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RESEARCH PAPER

A $(\star, *)$-BASED MINKOWSKI'S INEQUALITY FOR SUGENO FRACTIONAL INTEGRAL OF ORDER $\alpha > 0$

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Abstract

We first introduce the concept of Sugeno fractional integral based on the concept of g-seminorm. Then Minkowski's inequality for Sugeno fractional integral of the order $\alpha > 0$ based on two binary operations $\star, *$ is given. Our results significantly generalize the previous results in this field of fuzzy measure and fuzzy integral. Some examples are given to illustrate the results.

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Key Words and Phrases: Sugeno integral, Sugeno fractional integral, Minkowski's inequality, fuzzy integral, fuzzy measure

1. Introduction and motivation

Fractional calculus plays an important role in differential equations, probability and statistics, see e.g. [6]. In many non-deterministic problems, the assumption of additivity is not always plausible. So, it is necessary to extend the concept of additivity to non-additivity case. The theory of fuzzy measure and fuzzy integral was initially introduced by Sugeno [17] as a tool for modeling non-deterministic problems. The Sugeno integral with respect to a fuzzy measure is also called *fuzzy integral*. It is a useful tool in several problems of theoretical and applied mathematics which has been built on non-additive measure. The properties and applications of the Sugeno integral have been studied by many authors, see for example [9, 13, 16, 17, 18].

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Let us recall some notions and notations for the Sugeno integral used in the subsequent sections of this paper. A monotone measure μ on a measurable space $(\Omega, \mathcal{F})$ is a set function $\mu : \mathcal{F} \to B$ (where $B = [0,1]$ or $B = [0,\infty]$ or $B = [0,\infty)$) satisfying

(i) $\mu(\emptyset) = 0$;

(ii) $\mu(E) \le \mu(F)$ whenever $E \subseteq F$;

moreover, μ is called *real* if $\mu(\Omega) < \infty$. The triple $(\Omega, \mathcal{F}, \mu)$ is also called a *monotone measure space* if μ is a monotone measure on $\mathcal{F}$. Note that a real monotone measure μ satisfying $\mu(\Omega) = 1$, is also called *fuzzy measure* or *monotone probability*, [17, 18].

Now, given the monotone set function space $(\Omega, \mathcal{F}, \mu)$, the Sugeno integral of X over $A \in \mathcal{F}$ w.r.t. μ can be represented as:

$$(S)\!\!\fint_A X d\mu = \bigvee_{a \geqslant 0} (a \wedge \mu(A \cap \{X \ge a\}) := \sup_{a \geqslant 0} \min\left(a, \mu\left(A \cap \{X \ge a\}\right)\right).$$

A binary operator $T\colon [0,1] \times [0,1] \to [0,1]$ is called a t-seminorm [16] or semicopula [4], if it satisfies the following conditions:

(A) $T(x,1) = T(1,x) = x, \quad \forall x \in [0,1]$;

(B) $\forall x_1, x_2, y_1,\, y_2$ in $[0,1]$, if $x_1 \le x_2,\, y_1 \le y_2$,
then $T(x_1, y_1) \le T(x_2, y_2)$.

Let T be a t-seminorm, then the seminormed fuzzy integral of X over A with respect to T and the fuzzy measure μ is defined as

$$\fint_{TA} X d\mu = \bigvee_{a \in [0,1]} T\left(a, \mu(A \cap \{X \ge a\}\right), \tag{1.1}$$

which was independently proposed by Zhao [21] and Suárez and Gil [16]. Observe that this integral was also shown in [8] to be the smallest universal integral based on T as a pseudo-multiplication on $[0,1]$. Notice that, if the seminorm T is the standard product, then the Shilkret integral [12, 15] is obtained. For a fixed strict t-norm T [7], the corresponding seminormed fuzzy integral is the so-called Sugeno-Weber integral [19]. Note that the original Sugeno integral which was introduced by Sugeno [17] in 1974 is a special seminormed fuzzy integral when the seminorm T is min.

