(L,M)=\{(S,T): S\subseteq L, T\subseteq M\}$. For every $w_M\in\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$, we define the function $\Phi:\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)\to(\mathbb{R}^n)^{\mathcal{F}_B(N)}$ by

$$\begin{aligned}&\Phi_i\big(w_M(A)\big)\\&\quad=\sum_{K\subseteq(L\cup M)\setminus\{i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\\&\qquad\times\frac{1}{2}\Bigg[\sum_{(S_0,T_0)\in Q(L\cap(K\cup i),M\cap(K\cup i))}\left\{\prod_{t\in S_0}\mu_A(t)\prod_{r\in L\setminus S_0}\big(1-\mu_A(r)\big)+\prod_{t\in T_0}\nu_A(t)\prod_{r\in M\setminus T_0}\big(1-\nu_A(r)\big)\right\}v(S_0,T_0)\\&\qquad-\sum_{(S_0',T_0')\in Q(L\cap K,\ M\cap K)}\left\{\prod_{t\in S_0'}\mu_A(t)\prod_{r\in L\setminus S_0'}\big(1-\mu_A(r)\big)+\prod_{t\in T_0'}\nu_A(t)\prod_{r\in M\setminus T_0'}\big(1-\nu_A(r)\big)\right\}v\big(S_0',T_0'\big)\Bigg]\end{aligned} \tag{3.10}$$

Using (3.2) through (3.9), expression (3.10) can be simplified to the following:

$$\Phi_i\big(w_M(A)\big)=\sum_{K\subseteq(L\cup M)\setminus\{i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\big\{w_M(A_K\cup I)-w_M(A_K)\big\} \tag{3.11}$$

Now we have our main theorem as follows.

Theorem 3.1. *The function $\Phi : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ given by (3.10) (or equivalently by (3.11)) is an LG value for the class of fuzzy Bi-cooperative games in multilinear extension form given by (3.1).*

In order to prove Theorem 3.1, we need to show that the function Φ given in (3.10) (or equivalently in (3.11)) satisfies Axioms f1–f6.

Proof.
Axiom f1 (Efficiency). Take $L \cup M = \{i_1, i_2, i_3, \dots, i_p\}$ in (3.10). Note that here, we do not specify the roles of the players (positive or negative) in the formulation however it does not affect their bipolar nature.

For every i_j $(j = 1, 2, \dots p)$ and $K \subseteq (L \cup M) \setminus i_j$, in the following we list all the possible $(K \cup i_j)$s (where $K \cup i_j = (L \cap (K \cup i_j)) \cup (M \cap (K \cup i_j)))$ as entries in a matrix P_k of p clusters with $k + 1$ repeated entries.

$$P_k = \left(A_k^1 \ \vdots \ A_k^2 \ \vdots \ \cdots \ \vdots \ A_k^p\right)$$

where

$$A_k^p = \begin{pmatrix} \{i_1, i_2, \dots i_k, i_p\} & \{i_2, i_3, \dots i_{k+1}, i_p\} & \cdots \{i_{p-k}, i_{p-(k-1)}, \dots i_{p-1}, i_p\} \\ \{i_1, i_3 \cdots i_{k+1}, i_p\} & \{i_2, i_4 \cdots i_{k+2}, i_p\} & \cdots \times \\ \vdots & \vdots & \vdots \\ \{i_1, i_{p-k} \cdots i_{p-2}, i_p\} & \{i_2, i_{p-(k-1)} \cdots i_{p-1}, i_p\} & \cdots \times \\ \{i_2, i_{p-(k-1)} \cdots i_{p-1}, i_p\} & \times & \cdots \times \end{pmatrix}$$

In a similar way, for every i_j $(j = 1, 2, \dots p)$ and $K \subseteq (L \cup M) \setminus i_j$, we list all the possible Ks $(K = (L \cap K) \cup (M \cap K))$ as entries in a matrix Q_k of p clusters with $p - k$ repeated entries as follows.

$$Q_k = \left(B_k^1 \ \vdots \ B_k^2 \ \vdots \ \cdots \ \vdots \ B_k^p\right)$$

where the entries of B_k^l $(l = 1, 2, \dots, p)$ are exactly same as those of the corresponding A_k^l with only the exception that they do not contain i_l $(l = 1, 2, \dots, p)$.

