Generalizations of OWA Operators

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***Abstract*—OWA operators can be seen as symmetrized weighted arithmetic means, as Choquet integrals with respect to symmetric measures, or as comonotone additive functionals. Following these three different looks on OWAs, we discuss several already known generalizations of OWA operators, including GOWA, IOWA, OMA operators, as well as we propose new types of such generalizations.**

***Index Terms*—Choquet integral, comonotonicity, copula, generalized ordered weighted average (GOWA) operator, ordered modular average (OMA) operator, ordered weighted average (OWA) operator.**

I. Introduction

ORDERED Weighted Average (OWA) operators were introduced in 1988 by Yager [36]. Very soon they became an important tool in many domains, especially in decision problems. Rather early, several generalizations of OWAs appeared, such as generalized OWA (GOWA) [32], induced OWA (IOWA) [34], [37], 2-D OWA operators [2], etc. Recently, a survey of OWA literature using a citation network analysis was published in [8], including 537 OWA related sources in supplementary document. However, a systematic approach to OWA generalizations is still missing, at least to the best knowledge of authors. The aim of this paper is to fill the aforementioned gap. Note that following the basic introduction of OWA operators in [36], they can be seen as follows:

1) weighted arithmetic means of ordered inputs;
2) Choquet integrals with respect to symmetric capacities [11];
3) symmetric comonotone additive aggregation functions [3] (this axiomatic characterization follows from results of [27] and [11]).

Each of OWA generalizations known to us is related to a generalization linked to some of items summarized previously. Moreover, this way, several new kinds of generalized OWAs can be obtained, as shown in this paper. It is organized as follows. In the next section, more detailed look on OWA operators is offered. Section III reviews some known generalizations related to (1), i.e., when looking on OWAs as weighted arithmetic means applied to an ordered input n-tuple. Section IV deals with integral-based generalizations of OWAs, while in Section V, we generalize the axiomatic look on OWAs. Finally, some concluding remarks are given.

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II. Ordered Weighted Average Operators

Through this paper, we will deal with n-ary aggregation functions on $[0,1]$, i.e., functions $A:[0,1]^n \to [0,1]$ which are monotone and satisfy the boundary conditions $A(0,\dots,0) = 0,\ A(1,\dots,1) = 1$. For more details concerning the aggregation functions, we recommend monographs [1], [12]. Observe that the presented definitions and results can be mostly straightforwardly extended to the case of a general interval I, in particular for $I \in \{[0,\infty],\ [0,\infty[,\ [-1,1],\ [-\infty,\infty],]-\infty,\infty[\}$. OWA operators, as a special class of aggregation functions covering the standard $\min, \max$, and arithmetic mean operators, were introduced by Yager [36] in 1988.

Definition 2.1: Let $\mathbf{w} = (w_1,\dots,w_n) \in [0,1]^n$ be a normed weighting vector, i.e., $\sum_{i=1}^n w_i = 1$. A function $\mathrm{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$ given by

$$\mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^n w_i\, x_{\sigma(i)} \tag{1}$$

where $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ is a permutation of $\{1,\dots n\}$ such that $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$, is called an OWA operator.

Note that a normed weighting vector $\mathbf{w} \in [0,1]^n$ can be seen as a discrete probability distribution on the universe $X = \{1,\dots,n\}$, with $P(\{i\}) = w_i, i = 1,\dots,n$. This look allows to apply ideas known for discrete probabilities in OWA domain, too, such as the entropy concept [26], [35], for example. Observe also that the permutation σ applied in formula (1) need not be unique. However, the possible nonuniqueness may appear only in the case of ties within the input data $(x_1,\dots,x_n)$ and in such case, the formula (1) gives the same output $\mathrm{OWA}_{\mathbf{w}}(\mathbf{x})$, independently of permutation σ. It is evident that $\mathrm{OWA}_{\mathbf{w}}$ is an aggregation function. Moreover, it is idempotent (i.e., $\mathrm{OWA}_{\mathbf{w}}(c,\dots,c) = c$ for each $c \in [0,1]$) and symmetric (i.e., $\mathrm{OWA}_{\mathbf{w}}(x_1 \dots, x_n) = \mathrm{OWA}_{\mathbf{w}}(x_{\alpha(1)},\dots,x_{\alpha(n)})$ for an arbitrary permutation $\alpha : \{1,\dots,n\} \to \{1,\dots,n\}$).

Recall some basic OWA operators:

1) $\mathrm{OWA}_{(1,0,\dots,0)}(\mathbf{x}) = \max(x_1,\dots,x_n)$;
2) $\mathrm{OWA}_{(0,0,\dots,1)}(\mathbf{x}) = \min(x_1,\dots,x_n)$;
3) $\mathrm{OWA}_{(\frac{1}{n},\dots,\frac{1}{n})}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i$ (arithmetic mean);
4) $\mathrm{OWA}_{(\frac{1}{2},0,\dots,0,\frac{1}{2})}(\mathbf{x}) =$ $= \frac{1}{2}(\min(x_1,\dots,x_n) + \max(x_1,\dots,x_n))$;

5) $\mathrm{OWA}_{\left(0, \frac{1}{n-2}, \ldots, \frac{1}{n-2}, 0\right)}(\mathbf{x}) = = \frac{1}{n-2}\left(\sum_{i=1}^{n} x_i - \min(x_1, \ldots, x_n) - \max(x_1, \ldots, x_n)\right).$

Now, we will discuss OWA operators from different points of view.

1) Recall that, for a normed weighting vector $\mathbf{w} \in [0,1]^n$, the weighted arithmetic mean $W_{\mathbf{w}} : [0,1]^n \to [0,1]$ is an idempotent aggregation function given by

$$W_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i \, x_i. \tag{2}$$

Thus OWA operators, compare formula (1), can be seen as weighted arithmetic means applied to reordered input vectors,

$$\mathrm{OWA}_{\mathbf{w}}(x_1, \ldots, x_n) = W_{\mathbf{w}}\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$$

with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$. More, OWAs can be seen as symmetrized weighted arithmetic means. Note that following [3], [12], each aggregation function $A : [0,1]^n \to [0,1]$ can be symmetrized into a related aggregation function $A_s : [0,1]^n \to [0,1]$, $A_s(x_1, \ldots, x_n) = A\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$, where σ is a permutation as considered above, i.e., $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$. Hence $\mathrm{OWA}_{\mathbf{w}} = (W_{\mathbf{w}})_s$.