In 2014, Hu [5] introduced a new class of fuzzy integral called Sugeno-like integral, which is based on the concept of g-seminorm. Notice that, for a fixed measure μ with $\mu(\Omega)$ from B, a function $G\colon B \times B \to B$ is called a generalized seminorm (g-seminorm for short), if it satisfies:

(i) $G(x, \mu(\Omega)) \leqslant x$ for each $x \in B$;

(ii) If $x_1 \leqslant x_3, x_2 \leqslant x_4$ for each $x_1, x_2, x_3, x_4 \in B$,
then $G(x_1, x_2) \leqslant G(x_3, x_4)$.

Let G be a g-seminorm. The G-Sugeno integral of X over $A \in \mathcal{F}$ w.r.t. μ can be represented as [5]:

$$(SG)\!\!\!\!\!-\!\!\!\!\!\int_A X d\mu = \bigvee_{a \in B} G\left(a, \mu(A \cap \{X \geq a\}\right).$$

Clearly, if $G = \wedge$, then the Sugeno integral is obtained. If $G = T$ with $B = [0,1]$, then we obtain the seminormed fuzzy integral. Throughout this paper, we always consider the existence of all G-Sugeno integral.

Román-Flores et al. [14] were the first who studied several well-known inequalities for the fuzzy integral. Recently several papers have appeared on the study of inequalities for the Sugeno integral, as [1, 2, 3, 10, 11, 20]. For example, in 2010, Wu et al. [20] proved a Hölder inequality for the original Sugeno integral based on comonotone functions and a binary operation $\star : [0,1] \times [0,1] \to [0,1]$ whenever $\star \geq \max$. This inequality had two major limitations: First, in general, this inequality does not work when $\star < \max$ (especially when $\star$ is a seminorm). Second, it is based on the comonotonicity condition. Recently, Agahi and Mesiar [1] have proposed a new version of a Cauchy Schwarz type inequality for Sugeno integral without the comonotonicity condition based on the multiplication operator.

In this paper, we first introduce the concept of G-Sugeno fractional integral. Then we give a general version of Minkowski type inequality for G-Sugeno fractional integral of the order $\alpha > 0$ based on the concept of g-seminorm and two binary operations $(\star, *)$. In this inequality, the binary operation $\star$ includes not only some semiconorms but also some seminorms in special cases. If $\star = * = +$, we can obtain Minkowski type inequality for G-Sugeno fractional integral (see Corollary 2.1). If $* = \star = \min$, then we will generalize some results of [3] (see Remark 2.4). Also, a Cauchy Schwarz type inequality for G-Sugeno fractional integral is obtained (see Corollary 2.6).

The presentation of the paper is as follows. In Section 2, we introduce the G-Sugeno fractional integral of order $\alpha > 0$, including as a particular member the Sugeno fractional integral. Next, a new version of Minkowski-type inequality for G-Sugeno fractional integral is introduced. Furthermore, we also present several related interesting inequalities for G-Sugeno fractional integrals. Finally, some conclusions are added.

2. **Main results**

In this section, we introduce the following G-Sugeno fractional integral.

Definition 2.1. For a fixed $t \in (0,\infty)$, let $([0,t],\mathcal{B}([0,t]),\mu)$ be a monotone measure space, where $\mathcal{B}([0,t])$ is the Borel σ-algebra on $[0,t]$, and let G be a g-seminorm. For a measurable function $X:[0,t]\to B$ and $\alpha>0$, denote by X_α a function on $[0,t]$ given by

$$X_\alpha(\omega)=\frac{1}{\Gamma(\alpha)}(t-\omega)^{\alpha-1}X(\omega),$$

here $\Gamma(\alpha)$ is the Gamma function. Then the G-Sugeno fractional integral of X w.r.t. μ is given by

$$\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu,X]=(SG)\int_{[0,t]}X_\alpha d\mu. \tag{2.1}$$

Note that when dealing with a concrete function X, we will often express formula (2.1) in the form

$$\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu,X]=(SG)\int_0^t\frac{1}{\Gamma(\alpha)}(t-\omega)^{\alpha-1}X(\omega)\,d\mu(\omega),\quad 0\leqslant\omega\leqslant t, \tag{2.2}$$

as we use such form with ω in the next text.

Remark 2.1. (I) If $G=\min$ in Definition 2.1, then we obtain the Sugeno fractional integral.

