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On the Expected Value of Fuzzy Events

Erich Peter Klement
Department of Knowledge-Based Mathematical Systems,
Johannes Kepler University, Altenberger Straße 69, 4040 Linz, Austria
and JKU Softwarepark Hagenberg, Hauptstraße 90, 4232 Hagenberg, Austria
ep.klement@jku.at

Radko Mesiar
Department of Mathematics and Descriptive Geometry,
Faculty of Civil Engineering, Slovak University of Technology,
Radlinského 11, 810 05 Bratislava, Slovakia
and Institute of Information Theory and Automation, Czech Academy of Sciences,
Pod Vodárenskou věží 4, 182 08 Praha 8, Czech Republic
radko.mesiar@stuba.sk

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Generalizing a first approach by L. A. Zadeh (*J. Math. Anal. Appl.* 23, 1968), expected values of fuzzy events are studied which are (up to standard boundary conditions) only required to be monotone. They can be seen as an extension of capacities, i.e., monotone set functions satisfying standard boundary conditions. Some of these expected values can be characterized axiomatically, others are based on some distinguished integrals (Choquet, Sugeno, Shilkret, universal, and decomposition integral).

Keywords: Expected value; fuzzy event; capacity; Choquet integral; Sugeno integral; universal integral.

1. Introduction

The problem of determining the "size" of a fuzzy event is a challenging problem, going back to L. A. Zadeh,[60] who introduced and studied *probability measures of fuzzy events*. To be precise, he considered a fixed probability measure and proposed to apply the standard probabilistic expectation (i.e., the Lebesgue(-Stieltjes) integral with respect to the probability measure) to the membership function of the fuzzy event.

In the literature various more general types of measures (which are monotone non-decreasing, but not necessarily σ-additive or finitely additive) have been studied.[2,7–10,14,16,19,29,30,32–34,36–38,40,44–47,50–53,55–57] In the most general case, these measures are (up to some boundary conditions) only monotone non-decreasing with respect to set inclusion. In this paper, we shall consistently use the term *capacity* for a monotone non-decreasing set function (defined on a σ-algebra of crisp (or Cantorian) subsets of the universe of discourse) which also satisfies the standard boundary conditions. In this way, we shall avoid any confusion with similar notions like *monotone set function*, *monotone measure*, *non-additive measure* or *fuzzy measure* (even if these notions were used in the original sources).

For capacities with additional properties we shall either use their standard names (such as probability measure) or we will explicitly mention the additional properties: the Lebesgue measure on the unit interval and, more generally, each probability measure are examples of a σ-additive capacity.

The aim of this paper is to discuss several approaches to the expectation of fuzzy events. Considering capacities as *normed quantifications* of crisp sets, the expected values of fuzzy events presented in this paper generalize the ideas of Ref. 60.

Some of the expected values of fuzzy events can be defined axiomatically (see Sec. 3), and then the properties of the corresponding capacities should reflect these axioms. In Sec. 4 expected values of fuzzy sets based on some distinguished integrals and arbitrary capacities are discussed, including axiomatic approaches to these expectations. Section 5 is devoted to expectations with respect to universal integral, and in Sec. 6 we discuss expectations of fuzzy events based on decomposition integrals.

2. Preliminaries

Our basic setting is a *measurable space*,[1,17] i.e., a pair $(X, \mathcal{A})$ consisting of a non-empty set X (the *universe of discourse*) and a σ-algebra $\mathcal{A}$ of subsets of X. Recall that a *σ-algebra*[1] is a collection of subsets of X which contains the empty set $\emptyset$ and the universe X and which is closed under complementation and countable unions.

Definition 1. If $(X, \mathcal{A})$ is a measurable space, then a *capacity* on $(X, \mathcal{A})$ is a function $m\colon \mathcal{A} \to [0,1]$ which satisfies

(i) $m(\emptyset) = 0$ and $m(X) = 1$, (*boundary conditions*)
(ii) $m(A) \le m(B)$ whenever $A \subseteq B$. (*monotone non-decreasing*)

Note again that a capacity as given in Definition 1, is neither required to be (finitely or σ-)additive nor to be continuous in any sense (recall that a set function $m\colon \mathcal{A} \to [0,1]$ is said to be (*finitely*) *additive* whenever $m(A \cup B) = m(A) + m(B)$ for all disjoint subsets A, B of X, and *σ-additive* whenever

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

for each sequence $(A_1, A_2, \dots)$ of pairwise disjoint sunsets of X).

We shall write $\mathcal{D}$ for the class of all measurable spaces. Given a specific measurable space $(X, \mathcal{A})$, the set of all capacities $m\colon \mathcal{A} \to [0,1]$ will be denoted by $\mathcal{M}(X,\mathcal{A})$, and the set of all measurable functions $f\colon X \to [0,1]$ by $\mathcal{F}(X,\mathcal{A})$. Recall that a function $f\colon X \to [0,1]$ is *measurable* (with respect to the σ-algebra $\mathcal{A}$ on X and the σ-algebra $\mathcal{B}([0,1])$ of Borel subsets of $[0,1]$) if, for each $\alpha \in [0,1]$, we have $\{x \in X \mid f(x) \geq \alpha\} \in \mathcal{A}$.

To avoid any confusion, we will denote *fuzzy subsets*[42,59] of X by capital letters, and the membership function of the fuzzy set A by $\mu_A\colon X \to [0,1]$. A fuzzy set A is said to be a *subset* of a fuzzy set B (in symbols $A \subseteq B$) if and only if we have $\mu_A \leq \mu_B$, i.e., if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

In the case of a crisp set A we obviously have $\mu_A = \mathbf{1}_A$, i.e., the membership function coincides with the *characteristic function of* A.

If A is a fuzzy subset of X and $\alpha \in [0,1]$, then the *α-cut* (or *α-level set*) of A is the (crisp) set A^α defined by

$$A^\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}\,.$$

Note that we have $A^0 = X$ and $\mu_A(x) = \sup\{\min(\alpha, \mathbf{1}_{A^\alpha}(x)) \mid \alpha \in [0,1]\}$ for each $x \in X$.

Given a measurable space $(X,\mathcal{A})$, a fuzzy subset A of X with $\mu_A \in \mathcal{F}(X,\mathcal{A})$, i.e., with a measurable membership function (which is equivalent to $A^\alpha \in \mathcal{A}$ for each $\alpha \in [0,1]$), will be called a *fuzzy event.* The set of all fuzzy events will be denoted by $\widehat{\mathcal{A}}$, and it was called a (generated) *fuzzy σ-algebra* or *fuzzy tribe* on X in Refs. 4–6, 21 and 23.

The classical *expected value of a random variable* $Y: X \to \mathbb{R}$ is given by

$$\mathbf{E}_p(Y) = \int_X Y\, dp\,,$$

where $p\colon \mathcal{A} \to [0,1]$ is a probability measure on the measurable space $(X,\mathcal{A})$ and the integral is a Lebesgue(-Stieltjes) integral. For a fuzzy event $A \in \widehat{\mathcal{A}}$ the value $\mathbf{E}_p(\mu_A)$ was called the *probability of the fuzzy event* A in Ref. 60.

In the following text we shall briefly write $\mathbf{E}_p(A)$ rather than $\mathbf{E}_p(\mu_A)$, and we shall generalize this notion as follows:

Definition 2. Let $(X,\mathcal{A})$ be a measurable space. An *expected value of fuzzy events* is a function $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ which satisfies

(i) $\mathbf{E}(\emptyset) = 0$ and $\mathbf{E}(X) = 1$, (*boundary conditions*)
(ii) $\mathbf{E}(A) \leq \mathbf{E}(B)$ whenever $A \subseteq B$. (*monotone non-decreasing*)

Hence, each expected value of fuzzy events $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ as given in Definition 2 can be seen as an extension of some capacity $m\colon \mathcal{A} \to [0,1]$ given by $m(A) = \mathbf{E}(A)$. In such a case, the notation $\mathbf{E}_m$ will often be used to stress the link between the capacity m and the expectation of fuzzy events.