2) Choquet integral was introduced by Choquet in 1953 [5], although there are some earlier predecessors, see, e.g., Vitali's approach dated to 1925 [31]. When dealing with a finite universe $X = \{1, \ldots, n\}$, functions $f : X \to [0,1]$ can be identified with vectors $\mathbf{x} \in [0,1]^n$, $x_i = f(i), i = 1, \ldots, n$. A capacity (fuzzy measure) $m : 2^X \to [0,1]$ is a monotone set function constrained by the two boundary conditions, $m(\emptyset) = 0, m(X) = 1$.

Definition 2.2 ([12]): For a given vector $\mathbf{x} \in [0,1]^n$ and capacity m on X the corresponding Choquet integral is given by

$$\mathbf{Ch}_m(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)} \left(m(E_{\sigma,i}) - m(E_{\sigma,i-1})\right) \tag{3}$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}, E_{\sigma,0} = \emptyset$, and for $i = 1, \ldots, n,\ E_{\sigma,i} = \{\sigma(1), \ldots, \sigma(i)\}$.

As already observed in the case of formula (1), also in the case of formula (3) it may happen, that the permutation σ is not unique. This fact does not hurt the correctness of formula (3). As observed by Grabisch [11], formulae (1) and (3) may coincide for each $\mathbf{x} \in [0,1]^n$ if and only if $m(E_{\sigma,i})$ does not depend on the considered permutation σ. This means that only $\mathrm{card}, E_{\sigma,i} = i$ matters, i.e., $m(E) = m(\sigma(E))$ for any $E \in 2^X$ and permutation σ, $\sigma(E) = \{\sigma(i) |\, i \in E\}$. Such capacities are called symmetric. Now, it is enough to put $w_i = m(E_{\sigma,i}) - m(E_{\sigma,i-1})$ to see that

$$\mathrm{OWA}_{\mathbf{w}} = \mathbf{Ch}_m. \tag{4}$$

Vice-versa, for any normed weighting vector $\mathbf{w}$, it is enough to define a symmetric capacity $m : 2^X \to [0,1]$ by

$$m(E) = \sum_{i=1}^{\mathrm{card} E} w_i \tag{5}$$

to see the representation (4).

3) Schmeidler [27] has characterized the Choquet integral by means of the comonotone additivity. Recall that two vectors $\mathbf{x}, \mathbf{y} \in [0,1]^n$ are comonotone whenever there is a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ so that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ as well as $y_{\sigma(1)} \geq \cdots \geq y_{\sigma(n)}$.

Proposition 2.3: [27]

Let $A : [0,1]^n \to [0,1]$ be an aggregation function. Then, the following are equivalent.

1) A is comonotone additive, i.e., for each comonotone vectors $\mathbf{x}, \mathbf{y} \in [0,1]^n$ satisfying $\mathbf{x} + \mathbf{y} \in [0,1]^n$ it holds

$$A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}).$$

2) A is Choquet integral, $A = \mathbf{Ch}_m$, where the capacity $m : 2^X \to [0,1]$ is given by $m(E) = A(\mathbf{1}_E)$,

$$\mathbf{1}_E(i) = \begin{cases} 1, & \text{if } i \in E \\ 0, & \text{otherwise.} \end{cases}$$

Considering the symmetry of OWA operators and the representation (4), we have the next corollary, see also [12].

Corollary 2.4: Let $A : [0,1]^n \to [0,1]$ be an aggregation function. Then, the following are equivalent:

1) A is symmetric and comonotone additive;
2) A is OWA operator, $A = \mathrm{OWA}_{\mathbf{w}}$, where the normed weighting vector $\mathbf{w} \in [0,1]^n$ is given by $w_i = A(\underbrace{1, \ldots, 1}_{i-\text{times}}, 0, \ldots, 0) - A(\underbrace{1, \ldots, 1}_{(i-1)-\text{times}}, 0, \ldots, 0)$.

Observe also that the comonotone additivity of an aggregation function forces its positive homogeneity, and thus, the positive homogeneity can be considered as a genuine property of OWA operators.

III. Basic Generalizations of Ordered Weighted Average Operators

There are several classes of weighted aggregation functions, for an overview see, e.g., [12]. For any weighted aggregation function $A_{\mathbf{w}} : [0,1]^n \to [0,1]$, related to a weighting vector $\mathbf{w} \in \mathbb{R}_+^n$ (in general, $\mathbf{w}$ need not be normed), one can consider its symmetrization $(A_{\mathbf{w}})_s : [0,1]^n \to [0,1]$,

$$(A_{\mathbf{w}})_s(x_1, \ldots, x_n) = A_{\mathbf{w}}\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$$

as a generalization of OWA operators. Here, as also in the remainder of the paper, $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is an arbitrary permutation such that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ (i.e., σ is induced by the input vector $\mathbf{x} = (x_1, \ldots, x_n)$).

In particular, when considering weighted quasi-arithmetic mean $(A_g)_{\mathbf{w}} : [0,1]^n \to [0,1]$, defined, for a continuous strictly monotone function $g : [0,1] \to [-\infty, \infty]$ by

$$(A_g)_{\mathbf{w}}(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i\, g(x_i)\right)$$

the related OWA generalization $B_{g,\mathbf{w}} = \left((A_g)_{\mathbf{w}}\right)_s : [0,1]^n \to [0,1]$ is given by

$$B_{g,\mathbf{w}}(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i\, g(x_{\sigma(i)})\right) \tag{6}$$

$\mathbf{w} \in [0,1]^n$ being a normed weighting vector.

Note that, when considering the extension of classical $+$ and $\cdot$ operations to the extended real line, we adopt a convention $+\infty + (-\infty) = -\infty$ and $0.\infty = 0$.

Formally, one can extend the OWA operators to act on any interval $[a,b] \subseteq [-\infty, +\infty]$, just applying formula (1). Then, (6) can be rewritten into

$$B_{g,\mathbf{w}}(\mathbf{x}) = g^{-1}\left(\mathrm{OWA}_{\mathbf{w}}(g(\mathbf{x}))\right). \tag{7}$$

For example, considering $g(x) = -\ln x$, one obtains the ordered weighted geometric average $\mathrm{OWGA}_{\mathbf{w}}(\mathbf{x}) = \prod_{i=1}^{n} x_{\sigma(i)}^{w_i}$.