(II) If $t=1, G=T$ with $B=[0,1]$ in Definition 2.1, then we obtain the seminormed fuzzy fractional integral. Specially if $\alpha=1$, then we have the seminormed fuzzy integral (1.1).

(III) If G is the standard product on $[0,1]$, then we obtain the Shilkret fractional integral.

Now, we intend to give a version of Minkowski type inequality for G-Sugeno fractional integral of the order $\alpha>0$ based on two binary operations $(\star,*)$.

Theorem 2.1. *Let $t,s\in(0,\infty)$ and $c_1,c_2\in(0,1]$. Let two binary operations $\star,*\colon B\times B\to B$ be continuous and nondecreasing in both arguments and $X,Y:[0,t]\to B$ be two non-negative measurable functions such that*

$$X(\omega)\geqslant(X(\omega)*Y(\omega))\,c_1,\quad Y(\omega)\geqslant(X(\omega)*Y(\omega))\,c_2, \tag{2.3}$$

for any $0\leqslant\omega\leqslant t$. If the g-seminorm $G:B\times B\to B$ satisfies

$$G(a,b)\leqslant\min\left\{\frac{1}{c_1^s}G(c_1^s a,b),\frac{1}{c_2^s}G(c_2^s a,b)\right\} \tag{2.4}$$

for any a, b, then the inequality

$$\begin{aligned}&\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, X^s\right]\right)^{\frac{1}{s}} \star \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, Y^s\right]\right)^{\frac{1}{s}}\\ &\geqslant c_1 \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]\right)^{\frac{1}{s}} \star c_2 \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]\right)^{\frac{1}{s}} \end{aligned} \tag{2.5}$$

holds where the symbol $\mathbb{GSF}_{[0,t]}^{\alpha,G}$ is defined by (2.2).

P r o o f. Let $s \in (0, \infty)$ and $c_1, c_2 \in (0, 1]$. By (2.3), we have

$$(X(\omega))^s \geqslant c_1^s (X(\omega) * Y(\omega))^s. \tag{2.6}$$

Multiplying both sides of (2.6) by $\frac{(t-\omega)^{\alpha-1}}{\Gamma(\alpha)}, \omega \in (0, t), \alpha > 0$, we have

$$\frac{(t-\omega)^{\alpha-1}}{\Gamma(\alpha)} (X(\omega))^s \geqslant c_1^s \frac{(t-\omega)^{\alpha-1}}{\Gamma(\alpha)} (X(\omega) * Y(\omega))^s.$$

The monotonicity implies that

$$\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, X^s\right]\right)^{\frac{1}{s}} \geqslant \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, c_1^s (X * Y)^s\right]\right)^{\frac{1}{s}}. \tag{2.7}$$

By (2.4), we can see that

$$\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, c_1^s (X * Y)^s\right] \geqslant c_1^s \mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]. \tag{2.8}$$

Therefore, (2.7) and (2.8) imply that

$$\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, X^s\right]\right)^{\frac{1}{s}} \geqslant c_1 \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]\right)^{\frac{1}{s}}. \tag{2.9}$$

Also,

$$(Y(\omega))^s \geqslant c_2^s (X(\omega) * Y(\omega))^s, \tag{2.10}$$

and then, by multiplying both sides of (2.10) by $\frac{(t-\omega)^{\alpha-1}}{\Gamma(\alpha)}, \omega \in (0, t)$, we have

$$\begin{aligned}\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, Y^s\right]\right)^{\frac{1}{s}} &\geqslant \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, c_2^s (X * Y)^s\right]\right)^{\frac{1}{s}}\\ &\geqslant c_2 \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]\right)^{\frac{1}{s}}. \end{aligned} \tag{2.11}$$

It follows from (2.9) and (2.11) that

$$\begin{aligned}&\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, X^s\right]\right)^{\frac{1}{s}} \star \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, Y^s\right]\right)^{\frac{1}{s}}\\ &\geqslant c_1 \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]\right)^{\frac{1}{s}} \star c_2 \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X * Y)^s\right]\right)^{\frac{1}{s}}. \end{aligned}$$

This completes the proof. □

Let $\star = * = +$ in Theorem 2.1. Then the following result immediately follows from the previous theorem.