Since the jth cluster of Q_{k+1} corresponds to the jth cluster of P_k, it follows from (3.10)

$$\sum_{i_j \in (L \cup M)} \Phi_{i_j}\big(w_M(A)\big) = \sum_{k=0}^{p-1}\Bigg(\sum_{i_j \in (L \cup M)} C_k^{i_j}\Bigg) - \sum_{k=0}^{p-1}\Bigg(\sum_{i_j \in (L \cup M)} D_k^{i_j}\Bigg) \tag{3.12}$$

where

$$C_k^{i_j} = \sum_{K:|K|=k, i_j \notin K} \frac{k!(p-k-1)!}{p!} \times \frac{1}{2}$$
$$\times \sum_{(S_0,T_0) \in Q(L \cap (K \cup i_j), M \cap (K \cup i_j))} \Bigg\{\prod_{i_t \in S_0} \mu_A(i_t) \prod_{i_r \in L \setminus S_0} \big(1 - \mu_A(i_r)\big)$$
$$+ \prod_{i_t \in T_0} \nu_A(i_t) \prod_{i_r \in M \setminus T_0} \big(1 - \nu_A(i_r)\big)\Bigg\} v(S_0, T_0)$$

$$D_k^{i_j} = \sum_{K:|K|=k, i_j \notin K} \frac{k!(p-k-1)!}{p!} \times \frac{1}{2}$$
$$\times \sum_{(S_0',T_0') \in Q(L \cap K,\ M \cap K)} \Bigg\{\prod_{i_t \in S_0'} \mu_A(i_t) \prod_{i_r \in L \setminus S_0'} \big(1 - \mu_A(i_r)\big)$$
$$+ \prod_{i_t \in T_0'} \nu_A(i_t) \prod_{i_r \in M \setminus T_0'} \big(1 - \nu_A(i_r)\big)\Bigg\} v\big(S_0', T_0'\big)$$

Now

$$\sum_{i_j\in(L\cup M)} C_k^{i_j} = \sum_{i_j\in(L\cup M)} \sum_{K:|K|=k,i_j\notin K} \frac{k!(p-k-1)!}{p!} \times \frac{1}{2} \times \sum_{(S_0,T_0)\in Q(L\cap(K\cup\{i_j\}),M\cap(K\cup\{i_j\}))} \Bigg\{ \prod_{i_t\in S_0} \mu_A(i_t) \prod_{i_r\in L\setminus S_0} \big(1-\mu_A(i_r)\big) + \prod_{i_t\in T_0} \nu_A(i_t) \prod_{i_r\in M\setminus T_0} \big(1-\nu_A(i_r)\big) \Bigg\} v(S_0,T_0) \quad (3.13)$$

$$\sum_{i_j\in(L\cup M)} D_k^{i_j} = \sum_{i_j\in(L\cup M)} \sum_{K:|K|=k,i_j\notin K} \frac{k!(p-k-1)!}{p!} \times \frac{1}{2} \times \sum_{(S_0',T_0')\in Q(L\cap K,\ M\cap K)} \Bigg\{ \prod_{i_t\in S_0'} \mu_A(i_t) \prod_{i_r\in L\setminus S_0'} \big(1-\mu_A(i_r)\big) + \prod_{i_t\in T_0'} \nu_A(i_t) \prod_{i_r\in M\setminus T_0'} \big(1-\nu_A(i_r)\big) \Bigg\} v\big(S_0',T_0'\big) \quad (3.14)$$

It follows from the matrix P_k and Q_k the entries of P_k and Q_{k+1} are identical whereas each entry of P_k is repeated $k+1$ times and each entry of Q_k is repeated $p-k$ times. In view of this along with (3.13)–(3.14) and noting that the expressions of $\sum_{i_j\in(L\cup M)} C_{k-1}^{i_j}$ and $\sum_{i_j\in(L\cup M)} D_k^{i_j}$ are identical for $k=1,2,\dots p$, we see that in (3.12) the corresponding elements cancel each other. Thus we have,

$$\sum_{i_j\in(L\cup M)} \Phi_{i_j}\big(w_M(A)\big) = \sum_{i_j\in(L\cup M)} C_{p-1}^{i_j} = \frac{1}{2} \sum_{(S_0,T_0)\in Q(L\cap(K\cup\{i_j\}),M\cap(K\cup\{i_j\}))} \Bigg\{ \prod_{i_t\in S_0} \mu_A(i_t) \prod_{i_r\in L\setminus S_0} \big(1-\mu_A(i_r)\big) + \prod_{i_t\in T_0} \nu_A(i_t) \prod_{i_r\in M\setminus T_0} \big(1-\nu_A(i_r)\big) \Bigg\} v(S_0,T_0)$$

