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Normally Admissible Stratifications and Calculation of Normal Cones to a Finite Union of Polyhedral Sets

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Abstract This paper considers computation of Fréchet and limiting normal cones to a finite union of polyhedra. To this aim, we introduce a new concept of normally admissible stratification which is convenient for calculations of such cones and provide its basic properties. We further derive formulas for the above mentioned cones and compare our approach to those already known in the literature. Finally, we apply this approach to a class of time dependent problems and provide an illustration on a special structure arising in delamination modeling.

Keywords Union of polyhedral sets · Tangent cone · Fréchet normal cone · Limiting normal cone · Normally admissible stratification · Time dependent problems · Delamination model

Mathematics Subject Classifications (2010) 90C31 · 90C33 · 65K10 · 34D20 · 37C75

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1 Introduction

In the past few decades, applied mathematicians have paid a lot of attention to optimization and optimal control problems with various types of nonconvex constraints. In the variational geometry of nonconvex sets, the so-called tangent (Bouligand-Severi, contingent) cone, regular (Fréchet) normal cone and limiting (Mordukhovich) normal cone play important role in the study of optimization and optimal control, such as optimality conditions, related constraint qualifications, stability analysis etc., see [25] for theory in finite dimensions and [18, 19] for analysis in infinite-dimensional spaces. All cones mentioned above enjoy calculus rules that may simplify their calculations. However, in many cases, calculus provides only approximation (inclusion) which may not be useful for further analysis. Thus, exact computation for even trivial nonconvex set may become a very technical and lengthy procedure.

In this paper we focus on computation of normal cones to a finite–dimensional set Γ, which is a union of finitely many (convex) polyhedra. By polyhedron we understand a finite intersection of halfspaces, which is always closed and convex. Such sets naturally arise whenever a parameterized generalized equation

$$0 \in F(u, x) + G(u, x) \tag{1}$$

is considered with a continuously differentiable function $F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$, a polyhedral multifunction $G : \mathbb{R}^d \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a parameter or control variable u and a state variable x. Defining the solution map $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ associated with (1) as

$$S(u) = \{x \mid 0 \in F(u, x) + G(u, x)\},$$

one may intend to compute a generalized derivative of the solution map S. This is often connected with evaluation of some of the above mentioned cones to $\Gamma := \operatorname{gph} G$. Since G is a polyhedral multifunction, Γ is indeed a union of a finite number of polyhedra.

The computation of a generalized derivative is useful whenever we are interested in performing stability and sensitivity analysis of S or whenever we intend to solve a hierarchical problem constrained by system (1). This is the case of mathematical programs with equilibrium constraints such as the so-called disjunctive programs [10]. The latter class of (parameterized) programs includes, e.g., bilevel problems with linear constraints on the lower level [6], mathematical programs with complementarity constraints [17, 21] or mathematical programs with vanishing constraints [2].

Besides these particular applications, when we consider a polyhedral set C, the graph of the normal cone mapping $\mathrm{N}_C(\cdot)$ in the sense of convex analysis also enjoys the same polyhedral structure, as already observed in [23]. This is naturaly important in many aspects of variational analysis.

There has already been some attempts to provide formulas for normal cones to such sets Γ. In [7], the authors provide formula for the limiting normal cone to $\operatorname{gph} \mathrm{N}_C$, with C polyhedral, in terms of the so-called critical cones and their polars. This special case of a union of polyhedra has also been studied in [13]. In [12], the formula for the fully general case of a union of polyhedra has been provided utilizing the Motzkin's Theorem of the Alternative. There, the authors already build upon the well-known fact that the tangent and normal cones are constant on relative interior of a face of a polyhedral set, result that goes back to Robinson [23]. Additionally to simplified formulas for several special cases, a formula for normal cone to a particular case of a union of non-polyhedral sets is provided in [12]. In all the above mentioned papers, however, the resulting formulas are non-trivial with highly growing complexity with respect to the number of faces.

In this paper, we describe an alternative procedure for computation of full graph of normal cone mappings to Γ along with normal cones at a specific point. For this, we introduce the so-called *normally admissible stratification* of a union of polyhedra in order to generalize the observation of constant-valuedness of tangent and normal cone mappings on certain subsets of a polyhedra. Our results can be considered as a natural generalization of [5] where formulas for tangent and normal cones were derived for a special case of a union of polyhedra with each polyhedral set being a subset of $\{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}\}^n$. We obtain formulas which hold as equalities without any constraint qualification. This seems to be natural for the considered polyhedral setting. However, to the best of our knowledge, such a result cannot be achieved by applying general calculus rules without any additional information.

The article is organized as follows. In Section 2 we provide the definition of a normally admissible stratification of Γ and show that such stratification always exists. Further, we derive formulas for graphs of regular and limiting normal cones to Γ. In Section 3 we compare our procedure to those of Dontchev and Rockafellar [7] and Henrion and Outrata [12]. Finally, in Section 4 we consider an application arising in discretized time-dependent problems [1, 4]. We provide a theoretical background, specifying the form of normally admissible stratifications in this particular class of problems, and illustrate the benefits of our procedure on a special case arising in delamination modeling [26].