COROLLARY 2.1. *Let $t, s \in (0,\infty)$. Let $X, Y : [0,t] \to B$ be two non-negative measurable function such that*

$$0 < m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M$$

for any $0 \leqslant \omega \leqslant t$. If the g-seminorm $G : B \times B \to B$ satisfies

$$G(a,b) \leqslant \min\left\{ \left(\frac{m+1}{m}\right)^s G\left(\left(\frac{m}{m+1}\right)^s a, b\right), (M+1)^s G\left(\left(\frac{1}{M+1}\right)^s a, b\right)\right\}$$

for any a, b, then the inequality

$$\left(\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu, X^s]\right)^{\frac{1}{s}} + \left(\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu, Y^s]\right)^{\frac{1}{s}} \geqslant K\left(\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu, (X+Y)^s]\right)^{\frac{1}{s}} \tag{2.12}$$

holds, where $K = \frac{m(M+1)+(m+1)}{(m+1)(M+1)}$ and $\mathbb{GSF}^{\alpha,G}_{[0,t]}$ is defined by (2.2). In particular, if X, Y are proportional, then

$$\left(\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu, X^s]\right)^{\frac{1}{s}} + \left(\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu, Y^s]\right)^{\frac{1}{s}} \geqslant \left(\mathbb{GSF}^{\alpha,G}_{[0,t]}[\mu, (X+Y)^s]\right)^{\frac{1}{s}}. \tag{2.13}$$

P r o o f. Put $\star = * = +, c_1 = \frac{m}{m+1}$ and $c_2 = \frac{1}{M+1}$ in Theorem 2.1. Since $m \leqslant \frac{X(\omega)}{Y(\omega)}$, we have $X(\omega) \geqslant \frac{m}{m+1}(X(\omega) + Y(\omega))$. Also, if $M \geqslant \frac{X(\omega)}{Y(\omega)}$, we have $Y(\omega) \geqslant \frac{X(\omega)+Y(\omega)}{M+1}$. Then (2.3) holds readily. If X, Y are proportional, i.e., $X(\omega) = \beta Y(\omega), \beta > 0$, then $m = M = \beta$, $K = 1$ and (2.12) reduces on (2.13). This completes the proof. □

EXAMPLE 2.1. Let $X(\omega) = 4 - \omega$ and $Y(\omega) = 2 - \frac{\omega}{2}, \omega \in [0,4]$. Let $t = 4, s = 2, \alpha = 2$ and $\mu(A) = \lambda(A)$ where λ is the Lebesgue measure on $\mathbb{R}$. So,

$$\mathbb{GSF}^{2,\min}_{[0,4]}\left[\lambda, (X+Y)^2\right] = (S)\int_0^4 \frac{1}{\Gamma(2)}(4-\omega)\left(6 - \frac{3}{2}\omega\right)^2 d\lambda = 2.9104,$$

$$\mathbb{GSF}^{2,\min}_{[0,4]}\left[\lambda, X^2\right] = (S)\int_0^4 \frac{1}{\Gamma(2)}(4-\omega)^3 d\lambda = 2.6212,$$

$$\mathbb{GSF}^{2,\min}_{[0,4]}\left[\lambda, Y^2\right] = (S)\int_0^4 \frac{1}{4\Gamma(2)}(4-\omega)^3 d\lambda = 2.$$

Therefore,

$$\left(\mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, X^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, Y^2\right]\right)^{\frac{1}{2}} = 3.0332$$
$$\geqslant 1.7060 = \left(\mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, (X+Y)^2\right]\right)^{\frac{1}{2}}.$$

Remark 2.2. We can easily see that the inequality (2.12) in Corollary 2.1 (thus the inequality (2.5) in Theorem 2.1) is sharp (for example, when $X(\omega) = Y(\omega) \equiv 1,\ \omega \in [0,4], G = \min, \star = * = +, t = 4, s = 2, \alpha = 1, \mu(A) = \lambda(A)$ where λ is the Lebesgue measure on $\mathbb{R}$, then (2.12) becomes an equality).

Remark 2.3. As some special cases of $\star$ or $*$ in Theorem 2.1, we can find some interesting results. For example, we have the following results which works with a binary operation $*$ whenever $* \leqslant \min$ (Notice that in special case, if $B = [0,1]$ and $\mu(\Omega) = 1$, it works when $*$ is a seminorm).