Considering the power functions $g(x) = x^p, p \in \mathbb{R} \setminus \{0\}$, the corresponding generalizations of OWA operator form the class of so-called GOWA (generalized OWA) operators introduced and discussed in [32],

$$\mathrm{GOWA}_{p,\mathbf{w}}(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i\, x_{\sigma(i)}^{p}\right)^{1/p}.$$

To give another kind of example, recall the weighted maximum $\mathrm{WMax}_{\mathbf{w}} : [0,1]^n \to [0,1]$ defined for a weighting vector $\mathbf{w} \in [0,1]^n$ constrained by $\max\{w_1, \ldots, w_n\} = 1$, and given by

$$\mathrm{WMax}_{\mathbf{w}}(\mathbf{x}) = \max(\min(w_1, x_1), \ldots, \min(w_n, x_n)).$$

Then, the corresponding symmetrized aggregation function $(\mathrm{WMax}_{\mathbf{w}})_s : [0,1]^n \to [0,1]$ is given by

$$(\mathrm{WMax}_{\mathbf{w}})_s(\mathbf{x}) = \max(\min(w_1, x_{\sigma(1)}), \ldots, \min(w_n, x_{\sigma(n)})). \tag{8}$$

Note that if $w_1 \leq \cdots \leq w_n = 1$, then the $\mathrm{OWMax}_{\mathbf{w}}$ (Ordered Weighted Maximum) of Dubois and Prade [7] is recovered.

Till now, we have used the standard reordering procedure when transforming the input vector $(x_1, \ldots, x_n)$ into $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, replacing the projections $P_i, i = 1, \ldots, n, P_i(\mathbf{x}) = x_i$, by the order statistics $\mathrm{OS}_i, i = 1, \ldots, n$, $\mathrm{OS}_i(\mathbf{x}) = x_{\sigma(i)}$. Thus, formally, we have replaced, before aggregating step, the vector $(P_1(\mathbf{x}), \ldots, P_n(\mathbf{x}))$ by the related vector $(\mathrm{OS}_1(\mathbf{x}), \ldots, \mathrm{OS}_n(\mathbf{x}))$. However, we can consider different reordering procedures. For example, considering $(\mathrm{OS}_n(\mathbf{x}), \ldots, \mathrm{OS}_1(\mathbf{x}))$ reordering, and aggregating this new n-tuple by means of weighted arithmetic mean $W_{\mathbf{w}}$, it is immediate that

$$\begin{aligned} W_{\mathbf{w}}(\mathrm{OS}_n(\mathbf{x}), \ldots, \mathrm{OS}_1(\mathbf{x})) &= W_w(x_{\sigma(n)}, \ldots, x_{\sigma(1)}) = \\ &= \mathrm{OWA}_{\mathbf{w}'}(\mathbf{x}) \end{aligned}$$

where $\mathbf{w}' = (w_n, \ldots, w_1)$ is just the reversed normal weighting vector $\mathbf{w}$. A reordering procedure included by another n-tuple $\mathbf{u} = (u_1, \ldots, u_n)$, not dependent on $\mathbf{x}$, was proposed in [33] and [34]. Here, the permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ yielding $u_{\tau(1)} \geq \cdots \geq u_{\tau(n)}$ is considered and, the induced reordering of the original input vector $\mathbf{x}$ is then $(x_{\tau(1)}, \ldots, x_{\tau(n)})$. Then

$$W_{\mathbf{w}}(x_{\tau(1)}, \ldots, x_{\tau(n)}) = \sum_{i=1}^{n} w_i\, x_{\tau(i)}$$

is called an Induced OWA operator, IOWA, and it is well defined once there are no ties in the order inducing vector $\mathbf{u}$, i.e., when τ is unique. If there are more acceptable τ's, $\mathrm{IOWA}_{\mathbf{w}}$ is simply the arithmetic mean of all possible $W_{\mathbf{w}}(x_{\tau(1)}, \ldots, x_{\tau(n)})$ (i.e., for each possible τ we have to compute the weighted arithmetic mean of τ-reordered inputs, and these computed values are then aggregated by means of the standard arithmetic mean). Alternatively (but equivalently), one can modify input vector $\mathbf{x}$ considering the arithmetic mean of scores from the tied positions, and then to apply any permutation τ yielding $u_{\tau(1)} \geq \cdots \geq u_{\tau(n)}$.

Some scholars have proposed to combine the aforementioned approaches. Therefore, for example, combining the ideas of GOWA and IOWA operators, one can define Induced Generalized OWA (IGOWA) operators, see [19].

Another streaming in generalizing OWAs is linked to the penalty-based approach [4] to weighted arithmetic means. Indeed, for an arbitrary input vector $\mathbf{x}$, $W_{\mathbf{w}}(\mathbf{x})$ is the minimizer of the penalty function $L_{\mathbf{w},\mathbf{x}}(r) = \sum_{i=1}^{n} w_i (x_i - r)^2$. Generalized OWA operators, dealing with replacing of weights w_i by weighting functions $g_i : [0,1] \to [0, \infty[$, i.e., looking for minimizers of a penalty function

$$H_{\mathbf{g},\mathbf{x}}(r) = \sum_{i=1}^{n} g_i(x_{\sigma(i)})(x_{\sigma(i)} - r)^2$$

are considered, for example, in [22]. For several other proposals in this line, we recall [30].

IV. Integral-Based Generalizations of Ordered Weighted Average Operators

Recall that a symmetric capacity $m : 2^X \to [0,1]$ is given by $m(E) = v_{\mathrm{card} E}$, where $\mathbf{v} \in [0,1]^n$ is a vector of cumulative weights assigned to a normed weighting vector $\mathbf{w} \in [0,1]^n, v_1 = w_1, v_2 = w_1 + w_2, \ldots, v_n = w_1 + \cdots + w_n$. Recall once more that then $\mathrm{OWA}_{\mathbf{w}} = \mathbf{Ch}_m$. Fixing the symmetric capacity m (linked to $\mathbf{v}$), any integral with respect to m can be seen as a generalization of OWA operators. Therefore, for example, one can consider the Sugeno integral $\mathbf{Su}_m : [0,1]^n \to [0,1]$, see [29], which is given by

$$\mathbf{Su}_m(\mathbf{x}) = \max(\min(x_{\sigma(1)}, v_1), \ldots, \min(x_{\sigma(n)}, v_n)). \tag{9}$$

Comparing (9) and (8), we see that $\mathbf{Su}_m = \mathrm{OWMax}_{\mathbf{w}}$, i.e., Sugeno integral-based generalization of OWAs is just the ordered weighted maximum [7]. Observing that there are many different kinds of integrals, we restrict our considerations to integrals which are positively homogeneous only. Two typical classes of such integrals are decomposition integrals [9] and superdecomposition integrals [20]. Note that some of these integrals do not satisfy the boundary conditions for integrals. As an example, consider the concave integral $\mathbf{Cav}_m : [0,1]^n \to \mathbb{R}_+$ introduced by Lehrer in [18] as

$$\mathbf{Cav}_m(\mathbf{x}) = \max\left(\sum_{E\subseteq X} a_E\ m(E) \big| \sum_{E\subseteq X} a_E\ \mathbf{1}_E \le \mathbf{x} \right)$$

considering nonnegative values a_E only.