Our notation is basically standard. We use $\mathbb{R}_+, \mathbb{R}_-, \mathbb{R}_{++}$ and $\mathbb{R}_{--}$ to denote nonnegative, nonpositive, positive and negative real numbers, respectively. For a set Ω, $\operatorname{cl}\Omega$ and $\operatorname{rint}\Omega$ denote its closure and relative interior, respectively, where relative interior is defined as interior with respect to the smallest affine subspace which contains Ω. We say that Ω is relatively open if $\Omega = \operatorname{rint}\Omega$. For a cone A, A^* stands for its negative polar cone, $\operatorname{span} A$ and $\operatorname{con} A$ refer to the linear and convex conic hull of A, respectively. By $x \xrightarrow{\Omega} \bar{x}$ we mean that $x \to \bar{x}$ with $x \in \Omega$. For scalar product of x and y we use both $x^\top y$ and $\langle x, y\rangle$.

For the readers' convenience we now state the definitions of several basic notions from modern variational analysis. For a set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and some $\bar{x}$ we define Painlevé-Kuratowski upper (outer) limit by

$$\operatorname*{Limsup}_{x\to\bar{x}} M(x) := \{y \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, \exists y_k \to y \text{ with } y_k \in M(x_k)\}.$$

This concept allows us to define the *tangent (contingent, Bouligand-Severi) cone* to $\Omega \subset \mathbb{R}^n$ at $\bar{x}$ as

$$\mathrm{T}_\Omega(\bar{x}) := \operatorname*{Limsup}_{x\xrightarrow{\Omega}\bar{x}} \frac{\Omega - \bar{x}}{t}.$$

For a set Ω at $\bar{x} \in \Omega$ we define the *regular (Fréchet) normal cone* $\hat{\mathrm{N}}_\Omega(\bar{x})$ and *limiting (Mordukhovich) normal cone* $\mathrm{N}_\Omega(\bar{x})$ to Ω as

$$\hat{\mathrm{N}}_\Omega(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \;\middle|\; \operatorname*{limsup}_{x\xrightarrow{\Omega}\bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\| x - \bar{x} \|} \le 0 \right\} = (\mathrm{T}_\Omega(\bar{x}))^*,$$

$$\mathrm{N}_\Omega(\bar{x}) := \operatorname*{Limsup}_{x\xrightarrow{\Omega}\bar{x}} \hat{\mathrm{N}}_\Omega(x).$$

For a convex set Ω, both normal cones $\hat{\mathrm{N}}_\Omega$ and N_Ω amount to the normal cone of convex analysis which is usually denoted by N_Ω. Here, however, in order to stress out the possible generalization of some formulas developed in this manuscript to nonconvex sets, we use $\hat{\mathrm{N}}_\Omega$ even for convex sets Ω.

For a polyhedral set C and some $\bar{x} \in C$ and $\bar{y} \in \mathrm{N}_C(\bar{x})$, the *critical cone* to C at $\bar{x}$ for $\bar{y}$ is defined as

$$K_C(\bar{x}, \bar{y}) := \{w \in \mathrm{T}_C(\bar{x}) |\ w^\top \bar{y} = 0\}.$$

2 Main result

The main goal of this section is to compute $\hat{\mathrm{N}}_\Gamma$ and N_Γ, where $\Gamma \subset \mathbb{R}^n$ is a finite union of polyhedral sets Ω_r for $r = 1, \ldots, R$, that is

$$\Gamma = \bigcup_{r=1}^{R} \Omega_r. \tag{2}$$

In order to compute these normal cones, we will first introduce a convenient partition of Γ which satisfies certain suitable conditions. Next, we show existence of such partition. Finally, we derive formulas for both Fréchet and limiting normal cones to Γ.

Definition 1 We say that $\{\Gamma_s |\ s = 1, \ldots, S\}$ forms a *partition* of Γ if Γ_s are nonempty and pairwise disjoint for all $s = 1, \ldots, S$ and $\cup_{s=1}^{S} \Gamma_s = \Gamma$.

The following definition of normally admissible stratification is based on the strata theory [11, 22] which was developed for general manifolds. In the polyhedral case, we may add additional assumptions such as that stratas Γ_s are relatively open. Note that condition (3) is well–known as the so–called *frontier condition*. Similar partition was proposed in [27] under the term *polyhedral subdivision* with all the partition elements being closed polyhedra of the same dimension as Γ.

Definition 2 We say that $\{\Gamma_s |\ s = 1, \ldots, S\}$ forms a *normally admissible stratification* of Γ if it is a partition of Γ with Γ_s, $s = 1, \ldots, S$ relatively open, convex and $\operatorname{cl}\Gamma_s$ polyhedral such that the following property holds true for all $i, s = 1, \ldots, S$

$$\Gamma_s \cap \operatorname{cl}\Gamma_i \neq \emptyset \implies \Gamma_s \subset \operatorname{cl}\Gamma_i. \tag{3}$$

The term normally admissible stratification is coined in order to reflect the forthcoming Theorem 1 saying that normal cones are constant with respect to this stratification in a particular sense. Next, for a normally admissible stratification of Γ denoted by $\{\Gamma_s |\ s = 1, \ldots, S\}$ we define two index sets which are extensively used throughout the manuscript

$$I(s) := \{i \in \{1, \ldots, S\} |\ \Gamma_s \cap \operatorname{cl}\Gamma_i \neq \emptyset\}, \tag{4a}$$

$$\tilde{I}(s) := \{i \in I(s) |\ \nexists j \in I(s) : \operatorname{cl}\Gamma_i \subsetneq \operatorname{cl}\Gamma_j\} \subset I(s). \tag{4b}$$