Given a capacity $m \in \mathcal{M}(X,\mathcal{A})$, the monotonicity (ii) in Definition 2 implies that the smallest expectation of fuzzy events $(\mathbf{E}_m)_*$ with respect to m is given by $(\mathbf{E}_m)_*(A) = m(A^1)$, while the greatest expectation $(\mathbf{E}_m)^*$ of fuzzy events with respect to m is given by $(\mathbf{E}_m)^*(A) = m(\{x \in X \mid \mu_A(x) > 0\})$ (the sets A^1 and $\{x \in X \mid \mu_A(x) > 0\}$ are often called the *kernel* and the *support* of the fuzzy event A, abbreviated by $\ker(A)$ and $\operatorname{supp}(A)$, respectively).

3. Axiomatic Approaches to the Expected Value of a Fuzzy Event

A typical property of a probability measure p is the valuation property, i.e., we always have $p(A \cap B) + p(A \cup B) = p(A) + p(B)$. It is desirable to have some counterpart of this property also for fuzzy events and their expected values.

Note that the intersection and the union of fuzzy sets are usually defined by means of a pair consisting of a t-norm T and a its dual t-conorm S (see Refs. 26, 42 and 59), i.e., we have $S(x, y) = 1 - T(1 - x, 1 - y)$. Then $A \cap_T B$ denotes the T-based intersection of the fuzzy events A and B, and $A \cup_S B$ their S-based union.

Some basic t-norms are the *Gödel t-norm* $T_{\mathbf{M}} \colon [0,1]^2 \to [0,1]$, the *product t-norm* $T_{\mathbf{P}} \colon [0,1]^2 \to [0,1]$, and the *Łukasiewicz t-norm* $T_{\mathbf{L}} \colon [0,1]^2 \to [0,1]$ given by, respectively,

$$T_{\mathbf{M}}(x, y) = \min(x, y)\,, \tag{3.1}$$

$$T_{\mathbf{P}}(x, y) = \Pi(x, y) = x \cdot y\,, \tag{3.2}$$

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)\,. \tag{3.3}$$

An important family of t-norms is the family of *Frank t-norms*[13,26] $(T^{\mathbf{F}}_s)_{s \in [0,\infty]}$ given by

$$T^{\mathbf{F}}_s(x, y) = \begin{cases} T_{\mathbf{M}}(x, y) & \text{if } s = 0\,, \\ T_{\mathbf{P}}(x, y) & \text{if } s = 1\,, \\ T_{\mathbf{L}}(x, y) & \text{if } s = \infty, \\ \log_s\left(1 + \dfrac{(s^x - 1)(s^y - 1)}{s - 1}\right) & \text{otherwise.} \end{cases} \tag{3.4}$$

The Frank t-norms (and the *ordinal sums*[13,26] of Frank t-norms) together with their dual t-conorms are the only pairs (T, S) satisfying the functional equation

$$T(x, y) + S(x, y) = x + y$$

and, subsequently, for all fuzzy subsets A, B of X,

$$\mu_{A \cap_T B} + \mu_{A \cup_S B} = \mu_A + \mu_B\,,$$

thus generalizing the well-known valuation property for characteristic functions of crisp subsets U, V of X

$$\mathbf{1}_{U \cap V} + \mathbf{1}_{U \cup V} = \mathbf{1}_U + \mathbf{1}_V\,.$$

Definition 3. Let T be a t-norm and S the dual t-conorm. An expectation $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ of fuzzy events is said to be a *T-valuation* if for all fuzzy fuzzy events $A, B \in \widehat{\mathcal{A}}$ we have

$$\mathbf{E}(A \cap_T B) + \mathbf{E}(A \cup_S B) = \mathbf{E}(A) + \mathbf{E}(B)\,.$$

It is not difficult to see (compare Theorem 3.1 in Ref. 22) that, for a measurable t-norm T, each expectation of fuzzy events which is also a T-valuation as given in Definition 3, necessarily is a $T_{\mathbf{M}}$-valuation.

The following result for valuations with respect to the Łukasiewicz t-norm (3.3) provides an axiomatization of the concept of L.A. Zadeh,[60] and it is a consequence of Refs. 20, 21 and 25 and, in particular, of Theorem 2.6(c) in Ref. 3 (compare also Refs. 4, 5 and 6).

Theorem 1. *An expectation $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ of fuzzy events is a $T_{\mathbf{L}}$-valuation if and only if for each $A \in \widehat{\mathcal{A}}$ we have*

$$\mathbf{E}(A) = \int_X \mu_A \, dp_{\mathbf{E}}\,, \tag{3.5}$$

where the probability measure $p_{\mathbf{E}}\colon \mathcal{A} \to [0,1]$ is given by $p_{\mathbf{E}}(U) = \mathbf{E}(U)$.

In other words, Theorem 1 shows that the probability measures of fuzzy events studied in Ref. 60 are, in fact, expectations of fuzzy events which are $T_{\mathbf{L}}$-valuations.

For valuations with respect to a Frank t-norm $T_s^{\mathbf{F}}$ (3.4) with $s \in \,]0,\infty[$ we have the following result based on Theorem 4.1 in Ref. 22 (compare also Refs. 4–6 and 23).

Theorem 2. *If $s \in \,]0,\infty[$, then an expectation $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ of fuzzy events is a $T_s^{\mathbf{F}}$-valuation if and only if there exists a function $f \in \mathcal{F}(X,\mathcal{A})$ such that for each $A \in \widehat{\mathcal{A}}$ we have*

$$\mathbf{E}(A) = \int_{\{x \in X \mid \mu_A(x) > 0\}} [f + (1-f)\cdot \mu_A]\, dp_{\mathbf{E}}\,, \tag{3.6}$$

where the probability measure $p_{\mathbf{E}}\colon \mathcal{A} \to [0,1]$ is given by $p_{\mathbf{E}}(U) = \mathbf{E}(U)$

If $\mathbf{E}_1, \mathbf{E}_2\colon \widehat{\mathcal{A}} \to [0,1]$ are two $T_s^{\mathbf{F}}$-valuations with

$$\mathbf{E}_1(A) = \int_{\{x \in X \mid \mu_A(x) > 0\}} [f_1 + (1-f_1)\cdot \mu_A]\, dp_{\mathbf{E}_1}\,,$$
$$\mathbf{E}_2(A) = \int_{\{x \in X \mid \mu_A(x) > 0\}} [f_2 + (1-f_2)\cdot \mu_A]\, dp_{\mathbf{E}_2}\,,$$

respectively, then the equality $\mathbf{E}_1 = \mathbf{E}_2$ holds if and only if we have $p_{\mathbf{E}_1} = p_{\mathbf{E}_2}$ and $p_{\mathbf{E}_1}(\{x \in X \mid f_1(x) \neq f_2(x)\}) = 0$.

The following characterization for valuations with respect to the Gödel t-norm $T_{\mathbf{M}}$ (3.1) is the most general in this group of results. It goes back to the main theorem in Ref. 20 (compare also Refs. 4, 5 and 6). Its proof relies on the *Radon-Nikodým Theorem*[1] and involves the concept of a Markov kernel.[1]

In our context, an $\mathcal{A}$-*Markov kernel* is a two-place function $K\colon X \times \mathcal{B}([0,1[) \to [0,1]$ such that its first component $K(\boldsymbol{.}, U)\colon X \to [0,1]$ is $\mathcal{A}$-measurable for each $U \in \mathcal{B}([0,1[)$, and its second component $K(x, \boldsymbol{.})\colon \mathcal{B}([0,1[) \to [0,1]$ is a probability measure for each $x \in X$.