For the greatest capacity $m^* : 2^X \to [0,1]$ given by $m^*(E) = \begin{cases} 0, & \text{if } E = \emptyset \\ 1, & \text{otherwise} \end{cases}$, it is not difficult to check that $\mathbf{Cav}_{m^*}(\mathbf{x}) = \sum_{i=1}^n x_i$. Obviously, for ranking purposes, the possible defect $\mathbf{Cav}_m(1,\ldots,1) > 1$ does not cause any problem. Moreover, it would be possible to deal with a function $\frac{\mathbf{Cav}_m}{\mathbf{Cav}_m(1,\ldots,1)} : [0,1]^n \to [0,1]$ which is an aggregation function.

Example 4.1: Consider the concave integral and a normed weighting vector $\mathbf{w} = (0,1,0)$, i.e., $\mathbf{v} = (0,1,1)$. Obviously, the standard OWA operator $\mathrm{OWA}_{\mathbf{w}} : [0,1]^3 \to [0,1]$ is given by $\mathrm{OWA}_{\mathbf{w}}(x_1,x_2,x_3) = x_{\sigma(2)}$, i.e., it is the median operator. Considering the related $\mathbf{Cav}_m : [0,1]^3 \to \mathbb{R}_+$, it holds $\mathbf{Cav}_m(1,1,1) = \frac{3}{2}$, and for the aggregation function $\frac{2}{3}\mathbf{Cav}_m : [0,1]^3 \to [0,1]$ we have

$$\frac{2}{3}\mathbf{Cav}_m(x_1,x_2,x_3) = \\ = \begin{cases} \dfrac{x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)}}{3}, & \text{if } x_{\sigma(1)} \le x_{\sigma(2)} + x_{\sigma(3)} \\ \dfrac{2}{3}(x_{\sigma(2)} + x_{\sigma(3)}), & \text{otherwise.} \end{cases}$$

To avoid the necessity of normalization of decomposition (superdecomposition) integrals, we will consider only a hierarchical family of these integrals introduced in [17] and [23]; see also [9] and [24].

Definition 4.2: Let $m : 2^X \to [0,1]$ be a capacity and let $i \in \{1,2,\ldots,2n-1\}$. A product-based integral $I_m^{(i)} : [0,1]^n \to [0,1]$ is given by

1) *if* $i \in \{1,\ldots,n\}$ *then*

$$I_m^{(i)}(\mathbf{x}) = \max\left(\sum_{j=1}^{i} x_{\sigma(k_j)} \cdot (m(E_{\sigma,k_j}) - m(E_{\sigma,k_{j-1}})) \big| \\ 1 \le k_1 < \cdots < k_i = n \right) \tag{10}$$

with convention $k_0 = 0$ (for definition of E_{σ,k_j}, see the introduction of the Choquet integral in Section II);

2) *if* $i \in \{n,\ldots,2n-1\}$ *then*

$$I_m^{(i)}(\mathbf{x}) = \min\Big(\sum_{j=1}^{2n-i} x_{\sigma(k_j)} \cdot (m(E_{\sigma,k_{j+1}-1}) \\ - m(E_{\sigma,k_{j-1}}))\big| 1 = k_1 < \cdots < k_{2n-i} \le n \Big) \tag{11}$$

with convention

$$k_{2n-i+1} = \begin{cases} \min(r | x_{\sigma(r)} = 0), & \text{if } x_{\sigma(n)} = 0 \\ n+1, & \text{otherwise.} \end{cases}$$

Observe that, for $i = n$, applying any of formulae (10) or (11), the Choquet integral is recovered, $I_m^{(n)} = \mathbf{Ch}_m$. Moreover, $I_m^{(1)}$ is Shilkret integral [28],

$$I_m^{(1)}(\mathbf{x}) = \max(x_{\sigma(1)}\ m(E_{\sigma,1}), \ldots, x_{\sigma(n)}\ m(E_{\sigma,n})).$$

On the other hand,

$$I_m^{(2n-1)}(\mathbf{x}) = x_{\sigma(1)}\ m(\mathrm{supp}\ (\mathbf{x}))$$

where $\mathrm{supp}\ (\mathbf{x}) = \{i \in X | x_i > 0\}$. Observe that, in general, it holds

$$I_m^{(1)} \le I_m^{(2)} \le \cdots \le I_m^{(2n-1)}.$$

Definition 4.2 allows us to define a hierarchical family of OWAs generalizations as follows, see also [24]. In the next definition, $\mathrm{OWA}_{\mathbf{w}}{}^{(i)}$ is based on $I_m^{(i)}$, $i = 1,2,\ldots,2n-1$.