Clearly, $I(s)$ has a close connection with (3) and $\tilde{I}(s)$ is composed of such indices of $I(s)$ that correspond to maximal elements of $\{\operatorname{cl}\Gamma_i |\ i \in I(s)\}$ in the sense of subsets. We will often work with the following alternative representations of $\tilde{I}(s)$

$$\tilde{I}(s) = \{i \in I(s) |\ \forall j \in I(s) : \operatorname{cl}\Gamma_i \subset \operatorname{cl}\Gamma_j \implies i = j\} \tag{4c}$$

$$= \{i \in I(s) |\ j \in I(s) \cap I(i) \implies i = j\}. \tag{4d}$$

For a normally admissible stratification, formula (4b) is equivalent to (4c) due to [24, Theorem 6.3]. The equivalence of (4c) and (4d) follows from the fact that $j \in I(i)$ is equivalent to $\Gamma_i \subset \operatorname{cl}\Gamma_j$.

Next, we provide a constructive proof of existence of a normally admissible stratification to Γ.

Lemma 1 *Let $\Gamma \subset \mathbb{R}^n$ be a finite union of polyhedral sets. Then there exists a normally admissible stratification of Γ.*

Proof Consider Γ in the form (2) with Ω_r defined as

$$\Omega_r = \{x \,|\, \langle c_t^r, x\rangle \le b_t^r,\ t = 1, \dots, T(r)\}.$$

We now relabel all c_t^r to c_u, $u = 1, \dots, U$ with $U = \sum_{r=1}^R T(r)$ and similarly for b_u. For $I, J \subset \{1, \dots, U\}$ define the following sets

$$\Omega_{I,J} := \left\{ x \,\middle|\, \begin{array}{ll} \langle c_u, x\rangle < b_u & \text{for } u \in I \\ \langle c_u, x\rangle > b_u & \text{for } u \in J \\ \langle c_u, x\rangle = b_u & \text{for } u \in \{1, \dots, U\} \setminus (I \cup J) \end{array} \right\}, \tag{5}$$

$$\Theta := \left\{(I, J) \,\middle|\, \Omega_{I,J} \neq \emptyset,\ \Omega_{I,J} \subset \Gamma \right\}. \tag{6}$$

We claim that $\{\Omega_{I,J} \,|\, (I, J) \in \Theta\}$ is a normally admissible stratification of Γ.

First, we show that $\{\Omega_{I,J} \,|\, (I, J) \in \Theta\}$ is a partition of Γ. Indeed, if we restrict ourselves to $(I, J) \in \Theta$, then $\Omega_{I,J}$ are nonempty and pairwise disjoint by construction. Moreover, since $\Omega_{I,J} \subset \Gamma$, we have

$$\bigcup_{(I,J)\in\Theta} \Omega_{I,J} \subset \Gamma.$$

To show that the equality holds in the previous relation, choose any $x \in \Gamma$. By construction of sets $\Omega_{I,J}$, there exists exactly one couple (I, J) such that $x \in \Omega_{I,J}$. To show that $(I, J) \in \Theta$, it remains to realize that

$$\Omega_{I,J} \subset \bigcap_{\{r \,|\, x\in\Omega_r\}} \Omega_r \subset \Gamma.$$

Hence, we have shown that $\{\Omega_{I,J} \,|\, (I, J) \in \Theta\}$ is indeed a partition of Γ.

To prove that $\{\Omega_{I,J} \,|\, (I, J) \in \Theta\}$ is a normally admissible stratification of Γ, recall that for all $(I, J) \in \Theta$ we have $\Omega_{I,J}$ nonempty, which allows us to apply Lemma A1 to obtain that $\Omega_{I,J}$ is relatively open and

$$\operatorname{cl}\Omega_{I,J} = \left\{ x \,\middle|\, \begin{array}{ll} \langle c_u, x\rangle \le b_u & \text{for } u \in I \\ \langle c_u, x\rangle \ge b_u & \text{for } u \in J \\ \langle c_u, x\rangle = b_u & \text{for } u \in \{1, \dots, U\} \setminus (I \cup J) \end{array} \right\}.$$

Clearly, $\Omega_{I,J}$ is convex and $\operatorname{cl}\Omega_{I,J}$ polyhedral. Thus, it remains to show that property (3) holds. Assume that there is some $x \in \Omega_{I_1,J_1} \cap \operatorname{cl}\Omega_{I_2,J_2}$. This immediately means $I_1 \subset I_2$ and $J_1 \subset J_2$. But this implies that $\Omega_{I_1,J_1} \subset \operatorname{cl}\Omega_{I_2,J_2}$, which concludes the proof. □

Next we show a simple example with several possible partitions of a given set, where only some are normally admissible stratifications.

Example 1 Consider the following union of two polyhedral sets $\Gamma = (\mathbb{R}\times\{0\}) \cup (\{0\}\times\mathbb{R}_+)$. One possible partition of Γ to relatively open sets is $\Gamma = \Gamma_1 \cup \Gamma_2$ with

$$\Gamma_1 = \mathbb{R} \times \{0\},\ \Gamma_2 = \{0\} \times \mathbb{R}_{++}.$$

Since $(0, 0) \in \Gamma_1 \cap \operatorname{cl}\Gamma_2$, we have $I(1) = \{1, 2\}$. However, as $(1, 0) \in \Gamma_1$ and $(1, 0) \notin \operatorname{cl}\Gamma_2$ condition (3) is not satisfied for $s = 1$ and $i = 2$ and hence this partition is not normally admissible stratification. This situation is depicted on the left–hand side of Fig. 1.