Theorem 3. *An expectation* $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ *of fuzzy events is a* $T_{\mathbf{M}}$*-valuation if and only if there exists an* $\mathcal{A}$*-Markov kernel* $K\colon X \times \mathcal{B}([0,1[) \to [0,1]$ *such that for each* $A \in \widehat{\mathcal{A}}$*, we have*

$$\mathbf{E}(A) = \int_X K(x, [0, \mu_A(x)[)\, dp_{\mathbf{E}}(x)\,, \tag{3.7}$$

where the probability measure $p_{\mathbf{E}}\colon \mathcal{A} \to [0,1]$ *is given by* $p_{\mathbf{E}}(U) = \mathbf{E}(U)$.

If $\mathbf{E}_1, \mathbf{E}_2\colon \widehat{\mathcal{A}} \to [0,1]$ are two $T_{\mathbf{M}}$-valuations with

$$\mathbf{E}_1(A) = \int_X K_1(x, [0, \mu_A(x)[)\, dp_{\mathbf{E}_1}(x)\,,$$

$$\mathbf{E}_2(A) = \int_X K_2(x, [0, \mu_A(x)[)\, dp_{\mathbf{E}_2}(x)\,,$$

respectively, then the equality $\mathbf{E}_1 = \mathbf{E}_2$ holds if and only if we have $p_{\mathbf{E}_1} = p_{\mathbf{E}_2}$ and $p_{\mathbf{E}_1}(\{x \in X \mid K_1(x,U) \neq K_2(x,U)\}) = 0$ for each $U \in \mathcal{B}([0,1[)$.

4. Expected Value of Fuzzy Events Based on Some Special Integrals

Given an integral with respect to some capacity m, the expected value of a fuzzy event A can be defined in a straightforward way as the integral of the membership function μ_A with respect to m. This was the original idea in Ref. 60 for the case of a probability measure and the Lebesgue(-Stieltjes) integral, i.e., the classical expected value of a random variable, in this case of the membership function of a fuzzy event A.

The crucial property of an integral to allow the definition of an expected value, up to the monotonicity with respect to the integrand, is that it reproduces the capacity, i.e., yields $m(A)$, when applied to the characteristic function of a crisp subset A of X. There are several integrals having this property, sometimes, however, only for a restricted class of capacities.

This is the case for the Lebesgue(-Stieltjes) integral (with respect to a probability measure) and for *pseudo-additive integrals*[31,43,54,58] (with respect to pseudo-additive capacities).

Our aim is to avoid any constraints concerning the capacities under consideration, i.e., a certain universality of the integrals we work with.

4.1. *Choquet integral*

To the best of our knowledge, historically the first and also most applied integral of this type is the *Choquet integral*, formalized by G. Choquet in 1954,[8] based on some earlier ideas of G. Vitali.[56]

Definition 4. Given a measurable space $(X, \mathcal{A})$ and a capacity $m\colon \mathcal{A} \to [0,1]$, the *expected value* $(\mathrm{Ch})\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ *based on the Choquet integral* (*Choquet expectation*) is given by

$$(\mathrm{Ch})\mathbf{E}_m(A) = \int_0^1 m(A^\alpha)\, d\alpha\,, \tag{4.1}$$

where the integral on the right hand side is a Riemann integral.

In the case of a finite universe $X = \{x_1, x_2, \ldots, x_n\}$, where we usually put $\mathcal{A} = 2^X$, and a fuzzy event A with $\mu_A = \sum_{i=1}^n a_i \cdot \mathbf{1}_{\{x_i\}}$ we can apply Definition 4 and obtain its expected value $(\mathrm{Ch})\mathbf{E}_m(A)$ based on the Choquet integral by

$$(\mathrm{Ch})\mathbf{E}_m(A) = \sum_{i=1}^{n} a_{\sigma(i)} \cdot \big(m(\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\}) - m(\{x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\})\big)\,, \tag{4.2}$$

where the function $\sigma\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is a permutation such that $a_{\sigma(1)} \le a_{\sigma(2)} \le \cdots \le a_{\sigma(n)}$, and $\{x_{\sigma(n+1)}, x_{\sigma(n)}\} = \emptyset$ by convention.

Observe that $(\mathrm{Ch})\mathbf{E}_m$ is *positively homogeneous* and *comonotone additive*, i.e., for all fuzzy events $A, B \in \widehat{\mathcal{A}}$ which are comonotone and satisfy $A \cap_{T_{\mathbf{L}}} B = \emptyset$ and for all $c \in\,]0, \infty[$ satisfying $c \cdot \mu_A \le 1$ we have

$$(\mathrm{Ch})\mathbf{E}_m(c \cdot A) = c \cdot (\mathrm{Ch})\mathbf{E}_m(A)\,,$$
$$(\mathrm{Ch})\mathbf{E}_m(A \cup_{S_{\mathbf{L}}} B) = (\mathrm{Ch})\mathbf{E}_m(A) + (\mathrm{Ch})\mathbf{E}_m(B)\,,$$

where $\mu_{c\cdot A} = c \cdot \mu_A$. Recall that two fuzzy subsets A, B of X are *comonotone* if the union of the two chains of the α-cuts $(A^\alpha)_{\alpha\in[0,1]}$ and $(B^\alpha)_{\alpha\in[0,1]}$ is again a chain or, equivalently, if $(\mu_A(x) - \mu_A(y)) \cdot (\mu_B(x) - \mu_B(y)) \ge 0$ for all $x, y \in X$.

Obviously, if the underlying capacity m is a probability measure, then $(\mathrm{Ch})\mathbf{E}_m$ coincides with the expected value considered in Ref. 60.

The following characterization of the Choquet expectation of fuzzy events is a consequence of the results of D. Schmeidler in Refs. 48 and 49 (see also Ref. 24):

Theorem 4. *Let* $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ *be an expectation of fuzzy events with respect to some capacity* $m \in \mathcal{M}(X, \mathcal{A})$*. Then* $\mathbf{E} = (\mathrm{Ch})\mathbf{E}_m$ *if and only if* $\mathbf{E}$ *is comonotone additive.*

4.2. *Sugeno integral*

Another expected value of fuzzy events is based on the *Sugeno integral*, which had a predecessor, the *Ky Fan metric*[12] which was introduced in 1943 and measures the distance of a function from the zero function. Independently of this approach, the Sugeno integral was proposed in M. Sugeno's PhD thesis in 1974,[53] and it is based on the lattice operations $\vee$ (sup, max) and $\wedge$ (inf, min).

Definition 5. Given a measurable space $(X, \mathcal{A})$ and a capacity $m\colon \mathcal{A} \to [0,1]$, the *expected value* $(\mathrm{Su})\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ *based on the Sugeno integral*

(*Sugeno expectation*) is given by

$$(\mathrm{Su})\mathbf{E}_m(A) = \bigvee_{\alpha\in[0,1]} (\alpha \wedge m(A^\alpha))\,. \tag{4.3}$$

Applying Definition 5 to the case of a fuzzy subset A of a finite universe $X = \{x_1, x_2, \ldots, x_n\}$ with $\mu_A = \sum_{i=1}^n a_i \cdot \mathbf{1}_{\{x_i\}}$, we get

$$(\mathrm{Su})\mathbf{E}_m(A) = \bigvee_{i=1}^{n} a_{\sigma(i)} \wedge m(\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\})\,, \tag{4.4}$$

where the function $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is again a permutation such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)}$.