Corollary 2.2. *Let $t, s \in (0,\infty)$ and $c_1, c_2 \in (0,1]$. Let two binary operations $\star, *\colon B \times B \to B$ be continuous and nondecreasing in both arguments and $X, Y : [0,t] \to B$ be two non-negative measurable functions such that*

$$\min\left(X(\omega), Y(\omega)\right) \geqslant \left(X(\omega) * Y(\omega)\right)\max\left(c_1, c_2\right), \qquad 0 \leqslant \omega \leqslant t.$$

If the g-seminorm $G : B \times B \to B$ satisfies

$$G(a,b) \leqslant \min\left\{\frac{1}{c_1^s} G\left(c_1^s a, b\right), \frac{1}{c_2^s} G\left(c_2^s a, b\right)\right\} \tag{2.14}$$

for any a, b, then the inequality

$$\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, X^s\right]\right)^{\frac{1}{s}} \star \left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, Y^s\right]\right)^{\frac{1}{s}}$$
$$\geqslant c_1\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X*Y)^s\right]\right)^{\frac{1}{s}} \star c_2\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[\mu, (X*Y)^s\right]\right)^{\frac{1}{s}} \tag{2.15}$$

holds where the symbol $\mathbb{GSF}_{[0,t]}^{\alpha,G}$ is defined by (2.2).

Remark 2.4. (I) Let $B = [0,1], s \in [1,\infty),\ t = \alpha = 1, G(x,y) = *(x,y) = \star(x,y) = \min(x,y)$ for all $x, y \in [0,1]$ in Corollary 2.2. Then we have a Cauchy Schwarz type inequality which generalizes Theorem 2 in [3].

(II) Let $B = [0,1],\ t = \alpha = 1, G(x,y) = \min(x,y), \star(x,y) = *(x,y) = \frac{xy}{(x+y)}$ in Theorem 2.1. Then we have a version of Milne's integral inequality for the Sugeno integral.

(III) Clearly, the condition (2.14) works for $c_1 = c_2 = 1$. If $c_1, c_2 < 1$, for $G = \min$ (i.e., for Sugeno integral), the condition (2.14) is satisfied. In some special cases, for example, $G(x,y) = T_p(x,y) = xy, B = [0,1]$ (i.e., for Shilkret integral), the condition is also satisfied. But if $c_1, c_2 < 1, B = [0,1], G(x,y) = x^2 y$, we can see that condition (2.14) is not established. Furthermore, it can be easily shown that this condition in Corollary 2.2 (thus in Theorem 2.1) is essential.

Here, we would like to prove the general cases of Theorem 2.1.

Theorem 2.2. *Let $t, p, q, s \in (0,\infty)$ and $c_1, c_2 \in (0,1]$. Let two binary operations $\star, *\colon B \times B \to B$ be continuous and nondecreasing in both arguments and $X, Y, Z_1, Z_2 : [0,t] \to B$ be non-negative measurable functions such that*

$$X^p(\omega) \geqslant (X(\omega) * Y(\omega))\, c_1, \quad Y^q(\omega) \geqslant (X(\omega) * Y(\omega))\, c_2, \tag{2.16}$$

for any $0 \leqslant \omega \leqslant t$. If the g-seminorm G satisfies

$$G(a,b) \leqslant \min\left\{\frac{1}{c_1^s} G(c_1^s a, b), \frac{1}{c_2^s} G(c_2^s a, b)\right\} \tag{2.17}$$

for any a, b, then the inequality

$$\left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, X^{ps}]\right)^{\frac{1}{ps}} \star \left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, Y^{qs}]\right)^{\frac{1}{qs}}$$
$$\geqslant c_1^{\frac{1}{p}} \left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, (X*Y)^s]\right)^{\frac{1}{ps}} \star c_2^{\frac{1}{q}} \left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, (X*Y)^s]\right)^{\frac{1}{qs}}$$

holds, where the symbol $\mathbb{GSC}_{[0,t]}^{Z_i,\mu}, i = 1,2$ is defined by

$$\mathbb{GSC}_{[0,t]}^{Z_i,G}[\mu, X] = (SG)\int_0^t Z_i(t-\omega)\, X(\omega)\, d\mu(\omega), i = 1,2\,. \tag{2.18}$$