Definition 4.3: Let $\mathbf{w} \in [0,1]^n$ be a normed weighting vector and let $\mathbf{v} \in [0,1]^n$ be the related cumulative weighting vector. For $i \in \{1,2,\ldots,2n-1\}$, the OWA generalizations $\mathrm{OWA}_{\mathbf{w}}{}^{(i)} : [0,1]^n \to [0,1]$ are given as follows:

1) if $i \in \{1,\ldots,n\}$,

$$\mathrm{OWA}_{\mathbf{w}}{}^{(i)}(\mathbf{x}) = \\ = \max\left\{ \sum_{j=1}^{i} x_{(k_j)} \cdot (v_{k_j} - v_{k_{j-1}}) \big| \\ 1 \le k_1 < \cdots < k_i \le n \right\} \\ = \max\left\{ \sum_{j=1}^{i} \left(\sum_{r=k_{j-1}+1}^{k_j} w_r \right) \cdot x_{(k_j)} \big| \\ 1 \le k_1 < \cdots < k_i \le n \right\}$$

2) if $i \in \{n,\ldots,2n-1\}$,

$$\mathrm{OWA}_{\mathbf{w}}{}^{(i)}(\mathbf{x}) = \\ = \min\left\{ \sum_{j=1}^{2n-i} x_{(k_j)} \cdot (v_{k_{j+1}-1} - v_{k_j-1}) \big| \\ 1 = k_1 < \cdots < k_{2n-i} \le n \right\} \\ = \min\left\{ \sum_{j=1}^{2n-i} \left(\sum_{r=k_j}^{k_{j+1}-1} w_r \right) \cdot x_{(k_j)} \big| \\ 1 \le k_1 < \cdots < k_{2n-i} \le n \right\}.$$

Note that $\mathrm{OWA}_{\mathbf{w}}{}^{(n)} = \mathrm{OWA}_{\mathbf{w}}$ is the standard OWA operator and that

$$\mathrm{OWA}_{\mathbf{w}}{}^{(1)} \leq \mathrm{OWA}_{\mathbf{w}}{}^{(2)} \leq \cdots \leq \mathrm{OWA}_{\mathbf{w}}{}^{(2n-1)}.$$

All OWA generalizations introduced in Definition 4.3 can be straightforwardly introduced on any domain $[a,b] \subseteq [-\infty, \infty]$. Moreover, each $\mathrm{OWA}_{\mathbf{w}}{}^{(i)}$ is symmetric, positively homogeneous, and translation invariant.

Example 4.4: Consider the unique additive symmetric capacity $m : 2^X \to [0,1]$, $m(A) = \frac{|A|}{n}$, related to the constant weighting vector $\mathbf{w} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, linked to the cumulative vector $\mathbf{v} = \left(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right)$, and consider the input vector $\mathbf{x} = \left(1, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0\right)$. Then,

$$\mathrm{OWA}_{\mathbf{w}}{}^{(1)}(\mathbf{x}) = \max\left\{\frac{n-j}{n-1} \cdot \frac{j}{n} \,\middle|\, j \in \{1, \ldots, n\}\right\} = \;= \begin{cases} \dfrac{n}{4(n-1)}, & \text{if} \quad n \text{ is even} \\ \dfrac{n+1}{4n}, & \text{if} \quad n \text{ is odd} \end{cases}$$

$$\mathrm{OWA}_{\mathbf{w}}{}^{(2)}(\mathbf{x}) = \begin{cases} \dfrac{n}{3(n-1)}, & \text{if} \quad n = 3k \\ \dfrac{n+1}{3n}, & \text{else} \end{cases}$$

$$\mathrm{OWA}_{\mathbf{w}}{}^{(n)}(\mathbf{x}) = \frac{1}{2}$$

$$\mathrm{OWA}_{\mathbf{w}}{}^{(2n-2)}(\mathbf{x}) = \begin{cases} \dfrac{3n-4}{4n-4}, & \text{if } n \text{ is even} \\ \dfrac{3n-1}{4n}, & \text{if } n \text{ is odd} \end{cases}$$

and

$$\mathrm{OWA}_{\mathbf{w}}{}^{(2n-1)}(\mathbf{x}) = \frac{n-1}{n}.$$

Observe that in all cases, if $n \to \infty$, then the corresponding $\mathrm{OWA}_{\mathbf{w}}$ operators are approaching in limit the corresponding Π-based integral on $X = [0,1]$, see [17], with respect to the standard Lebesque measure λ, and from the identity function $f : X \to [0,1]$, $f(x) = x$. Thus

$$\lim_{n\to\infty} \mathrm{OWA}_{\mathbf{w}}{}^{(1)}(\mathbf{x}) = \frac{1}{4} \quad \text{(Shilkret integral)}$$

$$\lim_{n\to\infty} \mathrm{OWA}_{\mathbf{w}}{}^{(2)}(\mathbf{x}) = \frac{1}{3}$$

$$\lim_{n\to\infty} \mathrm{OWA}_{\mathbf{w}}{}^{(n)}(\mathbf{x}) = \frac{1}{2} \quad \text{(Choquet integral)}$$

$$\lim_{n\to\infty} \mathrm{OWA}_{\mathbf{w}}{}^{(2n-2)}(\mathbf{x}) = \frac{3}{4}$$

and

$$\lim_{n\to\infty} \mathrm{OWA}_{\mathbf{w}}{}^{(2n-1)}(\mathbf{x}) = 1.$$

V. Axiomatic Generalizations of Ordered Weighted Average Operators

As already observed in Section II, OWA operators can be characterized as comonotone additive symmetric aggregation functions on $[0,1]$. Moreover, each OWA operator is idempotent, $\mathrm{OWA}_{\mathbf{w}}(c, \ldots, c) = c$ for each $c \in [0,1]$ and for any normed weighted vector $\mathbf{w}$, as well as positively homogeneous, $\mathrm{OWA}_{\mathbf{w}}(c\,\mathbf{x}) = c\,\mathrm{OWA}_{\mathbf{w}}(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^n$ and $c \geq 0$ such that also $c\,\mathbf{x} \in [0,1]^n$.

Note that the additivity property of aggregation functions can be generalized into the modularity. We say that an aggregation function $A : [0,1]^n \to [0,1]$ is modular whenever

$$A(\mathbf{x} \vee \mathbf{y}) + A(\mathbf{x} \wedge \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}) \tag{12}$$

for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$, where the join

$$\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$$

and the meet

$$\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n)).$$

Each idempotent modular aggregation function $A : [0,1]^n \to [0,1]$ can be represented in the form

$$A(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i) \tag{13}$$

where $f_i : [0,1] \to [0,1]$, $i = 1, \ldots, n$, is a nondecreasing function and $\sum_{i=1}^{n} f_i(x) = x$ for each $x \in [0,1]$.

Hence, each f_i is necessarily 1- Lipschitz, $|f_i(x) - f_i(y)| \leq |x - y|$, and $f_i(0) = 0$. Evidently, adding the positive homogeneity, we have an alternative axiomatic definition of weighted arithmetic means (then $f_i(x) = w_i\,x$). Therefore, also the next alternative axiomatic characterization of OWAs holds.