The expected value $(\mathrm{Su})\mathbf{E}_m$ based on the Sugeno integral is *comonotone maxitive* and min-*homogeneous*, i.e., we have

$$(\mathrm{Su})\mathbf{E}_m(A \cup_{S_\mathbf{M}} B) = (\mathrm{Su})\mathbf{E}_m(A) \vee (\mathrm{Su})\mathbf{E}_m(B)$$

for all comonotone fuzzy events $A, B \in \widehat{\mathcal{A}}$ and

$$(\mathrm{Su})\mathbf{E}_m(A_c \cap_{T_\mathbf{M}} A) = c \wedge (\mathrm{Su})\mathbf{E}_m(A)$$

for all fuzzy events A and for all $c \in [0, 1]$ and the fuzzy event A_c with constant membership function $\mu_{A_c} = c \wedge \mathbf{1}_X$.

The following axiomatic characterization of the Sugeno expectation of fuzzy events follows from Ref. 35 (see also Ref. 24):

Theorem 5. *Let* $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0, 1]$ *be an expectation of fuzzy events with respect to some capacity* $m \in \mathcal{M}(X, \mathcal{A})$. *Then* $\mathbf{E} = (\mathrm{Su})\mathbf{E}_m$ *if and only if* $\mathbf{E}$ *is comonotone maxitive and* min-*homogeneous.*

4.3. *Shilkret integral*

The third integral we consider here is the *Shilkret integral*, originally introduced by N. Shilkret in 1971[51] for maxitive measures only. However, it is not difficult to see that only the monotonicity of the measure m (together with $m(\emptyset) = 0$) is needed, so it can be applied to each capacity.

Definition 6. Given a measurable space $(X, \mathcal{A})$ and a capacity $m\colon \mathcal{A} \to [0, 1]$, the *expected value* $(\mathrm{Sh})\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ *based on the Shilkret integral* (*Shilkret expectation*) is given by

$$(\mathrm{Sh})\mathbf{E}_m(A) = \bigvee_{\alpha\in[0,1]} (\alpha \cdot m(A^\alpha))\,. \tag{4.5}$$

Applying Definition 6 to the case of a fuzzy subset A of a finite universe $X = \{x_1, x_2, \ldots, x_n\}$ with $\mu_A = \sum_{i=1}^n a_i \cdot \mathbf{1}_{\{x_i\}}$, we obtain

$$(\mathrm{Sh})\mathbf{E}_m(A) = \bigvee_{i=1}^{n} a_{\sigma(i)} \cdot m(\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\})\,, \tag{4.6}$$

where the function $\sigma\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is again a permutation such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)}$.

The expected value $(\mathrm{Sh})\mathbf{E}_m$ based on the Shilkret integral is *comonotone maxitive* (as $(\mathrm{Su})\mathbf{E}_m$) and *positively homogeneous* (as $(\mathrm{Ch})\mathbf{E}_m$), i.e., we have

$$(\mathrm{Sh})\mathbf{E}_m(A \cup_{S_{\mathbf{M}}} B) = (\mathrm{Sh})\mathbf{E}_m(A) \vee (\mathrm{Sh})\mathbf{E}_m(B)$$

for all comonotone fuzzy events $A, B \in \widehat{\mathcal{A}}$ and

$$(\mathrm{Sh})\mathbf{E}_m(c \cdot A) = c \cdot (\mathrm{Sh})\mathbf{E}_m(A)$$

for all fuzzy events A and for all $c \in]0,\infty[$ satisfying $c \cdot \mu_A \leq 1$, where $\mu_{c\cdot A} = c \cdot \mu_A$.

In fact, these properties fully characterize the Shilkret expectation of fuzzy events:[24]

Theorem 6. *Let* $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ *be an expectation of fuzzy events with respect to some capacity* $m \in \mathcal{M}(X,\mathcal{A})$*. Then* $\mathbf{E} = (\mathrm{Sh})\mathbf{E}_m$ *if and only if* $\mathbf{E}$ *is comonotone maxitive and positively homogeneous.*

4.4. *First numerical examples*

Here are two examples: in the first one we use a σ-additive capacity (the Lebesgue measure on $[0,1]$), and in the second one a capacity which is not additive.

Example 1. Put $X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, and let $m \in \mathcal{M}(X,\mathcal{A})$ be the standard Lebesque measure (i.e., the uniform probability). Consider the fuzzy set $A \in \widehat{\mathcal{A}}$ whose membership function is given by $\mu_A = \mathrm{id}_X$, i.e. $\mu_A(x) = x$ for each $x \in X$. Clearly, for each $\alpha \in [0,1]$ we have $A^\alpha = [\alpha,1]$ and $m(A^\alpha) = 1-\alpha$.

Obviously, for the smallest expectation of fuzzy events $(\mathbf{E}_m)_*$ with respect to m and for the greatest expectation $(\mathbf{E}_m)^*$ we get $(\mathbf{E}_m)_*(A) = 0$ and $(\mathbf{E}_m)^*(A) = 1$, respectively.

Since m is σ-additive, the Choquet expectation $(\mathrm{Ch})\mathbf{E}_m$ coincides with the standard expectation $\mathbf{E}_m$, and we get

$$(\mathrm{Ch})\mathbf{E}_m(A) = \mathbf{E}_m(\mu_A) = \int_0^1 (1-\alpha)\, d\alpha = \frac{1}{2}.$$

For the Sugeno expectation $(\mathrm{Su})\mathbf{E}_m(A)$ and the Shilkret expectation $(\mathrm{Sh})\mathbf{E}_m(A)$ we obtain

$$(\mathrm{Su})\mathbf{E}_m(A) = \bigvee_{\alpha\in[0,1]} (\alpha \wedge (1-\alpha)) = \frac{1}{2},$$

$$(\mathrm{Sh})\mathbf{E}_m(A) = \bigvee_{\alpha\in[0,1]} (\alpha \cdot (1-\alpha)) = \frac{1}{4}.$$

Example 2. Let $(X,\mathcal{A}) = ([0,1],\mathcal{B}([0,1]))$ and consider the fuzzy event A whose trapezoidal membership function $\mu_A\colon X \to [0,1]$ is given by

$$\mu_A(x) = \max(\min(10x-2,-5x+4,1),0)$$

and the capacity $m \in \mathcal{M}(X, \mathcal{A})$ given by $m(B) = (\lambda(B))^2$, where λ is the standard Lebesgue measure on $\mathcal{B}([0,1])$. Clearly, for each $\alpha \in [0,1]$ we have $A^\alpha = [0.2 + 0.1\alpha, 0.8 - 0.2\alpha]$, and $m(A^\alpha) = (0.6 - 0.3\alpha)^2$. Then we obtain

$$\begin{aligned}
(\mathbf{E}_m)_*(A) &= 0.3^2 = 0.09\,,\\
(\mathbf{E}_m)^*(A) &= 0.6^2 = 0.36\,,\\
(\mathrm{Ch})\mathbf{E}_m(A) &= 0.21\,,\\
(\mathrm{Su})\mathbf{E}_m(A) &= \frac{0.68 - \sqrt{0.43}}{0.09} \approx 0.2695127\,,\\
(\mathrm{Sh})\mathbf{E}_m(A) &\approx 0.1066666\,.
\end{aligned}$$

5. Expected Value with Respect to Universal Integrals on $[0, 1]$

A unified approach for integrals defined on an arbitrary measurable space $(X, \mathcal{A}) \in \mathcal{D}$ and for an arbitrary capacity $m \in \mathcal{M}(X, \mathcal{A})$ reproducing the underlying capacity, i.e., for each $A \in \mathcal{A}$ the integral of $\mathbf{1}_A$ yields $m(A)$, was introduced by E.P. Klement, R. Mesiar & E. Pap in 2010[27] and studied under the name universal integral.