Following the fact that, when $|K|=p-1$ and $i_j\notin K$, $Q((L\cap(K\cup\{i_j\}),M\cap(K\cup\{i_j\})))=Q(L,M)$, we obtain,

$$\sum_{i_j\in(L\cup M)} \Phi_{i_j}\big(w_M(A)\big) = \frac{1}{2} \sum_{(S_0,T_0)\in Q(L,M)} \Bigg\{ \prod_{i_t\in S_0} \mu_A(i_t) \prod_{i_r\in L\setminus S_0} \big(1-\mu_A(i_r)\big) + \prod_{i_t\in T_0} \nu_A(i_t) \prod_{i_r\in M\setminus T_0} \big(1-\nu_A(i_r)\big) \Bigg\} v(S_0,T_0) = w_M(A).$$

As the player set $N=\{i_1,i_2,\dots,i_p\}$ is arbitrary the result holds for any players' set N. This proves the efficiency.

Axiom f2 (Linearity). Let, $\alpha,\beta\in\mathbb{R}$ and $w_M, w_M'\in\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$. Following (3.1) we have,

$$w_M(A) = \frac{1}{2} \sum_{(S,T)\subseteq \operatorname{Supp} A} \Bigg\{ \prod_{i\in S} \mu_A(i) \prod_{j\in L\setminus S} \big(1-\mu_A(j)\big) + \prod_{i\in T} \nu_A(i) \prod_{j\in M\setminus T} \big(1-\nu_A(j)\big) \Bigg\} v(S,T)$$

$$w_M'(A) = \frac{1}{2} \sum_{(S,T)\subseteq \operatorname{Supp} A} \Bigg\{ \prod_{i\in S} \mu_A(i) \prod_{j\in L\setminus S} \big(1-\mu_A(j)\big) + \prod_{i\in T} \nu_A(i) \prod_{j\in M\setminus T} (1-\nu_A)(j) \Bigg\} v'(S,T)$$

Now

$$\begin{aligned}&\big(\alpha w_M+\beta w'_M\big)(A)\\&\quad=\frac{1}{2}\sum_{(S,T)\subseteq \operatorname{Supp} A}\Big\{\prod_{i\in S}\mu_A(i)\prod_{j\in L\setminus S}\big(1-\mu_A(j)\big)+\prod_{i\in T}\nu_A(i)\prod_{j\in M\setminus T}\big(1-\nu_A(j)\big)\Big\}\big(\alpha v(S,T)+\beta v'(S,T)\big)\end{aligned}$$

Thus, $\Phi_i(\alpha w_M+\beta w'_M)(A)=\alpha\Phi_i(w_M)+\beta\Phi_i(w'_M)$.

Axiom f3 (Null Player Axiom). Let $i\in N$ be a Null Player. Then by Definition 2.11, for each $A\in\mathcal{F}_B(N)$, we have $w_M(B\cup I)=w_M(B)$, $\forall B\in\mathcal{F}_B(A)$ with $\mu_B(i)=\nu_B(i)=0$ and $\forall I\in\mathcal{F}_B(N)$ such that $\mu_I(j)=\nu_I(j)=0$, when $j\neq i$. Thus we obtain,

$$\begin{aligned}\Phi_i\big(w_M(A)\big)&=\sum_{K\subseteq(L\cup M)\setminus\{i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\big(w_M(A_K\cup I)-w_M(A_K)\big)\\&=0\end{aligned}$$

Axiom f4 (Intra-Coalition Symmetry). It follows from (3.1) that for any $A\in\mathcal{F}_B(N)$, permutation π such that $\pi A=A$ and $w_M\in\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$,

$$\begin{aligned}\Phi_{\pi i}\big(\pi w_M(\pi A)\big)&=\sum_{\pi K\subseteq\pi(L\cup M)\setminus\{\pi i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\big\{\pi w_M\big(\pi A_{\pi K}\cup(\pi I)\big)-\pi w_M(\pi A_{\pi K})\big\}\\&=\sum_{K'\subseteq(L\cup M)\setminus\{i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\big\{\pi w_M\big(\pi A_{\pi K}\cup(\pi I)\big)-\pi w_M(\pi A_{\pi K})\big\}\end{aligned}\tag{3.15}$$