Proposition 5.1: Let $A : [0,1]^n \to [0,1]$ be an aggregation function. Then, the following are equivalent:

1) A is an OWA operator;
2) A is comonotone modular, symmetric, and positively homogeneous.

Omitting the positive homogeneity results into our first axiomatic generalization of OWA operators, compare also [21].

Definition 5.2: Let A be a symmetric idempotent comonotone modular aggregation function. Then, A is called Ordered Modular Average operator (OMA).

Due to [21], we have the next important result.

Theorem 5.3: Let $A : [0,1]^n \to [0,1]$ be a function. Then, the following are equivalent:

1) A is an OMA operator;
2) There are 1-Lipschitz nondecreasing functions $f_1, \ldots, f_n : [0,1] \to [0,1]$, $\sum_{i=1}^{n} f_i(x) = x$ for each $x \in [0,1]$, and

$$A(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_{\sigma(i)}).$$

Moreover, there is a link between OMA operators and copula-based integrals with respect to symmetric capacities. We will not go more into details, for the interested readers we recommend [21].

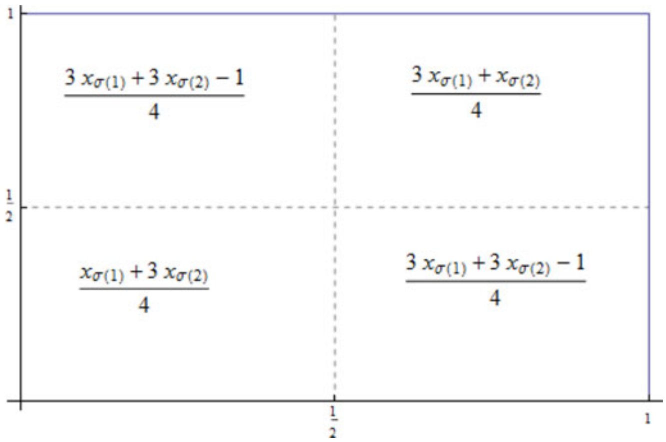

Fig. 1. Formulae for $\mathrm{OMA}_{(f_1,f_2)}$ from Example 5.4.

Example 5.4: For $n = 2$, define $f_1, f_2 : [0,1] \to [0,1]$ by

$$f_1(x) = \max\left(\frac{x}{4}, \frac{3x-1}{4}\right), f_2(x) = \min\left(\frac{3x}{4}, \frac{x+1}{4}\right).$$

Then, the corresponding OMA operator $\mathrm{OMA}_{(f_1,f_2)} : [0,1]^2 \to [0,1]$ is given by

$$\mathrm{OMA}_{(f_1,f_2)}(x_1,x_2) = f_1(x_{\sigma(1)}) + f_2(x_{\sigma(2)})$$

and it is depicted in Fig. 1 .

Observe that

$$\mathrm{OMA}_{(f_1,f_2)}|[0,0.5]^2 = \mathrm{OWA}_{(\frac{1}{4},\frac{3}{4})}|[0,0.5]^2$$

and

$$\mathrm{OMA}_{(f_1,f_2)}|[0.5,1]^2 = \mathrm{OWA}_{(\frac{3}{4},\frac{1}{4})}|[0.5,1]^2.$$

Note that $\mathrm{OMA}_{(f_1,f_2)}$ can be seen as an ordinal sum of two OWA operators, namely of $\mathrm{OWA}_{(\frac{1}{4},\frac{3}{4})}$ acting on $[0,0.5]$ and of $\mathrm{OWA}_{(\frac{3}{4},\frac{1}{4})}$ acting on $[0.5,1]$, as proposed by De Baets and Mesiar in [6]. Moreover, $\mathrm{OMA}_{(f_1,f_2)}$ can be seen as a level dependent capacity M-based Choquet integral introduced by Greco *et al.* [14], with level dependent capacity $M : [0,1] \times 2^{\{1,2\}}$ given by

$$M(t,E) = \begin{cases} m_1(E), & \text{if } t \in [0,0.5] \\ m_2(E), & \text{otherwise} \end{cases}$$

Here, m_1, m_2 are symmetric capacities related to normed weighting vectors $\left(\frac{1}{4},\frac{3}{4}\right)$ and $\left(\frac{3}{4},\frac{1}{4}\right)$, respectively.

Another possible axiomatic generalization of OWAs is based on the idea to replace the additivity by pseudoadditivity. Recall that an operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called a pseudoaddition whenever it is monotone, symmetric, associative, continuous, and 0 is its neutral element. More details concerning the pseudoadditions can be found in [16] and [25]. For our purpose, it is enough to observe that each pseudo-addition $\oplus$ can be represented as an ordinal sum of generated pseudoadditions, $\oplus = (\langle a_k, b_k, \varphi_k\rangle | k \in \mathcal{K})$, where $\{]a_k, b_k[\,|\, k \in \mathcal{K}\}$ is a disjoint system of open subintervals of $[0,\infty]$, and $\varphi_k : [a_k, b_k] \to [0,\infty]$, $k \in \mathcal{K}$, are continuous strictly monotone functions satisfying $\varphi_k(a_k) = 0$. Then,

$$x \oplus y =$$
$$= \begin{cases} \varphi_k^{-1}\left(\min(\varphi_k(b_k), \varphi_k(x) + \varphi_k(y))\right), & \text{if } (x,y) \in]a_k, b_k[^2 \\ & \text{for some } k \in \mathcal{K} \\ \max(x,y), & \text{otherwise.} \end{cases}$$

Definition 5.5: Let $\oplus : [0,\infty]^2 \to [0,\infty]$ be a given pseudoaddition, $\oplus = (\langle a_k, b_k, \varphi_k\rangle | k \in \mathcal{K})$. An idempotent symmetric aggregation function $A : [0,1]^2 \to [0,1]$ is called a $\oplus$-OWA operator whenever it is comonotone pseudoadditive, i.e., if for any comonotone pair $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \oplus \mathbf{y} = (x_1 \oplus y_1, \ldots, x_n \oplus y_n) \in [0,1]^n$ it holds

$$A(\mathbf{x} \oplus \mathbf{y}) = A(\mathbf{x}) \oplus A(\mathbf{y}). \tag{14}$$

For generated pseudoadditions, $\oplus = (\langle 0, \infty, \varphi\rangle)$, it is not difficult to check that OWA generalization given by formula (7) is obtained, i.e., A is a $\oplus$-OWA operator if and only if $A = B_{\varphi,\mathbf{w}}$, where the normed weighting vector $\mathbf{w}$ is given by

$$\begin{aligned} w_1 &= \frac{\varphi(A(1,0,\ldots,0))}{\varphi(1)} \\ w_2 &= \frac{\varphi(A(1,1,0,\ldots,0)) - \varphi(A(1,0,\ldots,0))}{\varphi(1)} \\ &\vdots \\ w_n &= \frac{\varphi(A(1,\ldots,1)) - \varphi(A(1,\ldots,1,0))}{\varphi(1)}. \end{aligned}$$

Thus, in this case, a generalization discussed already in Section III is recovered.