Fig. 1 Possible partitions of the set from Example 1. The *left* partition is not normally admissible while the *right* one is normally admissible

To remedy the situation, one may consider the following partition $\Gamma = \bigcup_{s=1}^{4} \tilde{\Gamma}_s$ with

$$\tilde{\Gamma}_1 = \mathbb{R}_{--} \times \{0\},\ \tilde{\Gamma}_2 = \{0\} \times \{0\},\ \tilde{\Gamma}_3 = \mathbb{R}_{++} \times \{0\},\ \tilde{\Gamma}_4 = \{0\} \times \mathbb{R}_{++},$$

see the right–hand side of Fig. 1. It is simple to verify that this is indeed a normally admissible stratification of Γ.

Now we present the main motivation for considering normally admissible stratification which states that the tangent and normal cone mappings are constant with respect to a particular component of this stratification.

Theorem 1 *Consider a finite union of polyhedral sets* Γ *and its normally admissible stratification* $\{\Gamma_s | s = 1, \dots, S\}$. *Then for any* $s \in \{1, \dots, S\}$, $i \in I(s)$ *and* $x, y \in \Gamma_s$ *we have*

$$\mathrm{T}_{\mathrm{cl}\,\Gamma_i}(x) = \mathrm{T}_{\mathrm{cl}\,\Gamma_i}(y) \qquad \textit{and} \qquad \hat{\mathrm{N}}_{\mathrm{cl}\,\Gamma_i}(x) = \hat{\mathrm{N}}_{\mathrm{cl}\,\Gamma_i}(y). \tag{7}$$

Proof From [24, Theorem 18.2] we know that Γ_s is contained in a relatively open face of $\mathrm{cl}\,\Gamma_i$, and so the statement follows from [9, Chapter 1, Lemma 4.11]. □

From Theorem 1 we know that for any s and $i \in I(s)$, tangent cone $\mathrm{T}_{\mathrm{cl}\,\Gamma_i}(x)$ does not depend on a choice of $x \in \Gamma_s$. To simplify notation, we denote this constant value by

$$\mathrm{T}_{\mathrm{cl}\,\Gamma_i}(\Gamma_s) := \mathrm{T}_{\mathrm{cl}\,\Gamma_i}(x_0) \qquad \text{for arbitrary} \qquad x_0 \in \Gamma_s.$$

In a similar way, we will use notation $\hat{\mathrm{N}}_{\mathrm{cl}\,\Gamma_i}(\Gamma_s)$ and $\mathrm{N}_{\mathrm{cl}\,\Gamma_i}(\Gamma_s)$. In the sequel it will become clear that formula (7) is the cornerstone of this paper.

In the following example we present a set and its several possible partitions. The first partition satisfies formula (7) even though one of its components is nonconvex, meaning that this partition is not normally admissible stratification. For the other two partitions considered, we show that neither condition (3) nor convexity can be dropped from Definition 2 in order to satisfy Theorem 1.

Example 2 Consider $\Gamma = \Omega_1 \cup \Omega_2$ to be union of $\Omega_1 = [0, 3] \times [0, 1]$ and $\Omega_2 = [0, 2] \times [1, 2]$. Then, one of the possible partitions of Γ, elements of which are relatively open and satisfy condition (3), contains a nonconvex plane segment

$$\Gamma_1 = \Big((0, 3) \times (0, 1)\Big) \cup \Big((0, 2) \times (0, 2)\Big),$$

six points and six line segments, see the left–hand side of Fig. 2. Since $\mathrm{cl}\,\Gamma_1$ is nonconvex, this partition is not normally admissible stratification. However, it is not difficult to verify that the statement of Theorem 1 holds true. To show an example, consider $s = 1$. Clearly,

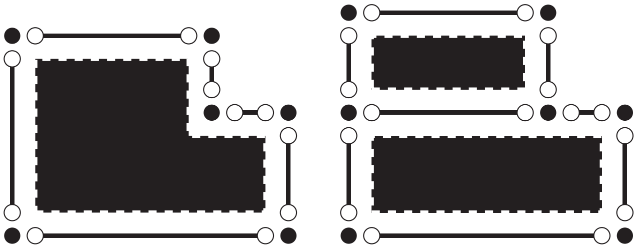

Fig. 2 Possible partitions of the set from Example 2. The figure on the *left–hand* side shows the need of convexity. The figure on the *right–hand* side shows a partition satisfying the result of Theorem 1 but not being normally admissible. Note that the *rectangles* are considered as one set

$I(1) = \{1\}$ and for all $x \in \Gamma_1$ we observe that $\mathrm{T}_{\mathrm{cl}\,\Gamma_1}(x) = \mathbb{R}^2$ and thus $\mathrm{T}_{\mathrm{cl}\,\Gamma_i}(\Gamma_1)$ is indeed well-defined for all $i \in I(1)$.