5.1. *Universal integrals*

Definition 7. A function

$$\mathbf{I}\colon \bigcup_{(X,\mathcal{A})\in\mathcal{D}} \big(\mathcal{M}(X,\mathcal{A}) \times \mathcal{F}(X,\mathcal{A})\big) \to [0,1]$$

is called a *universal integral on* $[0, 1]$ if there exists a semicopula $\otimes\colon [0,1]^2 \to [0,1]$ (i.e., a monotone non-decreasing binary operation on $[0, 1]$ with neutral element 1) such that the following two conditions hold:

(I1) for all $(X, \mathcal{A}) \in \mathcal{D}$, for all $m \in \mathcal{M}(X, \mathcal{A})$, for all $A \in \mathcal{A}$ and for all $c \in [0,1]$ we have

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A)\,,$$

(I2) for all $(X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2) \in \mathcal{D}$ and for all $(m_1, f_1) \in (\mathcal{M}(X_1, \mathcal{A}_1) \times \mathcal{F}(X_1, \mathcal{A}_1))$ and $(m_2, f_2) \in (\mathcal{M}(X_2, \mathcal{A}_2) \times \mathcal{F}(X_2, \mathcal{A}_2))$ satisfying, for each $\alpha \in [0,1]$, the inequality $m_1(\{f_1 \geq \alpha\}) \leq m_2(\{f_2 \geq \alpha\})$ we have

$$\mathbf{I}(m_1, f_1) \leq \mathbf{I}(m_2, f_2)\,.$$

Observe that, as a consequence of (I2) in Definition 7, for a given $(X, \mathcal{A}) \in \mathcal{D}$ the restriction $\mathbf{I}|_{\mathcal{M}(X,\mathcal{A})\times\mathcal{F}(X,\mathcal{A})}$ of the universal integral $\mathbf{I}$ to $\mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})$ is monotone non-decreasing in both arguments, i.e., with respect to the underlying capacities and to the integrands.

Furthermore, if m_1 and m_2 are probability measures (possibly on different measurable spaces) then the validity of the inequalities $m_1(\{f_1 \geq \alpha\}) \leq m_2(\{f_2 \geq \alpha\})$ for each $\alpha \in [0,1]$ is equivalent to the fact that the distribution functions F_1, F_2 of f_1 and f_2 satisfy $F_1 \geq F_2$ which means that for the standard expected values of the random variables f_1 and f_2 we have $\mathbf{E}_{m_1}(f_1) \leq \mathbf{E}_{m_2}(f_2)$. This shows that the universal integral generalizes the important monotonicity of the standard expected value (based on the Lebesgue(-Stieltjes) integral).

Observe that all three integrals considered in Sec. 4, the Choquet, the Sugeno and the Shilkret integral, are universal integrals on $[0,1]$. For the Choquet and the Shilkret integral the underlying semicopula is the standard product, and for the Sugeno integral it is the lattice meet.

Definition 8. Fixing a universal integral $\mathbf{I}$ on $[0,1]$, a measurable space $(X, \mathcal{A})$ and a capacity $m \in \mathcal{M}(X, \mathcal{A})$ then the *expected value* $(\mathrm{I})\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ *based on the universal integral* $\mathbf{I}$ (*universal expectation*) and the capacity m is given by

$$(\mathrm{I})\mathbf{E}_m(A) = \mathbf{I}(m, \mu_A)\,. \tag{5.1}$$

From property (I1) of the universal integral $\mathbf{I}$ given in Definition 8 it follows immediately that the expected value $(\mathrm{I})\mathbf{E}_m$ of fuzzy events based on $\mathbf{I}$ satisfies $(\mathrm{I})\mathbf{E}_m(A_c) = c$ for each $c \in [0,1]$ and the fuzzy event $A_c \in \widehat{\mathcal{A}}$ with constant membership function $\mu_{A_c} = c \wedge \mathbf{1}_X$.

Note that the universal integrals provide a framework for the definition of expected values of fuzzy events, but this approach is, up to some special cases, not constructive in nature.

If we fix a semicopula $\otimes$ then it is possible to compute the smallest universal integral $\mathbf{I}_\otimes$ with respect to $\otimes$ (see Eq. (7) in Ref. 27 for the explicit formula).

For example, this means that, for a fixed semicopula $\otimes$, a measurable space $(X, \mathcal{A})$ and a capacity $m \in \mathcal{M}(X, \mathcal{A})$, the *smallest expected value* $(\mathrm{I}_\otimes)\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ based on a universal integral related to $\otimes$ and the capacity m is given by

$$(\mathrm{I}_\otimes)\mathbf{E}_m(A) = \bigvee_{\alpha \in [0,1]} \big(\alpha \otimes m(A^\alpha)\big)\ . \tag{5.2}$$

Taking for $\otimes$ the standard product Π, this leads to the Shilkret expectation (4.5), i.e., $(\mathrm{I}_\Pi)\mathbf{E}_m = (\mathrm{Sh})\mathbf{E}_m$, while putting $\otimes = \wedge$ yields the Sugeno expectation (4.3), i.e., $(\mathrm{I}_\wedge)\mathbf{E}_m = (\mathrm{Su})\mathbf{E}_m$.

Note that an expected value $\mathbf{E}$ of fuzzy events related to a capacity $m \in \mathcal{M}(X, \mathcal{A})$ is described by (5.2), i.e., $\mathbf{E} = (\mathrm{I}_\otimes)\mathbf{E}_m$ if and only if $\mathbf{E}$ is comonotone maxitive and $\otimes$-homogeneous.

In a similar way, it is possible to compute the greatest universal integral $\mathbf{I}^\otimes$ with respect to a fixed semicopula $\otimes$ and, subsequently, the greatest expected value $(\mathrm{I}^\otimes)\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ based on a universal integral related to $\otimes$ and the capacity m (see again Ref. 27 for details).

5.2. *Copula-based universal integrals on* $[0,1]$

An important subclass of universal integrals on $[0,1]$ are those where the underlying semicopula is a *copula* $C\colon [0,1]^2 \to [0,1]$, i.e., a *supermodular* semicopula C which means that it satisfies also

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$ (here the operations $\vee$ and $\wedge$ are applied component-wise).

Each copula C is in a one-to-one correspondence to a probability measure P_C on the Borel subsets $\mathcal{B}([0,1]^2)$ of $[0,1]^2$ with uniform margins, i.e., for all $x \in [0,1]$ we have $P_C([0,x] \times [0,1]) = P_C([0,1] \times [0,x]) = x$. The correspondence between C and P_C is uniquely determined by

$$P_C([0,x] \times [0,y]) = C(x,y)$$

for all $(x,y) \in [0,1]^2$. For more details on copulas we recommend.[15,41]

Given a copula $C\colon [0,1]^2 \to [0,1]$, then the function

$$\mathbf{I}_{(C)}\colon \bigcup_{(X,\mathcal{A}) \in \mathcal{D}} \big(\mathcal{M}(X,\mathcal{A}) \times \mathcal{F}(X,\mathcal{A})\big) \to [0,1]$$

defined by

$$\mathbf{I}_{(C)}(m,f) = P_C(\{(x,y) \in [0,1]^2 \mid y \leq m(\{z \in X \mid f(z) \geq x\})\}) \tag{5.3}$$

is a universal integral on $[0,1]$.