Proof. Let $p, q, s \in (0,\infty)$ and $c_1, c_2 \in (0,1]$. By (2.16), we have

$$(X(\omega))^{ps} \geqslant c_1^s (X(\omega) * Y(\omega))^s\,. \tag{2.19}$$

Multiplying both sides of (2.19) by $Z_1(t-\omega), \omega \in (0,t)$. Then

$$Z_1(t-\omega)(X(\omega))^{ps} \geqslant c_1^s Z_1(t-\omega)(X(\omega) * Y(\omega))^s\,. \tag{2.20}$$

The monotonicity, (2.17) and (2.20) imply that

$$\left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, X^{ps}]\right)^{\frac{1}{ps}} \geqslant \left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, c_1^s (X*Y)^s]\right)^{\frac{1}{ps}}$$
$$\geqslant c_1^{\frac{1}{p}} \left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, (X*Y)^s]\right)^{\frac{1}{ps}}\,. \tag{2.21}$$

Also,

$$(Y(\omega))^{qs} \geqslant c_2^s (X(\omega) * Y(\omega))^s, \tag{2.22}$$

and then, by multiplying both sides of (2.22) by $Z_2(t-\omega), \omega \in (0,t)$, we have

$$\begin{aligned}\left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, Y^{qs}]\right)^{\frac{1}{qs}} &\geqslant \left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, c_2^s (X*Y)^s]\right)^{\frac{1}{qs}} \\ &\geqslant c_2^{\frac{1}{q}} \left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, (X*Y)^s]\right)^{\frac{1}{qs}}. \end{aligned} \tag{2.23}$$

By (2.21) and (2.23), the following result holds

$$\begin{aligned}&\left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, X^{ps}]\right)^{\frac{1}{ps}} \star \left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, Y^{qs}]\right)^{\frac{1}{qs}} \\ &\geqslant c_1^{\frac{1}{p}} \left(\mathbb{GSC}_{[0,t]}^{Z_1,G}[\mu, (X*Y)^s]\right)^{\frac{1}{ps}} \star c_2^{\frac{1}{q}} \left(\mathbb{GSC}_{[0,t]}^{Z_2,G}[\mu, (X*Y)^s]\right)^{\frac{1}{qs}}.\end{aligned}$$

This completes the proof. □

Let $Z_1 = Z_2 = Z$ in Theorem 2.2. Then we obtain the following result.

Corollary 2.3. *Let $t, p, q, s \in (0,\infty)$ and $c_1, c_2 \in (0,1]$. Let two binary operations $\star, *\colon B \times B \to B$ be continuous and nondecreasing in both arguments and $X, Y, Z : [0,t] \to B$ be non-negative measurable functions such that*

$$X^p(\omega) \geqslant (X(\omega) * Y(\omega))\, c_1, \quad Y^q(\omega) \geqslant (X(\omega) * Y(\omega))\, c_2, \tag{2.24}$$

for any $0 \leqslant \omega \leqslant t$. If the g-seminorm G satisfies

$$G(a,b) \leqslant \min\left\{\frac{1}{c_1^s} G(c_1^s a, b), \frac{1}{c_2^s} G(c_2^s a, b)\right\}$$

for any a, b, then the inequality

$$\begin{aligned}&\left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, X^{ps}]\right)^{\frac{1}{ps}} \star \left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, Y^{qs}]\right)^{\frac{1}{qs}} \\ &\geqslant c_1^{\frac{1}{p}} \left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, (X*Y)^s]\right)^{\frac{1}{ps}} \star c_2^{\frac{1}{q}} \left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, (X*Y)^s]\right)^{\frac{1}{qs}}\end{aligned}$$

holds where the symbol $\mathbb{GSC}_{[0,t]}^{Z,\mu}$ is defined by (2.18).