It is simple to find a normally admissible stratification of Γ. For example, it may consists of two rectangles, eight line segments and seven points as depicted on the right–hand side of Fig. 2. Now we illustrate the role of condition (3) in Theorem 1. Consider any partition of Γ containing the following sets

$$\tilde{\Gamma}_1 = (0, 3) \times \{1\},\ \tilde{\Gamma}_2 = (0, 2) \times (1, 2).$$

Since $(1, 1) \in \tilde{\Gamma}_1 \cap \mathrm{cl}\,\tilde{\Gamma}_2$, we have $2 \in I(1)$. However, it is clear that $\tilde{\Gamma}_1 \not\subset \mathrm{cl}\,\tilde{\Gamma}_2$ and thus (3) is violated. Moreover, we have

$$\mathrm{T}_{\mathrm{cl}\,\tilde{\Gamma}_2}((2, 1)) = \mathbb{R}_- \times \mathbb{R}_+,$$
$$\mathrm{T}_{\mathrm{cl}\,\tilde{\Gamma}_2}((1, 1)) = \mathbb{R} \times \mathbb{R}_+,$$

even though $(2, 1) \in \tilde{\Gamma}_1$ and $(1, 1) \in \tilde{\Gamma}_1$. Thus, formula (7) does not hold for $s = 1$ and $i = 2$.

Next, consider a partition of Γ with

$$\hat{\Gamma}_1 = [(0, 2) \times \{1\}] \cup [(2, 3) \times \{1\}],$$
$$\hat{\Gamma}_2 = [(0, 3) \times (0, 1)] \cup [(0, 2) \times (1, 2)],$$

and seven points and six line segments, see the left–hand side of Fig. 2. Then all the conditions for normally admissible stratification with the exception of convexity of $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ and the polyhedrality of $\mathrm{cl}\,\hat{\Gamma}_2$ are satisfied but Theorem 1 does not hold true. Finally, observe that indeed $\hat{\Gamma}_1 \subset \mathrm{cl}\,\hat{\Gamma}_2$.

We are now ready to provide the main result of this section which concerns the computation of normal cones to finite union of polyhedra.

Theorem 2 *Let Γ be a finite union of polyhedral sets and $\{\Gamma_s \,|\, s = 1, \ldots, S\}$ be its normally admissible stratification. Then for any $x \in \Gamma_s$ we have $\hat{\mathrm{N}}_\Gamma(x) = \hat{\mathrm{N}}_\Gamma(\Gamma_s)$ and further*

$$\hat{\mathrm{N}}_\Gamma(\Gamma_s) = \bigcap_{i \in I(s)} \hat{\mathrm{N}}_{\mathrm{cl}\,\Gamma_i}(\Gamma_s) = \bigcap_{i \in \tilde{I}(s)} \hat{\mathrm{N}}_{\mathrm{cl}\,\Gamma_i}(\Gamma_s). \tag{8}$$

Moreover, for graphs of Fréchet and limiting normal cones we have the following formulas

$$\operatorname{gph} \hat{\mathrm{N}}_\Gamma = \bigcup_{s=1}^{S} \left(\Gamma_s \times \hat{\mathrm{N}}_\Gamma(\Gamma_s)\right), \tag{9}$$

$$\operatorname{gph} \mathrm{N}_\Gamma = \bigcup_{s=1}^{S} \left(\operatorname{cl}\Gamma_s \times \hat{\mathrm{N}}_\Gamma(\Gamma_s)\right). \tag{10}$$

Proof Fix any $x \in \Gamma_s$. Then by simple calculus we obtain

$$\mathrm{T}_\Gamma(x) = \mathrm{T}_{\bigcup_{i\in I(s)} \operatorname{cl}\Gamma_i}(x) = \bigcup_{i\in I(s)} \mathrm{T}_{\operatorname{cl}\Gamma_i}(x),$$

$$\hat{\mathrm{N}}_\Gamma(x) = \bigcap_{i\in I(s)} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(x).$$

With regards to Theorem 1 we obtain the first equality in (8). The second equality in (8) follows from the fact that $\Gamma_s \subset \operatorname{cl}\Gamma_i \subset \operatorname{cl}\Gamma_j$ implies $\hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_s) \supset \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_j}(\Gamma_s)$.

Formula (9) is a direct consequence of (8). Since $\operatorname{gph}\mathrm{N}_\Gamma$ is a closure of $\operatorname{gph}\hat{\mathrm{N}}_\Gamma$ by definition, equation (10) follows as well. □

In some situations, computation of normal cone $\mathrm{N}_\Gamma(\bar{x})$ only at one particular point $\bar{x} \in \Gamma$ is required instead of computation of the whole graph of the normal cone mapping. The following corollary concerns such a case.

Corollary 1 *Under assumptions of Theorem 2, for any $\bar{x} \in \Gamma$ denote by $\bar{s}$ the index of the unique component $\Gamma_{\bar{s}}$ such that $\bar{x} \in \Gamma_{\bar{s}}$. Then*

$$\hat{\mathrm{N}}_\Gamma(\bar{x}) = \hat{\mathrm{N}}_\Gamma(\Gamma_{\bar{s}}) = \bigcap_{i\in I(\bar{s})} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_{\bar{s}}) = \bigcap_{i\in \tilde{I}(\bar{s})} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_{\bar{s}}), \tag{11}$$

$$\mathrm{N}_\Gamma(\bar{x}) = \bigcup_{s\in I(\bar{s})} \hat{\mathrm{N}}_\Gamma(\Gamma_s) = \bigcup_{s\in I(\bar{s})} \bigcap_{i\in I(s)} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_s) = \bigcup_{s\in I(\bar{s})} \bigcap_{i\in \tilde{I}(s)} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_s). \tag{12}$$