Definition 9. Fixing a copula C, a measurable space $(X,\mathcal{A})$ and a capacity $m \in \mathcal{M}(X,\mathcal{A})$ then the *expected value* $(\mathrm{I}_{(C)})\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ *based on the copula-based universal integral* $\mathbf{I}_{(C)}$ (*copula-based universal expectation*) and the capacity m is given by

$$(\mathrm{I}_{(C)})\mathbf{E}_m(A) = P_C(\{(x,y) \in [0,1]^2 \mid y \leq m(A^x)\})\,. \tag{5.4}$$

As an immediate consequence of Definition 9, note that for the product copula Π (i.e., for the standard product) the expected value $(\mathrm{I}_{(\Pi)})\mathbf{E}_m$ is the Choquet expectation (4.1), i.e., we have $(\mathrm{I}_{(\Pi)})\mathbf{E}_m = (\mathrm{Ch})\mathbf{E}_m$, and for the greatest copula M (which coincides with the lattice meet $\wedge$) the expected value $(\mathrm{I}_{(M)})\mathbf{E}_m$ equals the Sugeno expectation (4.3), i.e., $(\mathrm{I}_{(M)})\mathbf{E}_m = (\mathrm{Su})\mathbf{E}_m$.

In the case of a fuzzy subset A of a finite universe $X = \{x_1, x_2, \ldots, x_n\}$ with $\mu_A = \sum_{i=1}^n a_i \cdot \mathbf{1}_{\{x_i\}}$ we get

$$\begin{aligned}(\mathrm{I}_{(C)})\mathbf{E}_m(A) = \sum_{i=1}^{n} \big(&C(a_{\sigma(i)}, m(\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\})) \\ &- C(a_{\sigma(i)}, m(\{x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\}))\big)\,,\end{aligned} \tag{5.5}$$

where the function $\sigma\colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is a permutation such that $a_{\sigma(1)} \le a_{\sigma(2)} \le \dots \le a_{\sigma(n)}$, and $\{x_{\sigma(n+1)}, x_{\sigma(n)}\} = \emptyset$ by convention.

Remark 1. Expected values $\mathbf{E}\colon \widehat{\mathcal{A}} \to [0,1]$ of fuzzy events can be required to satisfy additional properties. One of them could be, for example, to satisfy

$$\mathbf{E}(\complement A) = 1 - \mathbf{E}(A)\,, \tag{5.6}$$

for each $A \in \widehat{\mathcal{A}}$, where the membership function of the *complement* $\complement A$ of $A \in \widehat{\mathcal{A}}$ is given by $\mu_{\complement A} = 1 - \mu_A$ (see Ref. 59).

This is possible if and only if the capacity $m = \mathbf{E}|_{\mathcal{A}}$ defined by $m(U) = \mathbf{E}(U)$ is *self-dual*. Note that the *dual capacity* m^{d} of m is given by $m^{\mathrm{d}}(A) = 1 - m(\complement A)$. Then self-duality of m means $m = m^{\mathrm{d}}$ or, equivalently, $m(A) + m(\complement A) = 1$ for each $A \in \mathcal{A}$ (and then m is sometimes called a *participation measure*).

Obviously, each probability measure p is self-dual, and the probability of fuzzy events proposed in Ref. 60 satisfies (5.6).

Other typical examples of expected values of fuzzy events satisfying (5.6) are related to copula-based integrals $\mathbf{I}_{(C)}$ with a *radially symmetric copula* C, i.e., satisfying $\widehat{C} = C$, where the *survival copula*[41] $\widehat{C}\colon [0,1]^2 \to [0,1]$ is given by

$$\widehat{C}(x,y) = x + y - 1 + C(1-x, 1-y)\,.$$

Note that this includes all copula-based universal expectations based on some Frank copula[13] (see (3.4)) and, subsequently, the Choquet and the Sugeno expectation. Notice also that, for each $m \in \mathcal{M}(X,\mathcal{A})$, the capacity $\widetilde{m} = \frac{m+m^{\mathrm{d}}}{2}$ is self-dual.

Example 3. Keeping the notations and the setting of Example 1, consider the *Hamacher product*[18,26] $H\colon [0,1]^2 \to [0,1]$ defined by

$$H(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0)\,, \\ \dfrac{x \cdot y}{x + y - x \cdot y} & \text{otherwise,} \end{cases}$$

which is a copula and, subsequently, also a semicopula. Then we get for the expected values $(\mathrm{I}_H)\mathbf{E}_m$ with respect to the universal integral $\mathbf{I}_H$ based on the semicopula H and for the copula-based universal expectation $(\mathrm{I}_{(H)})\mathbf{E}_m$, both with respect to the capacity m, respectively,

$$(\mathrm{I}_H)\mathbf{E}_m(A) = \bigvee_{\alpha \in [0,1]} H(\alpha, 1-\alpha) = \frac{1}{3}\,,$$

$$(\mathrm{I}_{(H)})\mathbf{E}_m(A) = P_H(\{(x,y) \in [0,1]^2 \mid x + y \le 1\}) = \frac{4\pi\sqrt{3} - 9}{27} \approx 0.4728\,.$$

6. Expected Value with Respect to Decomposition Integrals

For a fixed measurable space $(X,\mathcal{A})$, each finite subset H of $\mathcal{A} \setminus \{\emptyset\}$ is called a *collection*, and each non-empty system $\mathcal{H}$ of collections is called a *decomposition* *system* related to $(X, \mathcal{A})$. Recently, decomposition integrals were introduced by Y. Even & E. Lehrer in 2014[11] as follows:

Definition 10. Let $(X, \mathcal{A})$ be a measurable space and $\mathcal{H}$ a decomposition system related to $(X, \mathcal{A})$. then for each capacity $m \in \mathcal{M}(X, \mathcal{A})$ and for each function $f\colon X \to [0, \infty[$ which is measurable with respect to $\mathcal{A}$ and $\mathcal{B}([0, \infty[)$ the *decomposition integral* $\mathbf{I}_{\mathcal{H}}(m, f)$ is defined by

$$\mathbf{I}_{\mathcal{H}}(m, f) = \sup \left\{ \sum_{G \in H} a_G \cdot m(G) \;\middle|\; H \in \mathcal{H}, \right. \\ \left. a_G \geq 0 \text{ for each } G \in H \text{ and } \sum_{G \in H} a_G \cdot \mathbf{1}_G \leq f \right\}. \quad (6.1)$$

Not each decomposition integral given in Definition 10 reproduces the underlying capacity m when applied to a measurable characteristic function of a crisp subset of X, i.e., we do not have $\mathbf{I}_{\mathcal{H}}(m, \mathbf{1}_G) = m(G)$ for all $G \in \mathcal{A}$, in general. Necessary and sufficient conditions for decomposition systems satisfying $\mathbf{I}_{\mathcal{H}}(m, \mathbf{1}_G) = m(G)$ for all $G \in \mathcal{A}$ were given recently.[40] Only decomposition systems satisfying these conditions can be used to define expected values of fuzzy events with respect to decomposition integrals.

Definition 11. Fix a measurable space $(X, \mathcal{A})$, a decomposition system $\mathcal{H}$ related to $(X, \mathcal{A})$ satisfying the two conditions

(i) $\bigcup_{H \in \mathcal{H}} H = \mathcal{A} \setminus \{\emptyset\}$, (*completeness*)

(ii) $\bigcap_{G \in H} G \neq \emptyset$ for each $H \in \mathcal{H}$, (*independence*)

and a capacity $m \in \mathcal{M}(X, \mathcal{A})$. Then the *expected value* $(\mathrm{I}_{\mathcal{H}})\mathbf{E}_m(A)$ of a fuzzy event $A \in \widehat{\mathcal{A}}$ *based on the decomposition integral* $\mathbf{I}_{\mathcal{H}}$ and the capacity m (*decomposition expectation*) is given by

$$(\mathrm{I}_{\mathcal{H}})\mathbf{E}_m(A) = \mathbf{I}_{\mathcal{H}}(m, \mu_A)\,. \quad (6.2)$$

As a consequence of Definition 11, also the expected value $(\mathrm{I}_{\mathcal{H}})\mathbf{E}_m$ of fuzzy events based on the decomposition integral $\mathbf{I}_{\mathcal{H}}$ and a capacity m satisfies $(\mathrm{I}_{\mathcal{H}})\mathbf{E}_m(A_c) = c$ for each $c \in [0, 1]$ and the fuzzy event $A_c \in \widehat{\mathcal{A}}$ with constant membership function $\mu_{A_c} = c \wedge \mathbf{1}_X$.