Similar results are obtained when $\oplus$ possesses a summand

$$\langle a_k, b_k, \varphi_k\rangle = \langle 0, b_k, \varphi_k\rangle \text{ with } b_k \geq 1.$$

Another distinguished case is related to $\mathcal{K} = \emptyset$, i.e., $\oplus = \max = \vee$. Then, each $\vee$-OWA operator $A : [0,1]^n \to [0,1]$ is given as

$$A(\mathbf{x}) = \max\left(f_1(x_{\sigma(1)}), \ldots, f_n(x_{\sigma(n)})\right) \tag{15}$$

where $f_1, \ldots, f_n : [0,1] \to [0,1]$ are increasing functions satisfying $\max(f_1(x), \ldots, f_n(x)) = x$ for each $x \in [0,1]$. Note that formula (15) generalizes the symmetric weighted maximum $(\mathrm{WMax}_{\mathbf{w}})_s$ given in formula (8). Observe that the same results are obtained whenever $x \oplus y = \max(x,y)$ for all $x, y \in [0,1]$.

Example 5.6: For $n = 2$ and $k \in\,]0,1]$, consider $f_1, f_2 : [0,1] \to [0,1]$ given by

$$f_1(x) = \min(x,k),\ \ f_2(x) = \begin{cases} 0, & \text{if } x \leq k \\ x, & \text{otherwise} \end{cases}$$

Then, the $\vee$-OWA operator $A : [0,1]^2 \to [0,1]$ given by formula (15) is just k-median (idempotent nullnorm with annihilator k), see [10] and [12],

$$A(x_1,x_2) = \mathrm{med}(x_1,k,x_2).$$

Rather nontrivial generalizations of OWAs are obtained whenever there is a summand $\langle a_k, b_k, \varphi_k\rangle$ of $\oplus$ such that $a_k \in]0,1[$ or $b_k \in]0,1[$.

Theorem 5.7: Consider the pseudoaddition $\oplus = (\langle \frac{1}{2}, \infty, \varphi\rangle)$, where $\varphi : [\frac{1}{2}, \infty][0,\infty]$ is given by $\varphi(x) = x - \frac{1}{2}$. Then,

$$x \oplus y = \begin{cases} \max(x,y), & \text{if } \min(x,y) \le \dfrac{1}{2} \\ x + y - \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

An aggregation function $A : [0,1]^2 \to [0,1]$ is a $\oplus$-OWA operator if and only if there is a normed weighting vector $\mathbf{w} = (w_1, 1-w_1)$, nondecreasing functions $f_1, f_2, g : [0,\frac{1}{2}] \to [0,1]$ such that $\max(f_1(t), f_2(t)) = t$ for all $t \in [0, \frac{1}{2}]$, and

$$g(t) \ge \max\left(f_1\left(\frac{1}{2}\right), f_2(t)\right), \quad g\left(\frac{1}{2}\right) = \frac{w_1+1}{2}$$

and

$$A(x_1,x_2) = \\ = \begin{cases} \max\left(f_1(x_{\sigma(1)}), f_2(x_{\sigma(2)})\right), & \text{if } (x_1,x_2) \in [0,\frac{1}{2}]^2 \\ w_1\, x_{\sigma(1)} + (1-w_1)\, x_{\sigma(2)}, & \text{if } (x_1,x_2) \in [\frac{1}{2},1]^2 \\ g(x_{\sigma(2)}), & \text{if } x_{\sigma(2)} < \frac{1}{2} < x_{\sigma(1)} \\ & \text{and } g(x_{\sigma(2)}) \le \frac{1}{2} \\ 1 - g(x_{\sigma(2)}) + x_{\sigma(1)} & \\ \left(2g(x_{\sigma(2)}) - 1\right), & \text{otherwise.} \end{cases}$$

Proof: On domain $[0,\frac{1}{2}]^2$, the proof can be adopted from the representation of $\vee$-OWA operators, see formula (15), forcing the properties of functions f_1, f_2. Similarly, on domain $[\frac{1}{2},1]^2$ the standard OWA can be easily recognized.

Monotonicity of $\oplus$-OWAs forces the constraints for the function g. Now, consider, for example, $g(x) \le \frac{1}{2}$ for some $x \in [0, \frac{1}{2}[$. Then, $g(x) = A(x,1)$ and due to comonotone $\oplus$-additivity

$$A\left(x, \frac{3}{4}\right) \oplus A\left(x, \frac{3}{4}\right) \\ = \max\left(A\left(x, \frac{3}{4}\right), A\left(x, \frac{3}{4}\right)\right) = g(x)$$

i.e., $A\left(x, \frac{3}{4}\right) = g(x)$. Similarly

$$A\left(x, \frac{5}{8}\right) = A\left(x, \frac{9}{16}\right) = \cdots = A\left(x, \frac{2^n+1}{2^{n+1}}\right) = g(x)$$

and due to the monotonicity of A, $A(x,y) = g(x)$ for all $y \in]\frac{1}{2}, 1]$.