Remark 1 Relations similar to (11) and (12), see (13) and (14) below, can be obtained by simpler means. We present them to show the possible advantages of our approach. First, defining $J(x) := \{s|\, x \in \operatorname{cl}\Gamma_s\}$ we observe that $J(x) = I(t)$ where t is the unique index such that $x \in \Gamma_t$. Indeed, if $s \in J(x)$, then $x \in \operatorname{cl}\Gamma_s$, which together with assumed $x \in \Gamma_t$ implies $x \in \Gamma_t \cap \operatorname{cl}\Gamma_s$ and thus $s \in I(t)$. On the other hand, if $s \in I(t)$, then as the considered partition is normally admissible stratification, we have $x \in \Gamma_t \subset \operatorname{cl}\Gamma_s$ and thus $s \in J(x)$, which implies the desired equality. Formula (11) may then be derived in the following way

$$\hat{\mathrm{N}}_\Gamma(\bar{x}) = (\mathrm{T}_{\bigcup_{i\in J(\bar{x})} \operatorname{cl}\Gamma_i}(\bar{x}))^* = \left(\bigcup_{i\in J(\bar{x})} \mathrm{T}_{\operatorname{cl}\Gamma_i}(\bar{x})\right)^* = \bigcap_{i\in J(\bar{x})} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\bar{x}) = \bigcap_{i\in I(\bar{s})} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\bar{x}), \tag{13}$$

Similarly, for a sufficiently small neighborhood $\mathcal{X}$ of $\bar{x}$, one may obtain formula for the limiting normal cone directly from (13) as

$$\mathrm{N}_\Gamma(\bar{x}) = \bigcup_{x\in\mathcal{X}} \bigcap_{i\in J(x)} \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(x). \tag{14}$$

Although it is obvious that the union with respect to $x \in \mathcal{X}$ will reduce to a union with respect to a finite number of elements, it is not entirely clear how to obtain this reduction without the concept of a normally admissible stratification.

We conclude this section with a note that the computation of normal cones can be performed repeatedly, by which we mean that formula (10) provides a good background for computation of $\operatorname{gph} \mathrm{N}_{\operatorname{gph} \mathrm{N}_\Gamma}$.

Remark 2 Consider a normally admissible stratification $\{\Gamma_s \,|\, s = 1, \ldots, S\}$ of Γ. It follows from Lemma A3 that $\{\Gamma_t \,|\, s \in I(t)\}$ is a normally admissible stratification of $\operatorname{cl} \Gamma_s$ for any s.

Moreover, it is possible to show that

$$\{\Gamma_s \times D_{st} \,|\, s = 1, \ldots, S,\ t = 1, \ldots, T(s)\}$$

is a normally admissible stratification of $\operatorname{gph} \mathrm{N}_\Gamma$, where $\{D_{st} \,|\, t = 1, \ldots, T(s)\}$ are suitable normally admissible stratifications of $\mathrm{N}_\Gamma(\Gamma_s)$ for $s = 1, \ldots, S$. However, since the construction of D_{st} is not entirely simple and it is not used later in the text, we omit it here.

3 Relation to known results

This section revisits some notable results of other authors on computation of the limiting normal cone to a union of polyhedral sets and exploits the relationship between their results and those presented in the previous section. We firstly recall the result of Dontchev and Rockafellar in [7], where formula for the limiting normal cone to a special case of a union of polyhedral sets was given in terms of critical cones and then show that formulas from Corollary 1 coincide with those of Dontchev and Rockafellar. Secondly, we summarize the results of Henrion and Outrata in [12] who also considered a general union of polyhedral sets. Direct comparison yields that the explicit formula derived by Henrion and Outrata can be considered as a special case of our approach. We omit a detailed comparison with results of Červinka, Outrata and Pištěk in [5] due to the fact that their results are a special case of Theorem 2.

3.1 Normal cones to graph of a normal cone to a polyhedral set

To our knowledge, the first attempt to provide explicit formulas for computation of the limiting normal cone to a union of polyhedral sets can be found in [7]. It concerns a rather special case where $\Gamma = \operatorname{gph} \mathrm{N}_C \subset \mathbb{R}^{2n}$ with $C \subset \mathbb{R}^n$ being polyhedral. Due to polyhedrality of C, Γ is indeed a union of finitely many polyhedral sets. Interestingly, the formula for $\mathrm{N}_\Gamma(\bar{x}, \bar{y})$ was not given in [7] as a separate result but as a part of a proof of another result. We state it in the following proposition. Recall that $K_C(x, y)$ denotes the critical cone to C at x for y.

Proposition 1 ([7], part of the proof of Theorem 2) *Consider a polyhedral set C and some $\bar{x} \in C$ and $\bar{y} \in \mathrm{N}_C(\bar{x})$. Then*

$$\begin{aligned} \hat{\mathrm{N}}_{\operatorname{gph} \mathrm{N}_C}(\bar{x}, \bar{y}) &= K_C(\bar{x}, \bar{y})^* \times K_C(\bar{x}, \bar{y}), \\ \mathrm{N}_{\operatorname{gph} \mathrm{N}_C}(\bar{x}, \bar{y}) &= \bigcup_{(x,y) \in \mathcal{U}} K_C(x, y)^* \times K_C(x, y), \end{aligned} \tag{15}$$

for some sufficiently small neighborhood $\mathcal{U}$ of $(\bar{x}, \bar{y})$.