Consider a measurable space $(X, \mathcal{A})$, a capacity $m \in \mathcal{M}(X, \mathcal{A})$ and the decomposition system $\mathcal{H}_1 = \{\{G\} \mid G \in \mathcal{A}\}$, i.e., each collection in $\mathcal{H}_1$ is a singleton. Then $(\mathrm{I}_{\mathcal{H}_1})\mathbf{E}_m$ equals the Shilkret expectation (4.5), i.e., we have $(\mathrm{I}_{\mathcal{H}_1})\mathbf{E}_m = (\mathrm{Sh})\mathbf{E}_m$. Similarly, if the decomposition system $\mathcal{H}_\infty = \{H \mid H \text{ is a finite chain in } \mathcal{A}\}$ is considered, then $(\mathrm{I}_{\mathcal{H}_\infty})\mathbf{E}_m$ equals the Choquet expectation (4.1), i.e., we have $(\mathrm{I}_{\mathcal{H}_\infty})\mathbf{E}_m = (\mathrm{Ch})\mathbf{E}_m$.

Strictly speaking, the decomposition systems $\mathcal{H}_1$ and $\mathcal{H}_\infty$ heavily depend on the underlying measurable space $(X, \mathcal{A})$ (each change of $\mathcal{A}$ yields a different decomposition system). But their definition is universal enough that they can be defined in each measurable space $(X, \mathcal{A})$ in a generic way, and in such cases we can consider the (well-defined) function

$$\mathbf{I}_{\mathcal{H}}\colon \bigcup_{(X,\mathcal{A})\in\mathcal{D}} \big(\mathcal{M}(X,\mathcal{A}) \times \mathcal{F}(X,\mathcal{A})\big) \to [0,\infty]\,. \tag{6.3}$$

The only decomposition systems[40] $\mathcal{H}$ such that $\mathbf{I}_{\mathcal{H}}$ given by (6.3) is a universal integral on $[0,1]$ form a hierarchical system $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots \subseteq \mathcal{H}_\infty$ and are given by $\mathcal{H}_n = \{H \mid H \text{ is a chain in } \mathcal{A} \text{ and } \operatorname{card}(H) = n\}$ for each $n \in \{1, 2, \dots\}$. Then the corresponding expectations form a hierarchical system with the Shilkret and the Choquet expectations being the extremal cases:

$$(\mathrm{Sh})\mathbf{E}_m = (\mathrm{I}_{\mathcal{H}_1})\mathbf{E}_m \le (\mathrm{I}_{\mathcal{H}_2})\mathbf{E}_m \le \cdots \le (\mathrm{I}_{\mathcal{H}_\infty})\mathbf{E}_m = (\mathrm{Ch})\mathbf{E}_m\,.$$

A dual approach, the so-called *superdecomposition integral* was introduced by R. Mesiar, J. Li & E. Pap in 2015.[39]

Definition 12. Under the hypotheses of Definition 11, the *expected value* $(\mathrm{I}^{\mathcal{H}})\mathbf{E}_m(A)$ *of a fuzzy event* $A \in \widehat{\mathcal{A}}$*s based on the superdecomposition integral* $\mathbf{I}^{\mathcal{H}}$ and the capacity m (*superdecomposition expectation*) is defined by

$$\begin{aligned}(\mathrm{I}^{\mathcal{H}})\mathbf{E}_m(A) &= \mathbf{I}^{\mathcal{H}}(m, \mu_A)\\ &= \inf\Bigg\{ \sum_{G\in H} a_G \cdot m(G) \;\Bigg|\; H \in \mathcal{H}, a_G \ge 0 \text{ for each } G \in H\\ &\qquad \text{and } \sum_{G\in H} a_G \cdot \mathbf{1}_G \ge \mu_A \Bigg\}.\end{aligned} \tag{6.4}$$

As a consequence of Definition 12 we obtain

$$(\mathrm{I}^{\mathcal{H}_1})\mathbf{E}_m \ge (\mathrm{I}^{\mathcal{H}_2})\mathbf{E}_m \ge \cdots \ge (\mathrm{I}^{\mathcal{H}_\infty})\mathbf{E}_m = (\mathrm{Ch})\mathbf{E}_m\,,$$

where

$$(\mathrm{I}^{\mathcal{H}_1})\mathbf{E}_m(A) = \sup\{\mu_A(x) \mid x \in X\} \cdot m(\{x \in X \mid \mu_A(x) > 0\})\,.$$

For further generalizations based on copulas (where in the case of the product copula Π the decomposition/superdecomposition expectation is obtained) we refer to Ref. 28.

Example 4. Keeping the notations and settings of Example 1, we obtain for each $n \in \{1, 2, \dots\}$

$$(\mathrm{I}_{\mathcal{H}_n})\mathbf{E}_m(A) = \frac{n}{2n+2} \quad \text{and} \quad (\mathrm{I}^{\mathcal{H}_n})\mathbf{E}_m(A) = \frac{n+1}{2n}\,.$$

Observe that $\lim_{n\to\infty} (\mathrm{I}_{\mathcal{H}_n})\mathbf{E}_m(A) = \lim_{n\to\infty} (\mathrm{I}^{\mathcal{H}_n})\mathbf{E}_m(A) = (\mathrm{Ch})\mathbf{E}_m(A) = \frac{1}{2}$.

7. Concluding Remarks

We have discussed the challenging topic of expected values of fuzzy events, a study which originated in Ref. 60. In doing so, we have summarized known results (sometimes from a new perspective) and introduced new concepts. These parameters of fuzzy events can be seen as their "size", and different approaches allow us to consider different properties when measuring fuzzy events. So, for example, T-valuations were discussed in Sec. 3, while expectations of fuzzy events which are comonotone additive (comonotone maxitive, min-homogeneous, or positively homogeneous) were characterized in Sec. 4.

Note that the set $\mathcal{E}$ of all expected values of fuzzy events form a bounded lattice with top and bottom elements $\mathbf{E}_*$ and $\mathbf{E}^*$ given by

$$\mathbf{E}_*(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{E}^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

respectively, which can be written also in the form $\mathbf{E}_* = (\mathbf{E}_{m_*})_*$ and $\mathbf{E}^* = (\mathbf{E}_{m^*})^*$, see Sec. 2, where m_* is the smallest and m^* is the greatest capacity on $(X, \mathcal{A})$, respectively. Moreover, the set $\mathcal{E}$ is also a convex class.