On the other hand, if $g(x) > \frac{1}{2}$ for some $x \in [0, \frac{1}{2}[$, then

$$A\left(x, \frac{3}{4}\right) \oplus A\left(x, \frac{3}{4}\right) = 2A\left(x, \frac{3}{4}\right) - \frac{1}{2} = g(x) \text{ i.e.}$$

$$A\left(x, \frac{3}{4}\right) = \frac{1}{2}\left(g(x) + \frac{1}{2}\right) = 1 - g(x) \\ + \frac{3}{4}\left(2g(x) - 1\right) > \frac{1}{2}.$$

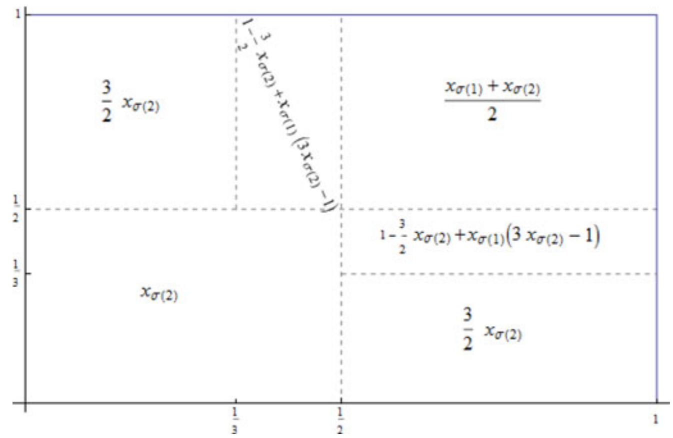

Fig. 2. Formulae describing the $\oplus$-OWA operator A from Example 5.8.

By induction, for any $n \in \mathbb{N}$ one can show that

$$A\left(x, \frac{2^n+1}{2^{n+1}}\right) = 1 - g(x) + \frac{2^n+1}{2^{n+1}}\left(2g(x) - 1\right) > \frac{1}{2}.$$

On the other hand

$$\left(x, \frac{2^n+1}{2^{n+1}}\right) \oplus \left(x, \frac{2^n+1}{2^{n+1}}\right) = \left(x, \frac{2^n+1}{2^{n+1}}\right) \text{ and thus}$$

$$A\left(x, \frac{2^n+1}{2^{n+1}}\right) = \left(x, \frac{2^n+1}{2^{n+1}}\right) \oplus \left(x, \frac{2^n+1}{2^{n+1}}\right) = \\ = \left(x, \frac{2^n+1}{2^{n+1}}\right) = 2 - 2g(x) + \frac{2^n+1}{2^n}\left(2g(x)-1\right) - \frac{1}{2} = \\ = 1 - g(x) + \frac{2^n+1}{2^{n+1}}\left(2g(x)-1\right).$$

By induction,

$$A\left(x, \frac{2^n+i}{2^{n+1}}\right) = 1 - g(x) + \frac{2^n+i}{2^{n+1}}\left(2g(x)-1\right)$$

for $i = 1, 2, \ldots, 2^n$, and due to the monotonicity of A,

$$A(x,y) = 1 - g(x) + y(2g(x) - 1) \text{ for all } y \in \left]\frac{1}{2}, 1\right].$$

■

Example 5.8. To illustrate Theorem 5.7, consider

$$f_1(t) = 0, f_2(t) = t, t \in \left[0, \frac{1}{2}\right], \text{ and } g(t) = \frac{3}{2}t, t \in \left[0, \frac{1}{2}\right].$$

The corresponding $\oplus$-OWA operator A is depicted in Fig. 2 . Observe that

$$g\left(\frac{1}{2}\right) = \frac{3}{4} = A\left(\frac{1}{2}, 1\right) = w_1 + \frac{1-w_1}{2} \\ = \frac{1+w_1}{2}, \text{ i.e., } w_1 = \frac{1}{2}$$

and fixing $x_{\sigma(2)} = \frac{1}{2}$,

$$\begin{aligned} 1 - \frac{3}{2}x_{\sigma(2)} + x_{\sigma(1)}(3x_{\sigma(2)} - 1) &= 1 - \frac{3}{4} + x_{\sigma(1)} \cdot \frac{1}{2} \\ &= \frac{x_{\sigma(1)} + x_{\sigma(2)}}{2}. \end{aligned}$$

Remark 5.9: Observe that the $\oplus$-OWA operator A introduced in Example 5.8 is not continuous. To guarantee the continuity of $\oplus$-OWAs characterized in Theorem 5.7, one should consider $f_1\left(\frac{1}{2}\right) = \frac{1}{2}$ (then $A\left(\frac{1}{2}, t\right) = A\left(t, \frac{1}{2}\right) = \frac{1}{2}$ for $t \in \left[0, \frac{1}{2}\right]$. Then, $w_1 = 0$ and the corresponding $\oplus$-OWA operator $B : [0,1] \to [0,1]$ is given by

$$B(x_1, x_2) = \begin{cases} x_{\sigma(1)}, & \text{if } (x_1, x_2) \in \left[0, \frac{1}{2}\right]^2 \\ x_{\sigma(2)}, & \text{if } (x_1, x_2) \in \left[\frac{1}{2}, 1\right]^2 \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

i.e., $B(x_1, x_2) = \mathrm{med}\left(x_1, \frac{1}{2}, x_2\right)$ is the $\frac{1}{2}$-median, see [10].

VI. Concluding Remarks

Based on three different looks on OWAs, we have discussed possible generalizations of OWA operators. In several cases, already known generalizations were obtained. Our approach has shown the roots of these generalizations. In some cases, completely new types of generalizations were introduced, see for example Theorem 5.7. Our approach opens the doors for several new kinds of OWA generalizations, considering, for example, new kinds of integrals, such as symmetric level dependent capacities based integrals [15], or recent generalizations of integrals based on (symmetric) aggregation functions as proposed in [13]. As an interesting example from [13], recall the superadditive integral based on median operator $\mathrm{med} : [0,1]^3 \to [0,1]$ (i.e., an OWA operator related to the normed weighting vector $\mathbf{w} = (0,1,0)$), which yields an OWA generalization $\mathrm{med}^* : [0,1]^3 \to \mathbb{R}_+$ given by

$$\mathrm{med}^*(x_1, x_2, x_3) = \min\left(\frac{x_{(1)} + x_{(2)} + x_{(3)}}{2}, x_{(2)} + x_{(3)}\right)$$

compare also Example 4.1.

Note that similarly as in the case of some integrals discussed in Section IV, one can normalize med^* by a factor $\mathrm{med}^*(1,1,1) = \frac{3}{2}$, and then the related aggregation function $D : [0,1]^3 \to [0,1]$ is given by

$$D(x_1, x_2, x_3) = \min\left(\frac{x_{(1)} + x_{(2)} + x_{(3)}}{3}, \frac{2(x_{(2)} + x_{(3)})}{3}\right).$$

Observe that for any $\mathbf{x} \in [0,1]^3$, $|D(\mathbf{x}) - \mathrm{med}(\mathbf{x})| \leq \frac{1}{3}$.

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