The original proof of Proposition 1 by Dontchev and Rockafellar is based on the application of the so-called Reduction Lemma, cf. [8, Lemma 2E.4]. To illuminate the relation between Proposition 1 and Corollary 1, we provide an alternative proof exploiting the properties of relatively open faces forming a partition of a polyhedral set, see [24, Theorem 18.2]. To this end, we recall the definition of faces of a convex set, see [16].

Definition 3 A subset F of a convex set P is called a face of P provided the following implication holds true: if x_1 and x_2 belong to P and $\lambda x_1+(1-\lambda)x_2 \in F$ for some $\lambda \in (0, 1)$, then x_1 and x_2 belong to F as well. We say that $\tilde{F}$ is a relatively open face of P if there exists a face F of P such that $\tilde{F} = \operatorname{rint} F$.

Consider all nonempty faces of a polyhedral set C and let us denote them $\tilde{C}_s$ with $s = 1, \dots, S$. We shall call $C_s := \operatorname{rint} \tilde{C}_s$ relatively open faces of C. By virtue of Lemma A4 we obtain that $\{C_s | \, s = 1, \dots, S\}$ form a normally admissible stratification of C. Thus, Theorem 1 implies that $\mathrm{N}_C(x)$ has the same value for all $x \in C_s$. Following the notation developed in previous sections, let us denote it by $\mathrm{N}_C(C_s)$. Since $\mathrm{N}_C(C_s)$ is also a polyhedral set, we can as well find its relatively open faces D_{st}. Again, let $\{D_{st} | \, t = 1, \dots, T(s)\}$ form a normally admissible stratification of $\mathrm{N}_C(C_s)$. This results in the following representation of Γ:

$$\Gamma := \operatorname{gph} \mathrm{N}_C = \bigcup_{s=1}^{S} \bigcup_{t=1}^{T(s)} C_s \times D_{st}.$$

It follows from Lemma A4 that $\{C_s \times D_{st} | \, s = 1, \dots, S, \ t = 1, \dots, T(s)\}$ forms a normally admissible stratification of Γ.

As a consequence, for a given pair $\bar{x} \in C$ and $\bar{y} \in \mathrm{N}_C(\bar{x})$ there is a unique couple of indices $(\bar{s}, \bar{t})$ such that $(\bar{x}, \bar{y}) \in C_{\bar{s}} \times D_{\bar{s}\bar{t}}$. By application of Corollary 1 to $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathrm{N}_C$, we immediately obtain

$$\begin{aligned} \hat{\mathrm{N}}_{\operatorname{gph} \mathrm{N}_C}(\bar{x}, \bar{y}) &= \bigcap_{(i,j)\in I(\bar{s},\bar{t})} \mathrm{N}_{\operatorname{cl}(C_i \times D_{ij})}(C_{\bar{s}} \times D_{\bar{s}\bar{t}}), \\ \mathrm{N}_{\operatorname{gph} \mathrm{N}_C}(\bar{x}, \bar{y}) &= \bigcup_{(s,t)\in I(\bar{s},\bar{t})} \bigcap_{(i,j)\in I(s,t)} \mathrm{N}_{\operatorname{cl}(C_i \times D_{ij})}(C_s \times D_{st}). \end{aligned} \tag{16}$$

Since Γ is the union of finitely many polyhedral sets, only finitely many cones can be manifested as $\hat{\mathrm{N}}_\Gamma(x, y)$ at points $(x, y) \in \Gamma$ near $(\bar{x}, \bar{y})$. It is not difficult to see that each of such cones corresponds to $\hat{\mathrm{N}}_\Gamma(C_s, D_{st})$ with $(s, t) \in I(\bar{s}, \bar{t})$. Invoking Remark 1, this establishes the correspondence of union in (16) with union in (15). In order to show the equivalence of (15) and (16), consider a fixed pair of indices $(s, t) \in I(\bar{s}, \bar{t})$ and let us simplify the intersection in (16). By elementary operations and [25, Proposition 6.41] we obtain

$$\bigcap_{(i,j)\in I(s,t)} \mathrm{N}_{\operatorname{cl}(C_i \times D_{ij})}(C_s \times D_{st}) = \bigcap_{\{(i,j)| \ C_s \subset \operatorname{cl} C_i, \ D_{st} \subset \operatorname{cl} D_{ij}\}} \left[\mathrm{N}_{\operatorname{cl} C_i}(C_s) \times \mathrm{N}_{\operatorname{cl} D_{ij}}(D_{st})\right]. \tag{17}$$

Note that for any i there exists an index $l \in \{1, \dots, T(i)\}$ such that $\operatorname{cl} D_{il} = \mathrm{N}_C(C_i)$. This means that for every $j \in \{1, \dots, T(i)\}$ such that $D_{st} \subset \operatorname{cl} D_{ij}$ we have $\operatorname{cl} D_{ij} \subset$

$\operatorname{cl} D_{il} = \mathrm{N}_C(C_i)$. This, in turn, implies that $\mathrm{N}_{\operatorname{cl} D_{ij}}(D_{st}) \supset \mathrm{N}_{\mathrm{N}_C(C_i)}(D_{st})$. In particular, we have

$$\bigcap_{(i,j)\in I(s,t)} \mathrm{N}_{\operatorname{cl}(C_i\times D_{ij})}(C_s \times D_{st}) = \bigcap_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \left[\mathrm{N}_{\operatorname{cl} C_i}(C_s) \times \mathrm{N}_{\mathrm{N}_C(C_i)}(D_{st})\right]$$
$$= \left[\bigcap_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \mathrm{N}_{\operatorname{cl} C_i}(C_s)\right] \times \left[\bigcap_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \mathrm{N}_{\mathrm{N}_C(C_i)}(D_{st})\right] \tag{18}$$

It suffices to show that both parts of the Cartesian product in (15) correspond to those of (18). To verify that, we present the following two lemmas. Note that a result similar to the first lemma was proved in [15, Theorem 5.2].