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References

1. H. Bauer, *Probability Theory and Elements of Measure Theory* (Academic Press, London, 1981).
2. K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges. A Study of Finitely Additive Measures* (Academic Press, New York, 1983).
3. D. Butnariu, Fuzzy measurability and integrability, *J. Math. Anal. Appl.* **117** (1986) 385–410.
4. D. Butnariu and E. P. Klement, Triangular norm-based measures and their Markov kernel representation, *J. Math. Anal. Appl.* **162** (1991) 111–143.
5. D. Butnariu and E. P. Klement, *Triangular Norm-Based Measures and Games with Fuzzy Coalitions* (Kluwer Academic Publishers, Dordrecht, 1993).
6. D. Butnariu and E. P. Klement, Triangular norm-based measures, in *Handbook of Measure Theory*, Chapter 23, ed. E. Pap (Elsevier Science, Amsterdam, 2002), pp. 947–1010.
7. C. Carathéodory, *Vorlesungen über reelle Funktionen* (Teubner, Berlin, 1918).
8. G. Choquet, Theory of capacities, *Ann. Inst. Fourier* (*Grenoble*) **5** (1954) 131–295.
9. A. P. Dempster, Upper and lower probabilities induced by a multivalued mapping, *Ann. Math. Stat.* **38** (1967) 325–339.
10. D. Denneberg, *Non-Additive Measure and Integral* (Kluwer Academic Publishers, Dordrecht, 1994).
11. Y. Even and E. Lehrer, Decomposition-integral: unifying Choquet and the concave integrals, *Econom. Theory* **56** (2014) 33–58.

12. K. Fan, Entfernung zweier zufälligen Größen und die Konvergenz nach Wahrscheinlichkeit, *Math. Z.* **49** (1944) 681–683.
13. M. J. Frank, On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, *Aequationes Math.* **19** (1979) 194–226.
14. M. Grabisch, k-order additive discrete fuzzy measures and their representation, *Fuzzy Sets and Systems* **92** (1997) 167–189.
15. M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation Functions* (Cambridge University Press, Cambridge, 2009).
16. M. Grabisch, T. Murofushi and M. Sugeno (eds.), *Fuzzy Measures and Integrals. Theory and Applications* (Physica-Verlag, Heidelberg, 2000).
17. P. R. Halmos, *Measure Theory* (Van Nostrand Reinhold, New York, 1950).
18. H. Hamacher, *Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien* (Rita G. Fischer Verlag, Frankfurt, 1978).
19. U. Höhle and S. E. Rodabaugh (eds.), *Mathematics of Fuzzy Sets. Logic, Topology, and Measure Theory* (Kluwer Academic Publishers, Boston, 1999).
20. E. P. Klement, Characterization of finite fuzzy measures using Markoff-kernels, *J. Math. Anal. Appl.* **75** (1980) 330–339.
21. E. P. Klement, Fuzzy σ-algebras and fuzzy measurable functions, *Fuzzy Sets and Systems* **4** (1980) 83–93.
22. E. P. Klement, Characterization of fuzzy measures constructed by means of triangular norms, *J. Math. Anal. Appl.* **86** (1982) 345–358.
23. E. P. Klement, Construction of fuzzy σ-algebras using triangular norms. *J. Math. Anal. Appl.* **85** (1982) 543–565.
24. E. P. Klement, J. Li, R. Mesiar and E. Pap, Integrals based on monotone set functions, *Fuzzy Sets and Systems* **281** (2015) 88–102.
25. E. P. Klement, R. Lowen and W. Schwyhla, Fuzzy probability measures, *Fuzzy Sets and Systems* **5** (1981) 21–30.
26. E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms* (Kluwer Academic Publishers, Dordrecht, 2000).
27. E. P. Klement, R. Mesiar and E. Pap, A universal integral as common frame for Choquet and Sugeno integral, *IEEE Trans. Fuzzy Systems* **18** (2010) 178–187.
28. E. P. Klement, R. Mesiar, F. Spizzichino and A. Stupňanová, Universal integrals based on copulas, *Fuzzy Optim. Decis. Mak.* **13** (2014) 273–286.
29. E. P. Klement and S. Weber, Generalized measures, *Fuzzy Sets and Systems* **40** (1991) 375–394.
30. E. P. Klement and S. Weber, Fundamentals of a generalized measure theory, in *Mathematics of Fuzzy Sets. Logic, Topology, and Measure Theory*, Chapter 11, eds. U. Höhle and S. E. Rodabaugh (Kluwer Academic Publishers, Boston, 1999), pp. 633–651.
31. R. Kruse, Fuzzy integrals and conditional fuzzy measures, *Fuzzy Sets and Systems* **10** (1983) 309–313.
32. E. Lehrer, A new integral for capacities, *Econom. Theory* **39** (2009) 157–176.
33. J. Li and R. Mesiar, Lusin's theorem on monotone measure spaces, *Fuzzy Sets and Systems* **175** (2011) 75–86.
34. J. Li, R. Mesiar, and E. Pap, Atoms of weakly null-additive monotone measures and integrals, *Inform. Sci.* **257** (2014) 183–192.
35. J.-L. Marichal, An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework, *IEEE Trans. Fuzzy Systems* **9** (2001) 164–172.

36. P. Medolaghi, La logica matematica ed il calcolo delle probabilità, *Bollettino dell'Associazione Italiana per l'Incremento delle Scienze Attuariali* **18** (1907) 20–39.
37. R. Mesiar, k-order additive measures, *Int. J. Uncertain. Fuzziness Knowledge-Based Systems* **7** (1999) 561–568.
38. R. Mesiar, Fuzzy measures and integrals, *Fuzzy Sets and Systems* **156** (2005) 365–370.
39. R. Mesiar, J. Li and E. Pap, Superdecomposition integrals, *Fuzzy Sets and Systems* **259** (2015) 3–11.
40. R. Mesiar and A. Stupňanová, Decomposition integrals, *Int. J. Approx. Reason.* **54** (2013) 1252–1259.
41. R. B. Nelsen, *An Introduction to Copulas*, 2nd edn. (Springer, New York, 2006).
42. H. T. Nguyen and E. Walker, *A First Course in Fuzzy Logic* (CRC Press, Boca Raton, 1997).
43. E. Pap, An integral generated by a decomposable measure, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.* **20** (1990) 135–144.
44. E. Pap, *Null-Additive Set Functions* (Kluwer Academic Publishers, Dordrecht, 1995).
45. E. Pap (ed.), *Handbook of Measure Theory* (Elsevier Science, Amsterdam, 2002).
46. E. Pap, Generalized real analysis and its applications, *Int. J. Approx. Reason.* **47** (2008) 368–386.
47. D. Ralescu and G. Adams, The fuzzy integral, *J. Math. Anal. Appl.* **75** (1980) 562–570.
48. D. Schmeidler, Integral representation without additivity, *Proc. Amer. Math. Soc.* **97** (1986) 255–261.
49. D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica* **57** (1989) 571–587.
50. G. Shafer, *A Mathematical Theory of Evidence* (Princeton University Press, Princeton, 1976).
51. N. Shilkret, Maxitive measure and integration, *Indag. Math.* **33** (1971) 109–116.
52. O. Stolz, Über einen zu einer unendlichen Punktmenge gehörigen Grenzwert, *Math. Ann.* **23** (1883) 152–156.
53. M. Sugeno, *Theory of fuzzy integrals and its applications*, PhD thesis, Tokyo Institute of Technology, 1974.
54. M. Sugeno and T. Murofushi, Pseudo-additive measures and integrals, *J. Math. Anal. Appl.* **122** (1987) 197–222.
55. V. Torra, Y. Narukawa, and M. Sugeno (eds.), *Non-Additive Measures. Theory and Applications* (Springer, 2014).
56. G. Vitali, Sulla definizione di integrale delle funzioni di una variabile, *Ann. Mat. Pura Appl.* **2** (1925) 111–121.
57. Z. Wang and G. J. Klir, *Generalized Measure Theory* (Springer, New York, 2009).
58. S. Weber, $\perp$-decomposable measures and integrals for Archimedean t-conorms $\perp$, *J. Math. Anal. Appl.* **101** (1984) 114–138.
59. L. A. Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338–353.
60. L. A. Zadeh, Probability measures of fuzzy events, *J. Math. Anal. Appl.* **23** (1968) 421–427.