Following (3.2) and (3.3) we have,

$$\begin{aligned}\mu_{\pi A_{\pi K}}(j)&=\mu_{\pi A}(j)\wedge\chi_{\pi K}(j),\quad\forall j\in N\\&=\mu_A\big(\pi^{-1}j\big)\wedge\chi_K\big(\pi^{-1}j\big),\quad\forall j\in N\\&=\mu_{A_K}\big(\pi^{-1}j\big)\\&=\mu_{\pi A_K}(j)\end{aligned}\tag{3.16}$$

It follows that,

$$\nu_{\pi A_{\pi K}}(j)=\nu_{\pi A_K}(j),\quad\forall j\in N\tag{3.17}$$

$$\mu_{(\pi A_{\pi K}\cup(\pi I))}(j)=\mu_{\pi(A_K\cup I)}(j),\quad\forall j\in N\tag{3.18}$$

$$\nu_{(\pi A_{\pi K}\cup(\pi I))}(j)=\nu_{\pi(A_K\cup I)}(j),\quad\forall j\in N\tag{3.19}$$

Putting (3.16) through (3.19) in (3.15), we get,

$$\Phi_{\pi i}\big(\pi w_M(\pi A)\big)=\sum_{K'\subseteq(L\cup M)\setminus\{i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\big\{\pi w_M\big(\pi\big(A_K\cup(I)\big)\big)-\pi w_M\big(\pi(A_K)\big)\big\}\tag{3.20}$$

Using the fact that $(\pi w_M)(\pi A)=w_M(A)$ and $\pi A=A$, (3.20) becomes

$$\begin{aligned}\Phi_{\pi i}\big(\pi w_M(A)\big)&=\sum_{K'\subseteq(L\cup M)\setminus\{i\}}\frac{k!(l+m-k-1)!}{(l+m)!}\big\{w_M(A_K\cup I)-w_M(A_K)\big\}\\&=\Phi_i\big(w_M(A)\big)\end{aligned}$$

Axiom f5 (Inter-Coalition Symmetry). Let w^i_M and w^j_M be two Bi-cooperative games with fuzzy bi-coalitions such that for every $B\in\mathcal{F}_B(A)$ with $i,j\notin\operatorname{Supp}B$ and every pair $I,J\in\mathcal{F}_B(N)$ such that $\mu_I(i)=\nu_J(j)>0$ or $\mu_J(j)=\nu_I(i)>0$ and $\mu_I(k)=\mu_J(k)=0=\nu_I(k)=\nu_J(k)$, $\forall k\in N\setminus\{i,j\}$, it holds that,

(i) $w_M^i(B \cup I) - w_M^i(B) = w_M^j(B) - w_M^j(B \cup J)$
(ii) $w_M^i(B \cup J \cup I) - w_M^i(B \cup J) = w_M^j(B \cup I) - w_M^j(B \cup I \cup J)$

Given $K \subseteq N$ and $i, j, l \in N$, we construct A_K^{ij} as follows.

$$\mu_{A_K^{ij}}(l) = \begin{cases} \mu_A(l) \wedge \chi_K(l), & \text{if } l \in N \setminus \{i, j\} \\ 0, & \text{otherwise} \end{cases} \tag{3.21}$$

$$\nu_{A_K^{ij}}(l) = \begin{cases} \nu_A(l) \wedge \chi_K(l), & \text{if } l \in N \setminus \{i, j\} \\ 0, & \text{otherwise} \end{cases} \tag{3.22}$$

where the function χ_K is given by (3.4).