Corollary 2.4. *Let $t, s \in (0,\infty)$. Let $X, Y, Z : [0,t] \to B$ be non-negative measurable functions such that*

$$0 < m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M$$

for any $0 \leqslant \omega \leqslant t$. *If the g-seminorm* G *satisfies*

$$G(a,b) \leqslant \min\left\{\left(\frac{m+1}{m}\right)^s G\left(\left(\frac{m}{m+1}\right)^s a, b\right), (M+1)^s G\left(\left(\frac{1}{M+1}\right)^s a, b\right)\right\}$$

for any a, b, *then the inequality*

$$\left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, X^s]\right)^{\frac{1}{s}} + \left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, Y^s]\right)^{\frac{1}{s}} \geqslant K\left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, (X+Y)^s]\right)^{\frac{1}{s}}$$

holds, where $K = \frac{m(M+1)+(m+1)}{(m+1)(M+1)}$ *and* $\mathbb{GSC}_{[0,t]}^{Z,G}$ *is defined by (2.18) for all* $\alpha > 0$. *In particular, if* X, Y *are proportional, then*

$$\left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, X^s]\right)^{\frac{1}{s}} + \left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, Y^s]\right)^{\frac{1}{s}} \geqslant \left(\mathbb{GSC}_{[0,t]}^{Z,G}[\mu, (X+Y)^s]\right)^{\frac{1}{s}}.$$

COROLLARY 2.5. *Let* $t \in (0, \infty)$. *Let* $X, Y, Z : [0,t] \to B$ *be non-negative measurable functions such that*

$$0 < m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M \quad \forall \omega \in [0,t].$$

If the g-seminorm G *satisfies*

$$G(a,b) \leqslant \min\left\{\frac{1}{m} G(ma, b), MG\left(\frac{a}{M}, b\right)\right\}$$

for any a, b, *such that* $\max\left\{\frac{1}{M}, m\right\} \leqslant 1$, *then the inequality*

$$\left(\mathbb{GSC}_{[0,t]}^{Z,G}\left[X^2\right]\right)^{\frac{1}{2}} \left(\mathbb{GSC}_{[0,t]}^{Z,G}\left[Y^2\right]\right)^{\frac{1}{2}} \geqslant K\left(\mathbb{GSC}_{[0,t]}^{Z,G}[XY]\right)$$

holds, where $K = \sqrt{\frac{m}{M}}$ *and the symbol* $\mathbb{GSC}_{[0,t]}^{Z,\mu}$ *is defined by (2.18).*

P r o o f. Put $s = 1, p = q = 2, \star = * = \times, c_1 = m$ and $c_2 = \frac{1}{M}$ where $\max\left\{\frac{1}{M}, m\right\} \leqslant 1$ in Corollary 2.3. Since $m \leqslant \frac{X(\omega)}{Y(\omega)}$, we have $X^2(\omega) \geqslant m(X(\omega)Y(\omega))$. Also, if $M \geqslant \frac{X(\omega)}{Y(\omega)}$, we have $Y^2(\omega) \geqslant \frac{1}{M}X(\omega)Y(\omega)$. Then (2.24) holds readily. This completes the proof. □

If we take $Z = (\Gamma(\alpha))^{-1}(t-\omega)^{\alpha-1}, \alpha > 0$ in Corollary 2.5, then we have the following result (notice that for $\alpha = 1$, the results of [1] is obtained).

COROLLARY 2.6. *Let* $t \in (0, \infty)$. *Let* $X, Y, Z : [0,t] \to B$ *be non-negative measurable functions such that*

$$0 < m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M, \quad \forall \omega \in [0,t].$$

If the g-seminorm G *satisfies*

$$G(a,b) \leqslant \min\left\{\frac{1}{m} G(ma, b), MG\left(\frac{a}{M}, b\right)\right\} \tag{2.25}$$

for any a, b, *such that* $\max\left\{\frac{1}{M}, m\right\} \leqslant 1$, *then the inequality*

$$\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[X^2\right]\right)^{\frac{1}{2}}\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[Y^2\right]\right)^{\frac{1}{2}} \geqslant K\left(\mathbb{GSF}_{[0,t]}^{\alpha,G}\left[XY\right]\right)$$

holds, where $K = \sqrt{\frac{m}{M}}$ *and the symbol* $\mathbb{GSF}_{[0,t]}^{\alpha,G}$ *is defined by (2.2).*

Remark 2.5. Clearly, min on any B, usual product on $[0,1]$, etc., satisfy the constraint (2.25) of this corollary.

The following example easily shows that the condition $\max\left\{\frac{1}{M}, m\right\} \leqslant 1$ in Corollary 2.6 is essential.

Example 2.2. Using Example 2.1, then

$$\mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, XY\right] = (S)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\int_0^4 \frac{1}{\Gamma(2)}(4-\omega)^2\left(2-\frac{\omega}{2}\right)d\lambda = 2.3298.$$

Therefore,

$$\begin{aligned}\left(\mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, X^2\right]\right)^{\frac{1}{2}}\left(\mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, Y^2\right]\right)^{\frac{1}{2}} &= 2.2896 \\ &\ngeqslant 2.3298 = \mathbb{GSF}_{[0,4]}^{2,\min}\left[\lambda, XY\right].\end{aligned}$$

3. Conclusion

We have introduced the G-Sugeno fractional integral, extending the Sugeno fractional integral and the seminormed fuzzy fractional integral. We have proved a general version of Minkowski type inequality for G-Sugeno fractional integral of the order $\alpha > 0$ based on two binary operations $(\star, *)$. Also, a Cauchy Schwarz type inequality for G-Sugeno fractional integral is obtained. For further investigation, it would be a challenging problem to determine the conditions under which (2.5) becomes an equality.

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References

[1] H. Agahi, R. Mesiar, On Cauchy-Schwarz's inequality for Choquet-like integrals without the comonotonicity condition. *Soft Computing* **19** (2015), 1627–1634.

[2] H. Agahi, R. Mesiar, Y. Ouyang, New general extensions of Chebyshev type inequalities for Sugeno integrals. *International Journal of Approximate Reasoning* **51** (2009), 135–140.

[3] J. Caballero, K. Sadarangani, A Cauchy Schwarz type inequality for fuzzy integrals. *Nonlinear Analysis* **73** (2010), 3329–3335.

[4] F. Durante, C. Sempi, Semicopulae. *Kybernetika* **41** (2005), 315–328.

[5] Y. Hu, Chebyshev type inequalities for general fuzzy integrals. *Information Sciences* **278** (2014), 822–825.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Elsevier, New York (2006).

[7] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms, Trends in Logic*. Studia Logica Library, Vol. 8, Kluwer Academic Publishers, Dodrecht (2000).

[8] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral. *IEEE Trans. Fuzzy Systems* **18** (2010), 178–187.

[9] R. Mesiar, Choquet-like integrals. *Journal of Mathematical Analysis and Applications* **194** (1995), 477–488.

[10] R. Mesiar, Y. Ouyang, General Chebyshev type inequalities for Sugeno integrals. *Fuzzy Sets and Systems* **160** (2009), 58–64.

[11] Y. Ouyang, R. Mesiar, H. Agahi, An inequality related to Minkowski type for Sugeno integrals. *Information Sciences* **180** (2010), 2793–2801.

[12] E. Pap, *Null-Additive Set Functions*. Kluwer, Dordrecht (1995).

[13] D. Ralescu, G. Adams, The fuzzy integral. *Journal of Mathematical Analysis and Applications* **75** (1980), 562–570.

[14] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, A Jensen type inequality for fuzzy integrals. *Information Sciences* **177** (2007), 3192–3201.

[15] N. Shilkret, Maxitive measure and integration. *Indagationes Mathematicae* **8** (1971), 109–116.

[16] F. Suárez García, P. Gil Álvarez, Two families of fuzzy integrals. *Fuzzy Sets and Systems* **18** (1986), 67–81.

[17] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, Ph.D. Dissertation, Tokyo Institute of Technology (1974).

[18] Z. Wang, G. Klir, *Generalized Measure Theory*. Springer Verlag, New York (2008).

[19] S. Weber, Two integrals and some modified versions: critical remarks. *Fuzzy Sets and Systems* **20** (1986), 97–105.

[20] L. Wu, J. Sun, X. Ye, L. Zhu, Hölder type inequality for Sugeno integrals. *Fuzzy Sets and Systems* **161** (2010), 2337–2347.

[21] R. Zhao, (N) fuzzy integral. *J. Mathematical Research and Exposition* **2** (1981), 55–72.

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