Let $J \in \mathcal{F}_B(N)$ be such that,

$$\mu_J(j) = \mu_{A_K}(j), \qquad \nu_J(j) = \nu_{A_K}(j) \tag{3.23}$$

$$\mu_J(k) = \nu_J(k) = 0, \quad \text{if } k \neq j \tag{3.24}$$

Therefore using the facts that $A_K = A_K^{ij} \cup J$ when $K \subseteq (L \cup M) \setminus \{i\}$ and $A_K = A_K^{ij} \cup I$ when $K \subseteq (L \cup M) \setminus \{j\}$, we have from (3.10),

$$\begin{aligned} \Phi_i\big(w_M^i(A)\big) &= \sum_{K \subseteq (L\cup M)\setminus\{i\}} \frac{k!(l+m-k-1)!}{(l+m)!}\big(w_M^i(A_K \cup I) - w_M^i(A_K)\big), \quad i, j \notin K \\ &= \sum_{K \subseteq (L\cup M)\setminus\{i\}} \frac{k!(l+m-k-1)!}{(l+m)!}\big(w_M^i\big(A_K^{ij} \cup J \cup I\big) - w_M^i\big(A_K^{ij} \cup J\big)\big) \end{aligned}$$

and

$$\begin{aligned} \Phi_j\big(w_M^j(A)\big) &= \sum_{K \subseteq (L\cup M)\setminus\{j\}} \frac{k!(l+m-k-1)!}{(l+m)!}\big(w_M^j\big(A_K \cup I\big) - w_M^j\big(A_K\big)\big) \\ &= \sum_{K \subseteq (L\cup M)\setminus\{j\}} \frac{k!(l+m-k-1)!}{(l+m)!}\big(w_M^j\big(A_K^{ij} \cup I \cup J\big) - w_M^j\big(A_K^{ij} \cup I\big)\big) \end{aligned}$$

Using (i) and (ii), this further simplifies to, $\Phi_i(w_M^i(A)) = -\Phi_j(w_M^j(A))$.

Axiom f6 (Monotonicity). Let w_M and w_M' be two Bi-cooperative games with fuzzy bi-coalitions and $A \in \mathcal{F}_B(N)$. Let further that $\exists$ an $i \in N$, such that for every $I \in \mathcal{F}_B(N)$ with $\mu_I(i) > 0$ or $\nu_I(i) > 0$, $\mu_I(j) = \nu_I(j) = 0$, $\forall j \neq i$ and $\forall B \in \mathcal{F}_B(A)$ such that $\mu_B(i) = \nu_B(i) = 0$, it holds that

(i) $w_M'(B) = w_M(B)$
(ii) $w_M'(B \cup I) \geq w_M(B \cup I)$

The result follows from (3.11) along with (i) and (ii). This completes the proof. □

Example 3.2. Let $N = \{1, 2, 3\}$. Take an illustrative example where the crisp Bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$ is given by $v(\emptyset, \emptyset) = 0$, $v(1, \emptyset) = 1$, $v(3, \emptyset) = 2$, $v(\emptyset, 2) = 3$, $v(1, 2) = 4$, $v(3, 2) = 2$, $v((1, 3), 2) = 5$, $v((1, 3), \emptyset) = 6$ and $v(S, T) = 0$ for any other $(S, T) \in \mathcal{Q}(N)$. Let A be a fuzzy bi-coalition over N given by

$$A = \big\{\langle 1, 0.1, 0\rangle, \langle 2, 0, 0.2\rangle, \langle 3, 0.3, 0\rangle\big\}$$

Thus using (3.1), $w_M(A) = 2.2$. After some computations, the LG value of w_M for A is obtained as (0.5497, 0.9628, 0.7123). A close look at the values of the crisp Bi-cooperative game reveals that player 2 is a negative contributor and both players 1 and 3 are positive contributors, yet 3 is more influential than 1 in generating a value. Moreover, player 3 has more membership as a positive contributor than 1 in the fuzzy bi-coalition A also. Keeping all these into account, the LG value divides the value $w_M(A)$ among the three players accordingly.

4. Conclusion

This paper proposes a new class of fuzzy Bi-cooperative games, namely the fuzzy Bi-cooperative games in multilinear extension form. An LG value is obtained as a solution concept to those games. The class of games we defined here generalizes Owen's multilinear extension. Similar extensions incorporating bipolarity exist in the literature and are related to various forms of the Choquet integrals [5,6,11]. Basically they are distinguished by the order relation on the class of bi-coalitions. Since the product order naturally endorses bipolarity among the players in a game, we focus only on extensions with product order, see [9]. In a future work, we aim to focus on alternative extensions based on principles of symmetric extensions of Choquet integral as discussed in [11].

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