Lemma 2 *For any $x \in C_s$ and $y \in D_{st}$ the following equality holds*

$$K(x, y) = \bigcap_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \mathrm{N}_{\mathrm{N}_C(C_i)}(D_{st}). \tag{19}$$

Proof In order to verify (19), note first that for any i such that $C_s \subset \operatorname{cl} C_i$ and $D_{st} \subset \mathrm{N}_C(C_i)$ we have $\mathrm{N}_C(C_i) \subset \mathrm{N}_C(C_s)$. This, in turn, yields $\mathrm{N}_{\mathrm{N}_C(C_i)}(D_{st}) \supset \mathrm{N}_{\mathrm{N}_C(C_s)}(D_{st})$. This implies that

$$\bigcap_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \mathrm{N}_{\mathrm{N}_C(C_i)}(D_{st}) = \mathrm{N}_{\mathrm{N}_C(C_s)}(D_{st}). \tag{20}$$

Since the set $\mathrm{N}_C(C_s)$ is a cone, from Theorem 1 and [25, Example 11.4 (b)] we obtain

$$\mathrm{N}_{\mathrm{N}_C(C_s)}(D_{st}) = \mathrm{N}_{\mathrm{N}_C(x)}(y) = \left\{u \in (\mathrm{N}_C(x))^* \,\middle|\, u^\top y = 0\right\} = K(x, y), \tag{21}$$

which concludes the proof. □

Lemma 3 *For any $x \in C_s$ and $y \in D_{st}$ the following equality holds*

$$K(x, y)^* = \bigcap_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \mathrm{N}_{\operatorname{cl} C_i}(C_s). \tag{22}$$

Proof Recall first that due to [16, relation (42)] one has $\mathrm{T}_P(x_0) = \operatorname{con}(P - x_0)$ for any polyhedral set P and any $x_0 \in P$. This, by virtue of Theorem 1 implies

$$\mathrm{T}_C(C_s) = \operatorname{con}(C - \operatorname{cl} C_s). \tag{23}$$

Similarly, from the definition of normal cone and Theorem 1 one has

$$\mathrm{N}_{\operatorname{cl} C_i}(C_s) = \{y|\ y^\top(\operatorname{cl} C_i - \operatorname{cl} C_s) \le 0\} = (\operatorname{con}(\operatorname{cl} C_i - \operatorname{cl} C_s))^*.$$

Since the equality of two sets implies equality of their polars, to prove the desired equality (22) it is enough to show that

$$K(x, y) = \bigcup_{\{i|\ C_s\subset \operatorname{cl} C_i,\ D_{st}\subset \mathrm{N}_C(C_i)\}} \operatorname{con}(\operatorname{cl} C_i - \operatorname{cl} C_s).$$

Suppose that $u \in \operatorname{con}(\operatorname{cl} C_i - \operatorname{cl} C_s)$ for some i such that $C_s \subset \operatorname{cl} C_i,\ D_{st} \subset \mathrm{N}_C(C_i)$. To show that $u \in K(x, y)$ we need to prove that $u \in \mathrm{T}_C(C_s)$ and that $y^\top u = 0$. The first relation follows immediately from (23) and the second one from the following chain of implications

$$\begin{array}{l} y \in D_{st} \subset \mathrm{N}_C(C_s) \implies y^\top(C - \mathrm{cl}\, C_s) \le 0 \\ y \in D_{st} \subset \mathrm{N}_C(C_i) \implies y^\top(C - \mathrm{cl}\, C_i) \le 0 \end{array} \implies y^\top(\mathrm{cl}\, C_s - \mathrm{cl}\, C_i) \le 0.$$

To show the opposite inclusion, we obtain first from [16, Lemma 4] and [16, relation (44)] that there exists an index i such that $C_s \subset \mathrm{cl}\, C_i$ and such that

$$K(x, y) = \mathrm{T}_{\mathrm{cl}\, C_i}(C_s) = \mathrm{con}(\mathrm{cl}\, C_i - \mathrm{cl}\, C_s). \tag{24}$$

To finish the proof, it remains to show that $D_{st} \subset \mathrm{N}_C(C_i)$. From (24) we immediately obtain $y^\top(\mathrm{cl}\, C_i - \mathrm{cl}\, C_s) = 0$. Due to Theorem 1, $K(x, y)$ does not depend on the particular choice of $y \in D_{st}$ and thus we obtain $D_{st}^\top(\mathrm{cl}\, C_i - \mathrm{cl}\, C_s) = 0$. As already stated above, $D_{st} \subset \mathrm{N}_C(C_s)$ implies $D_{st}^\top(C - \mathrm{cl}\, C_s) \le 0$. Together, this shows that $D_{st}^\top(C - \mathrm{cl}\, C_i) \le 0$, which in turn implies $D_{st} \subset \mathrm{N}_C(C_i)$. This concludes the proof. □

Summarizing this special case, the relatively open faces of polyhedral sets appear to be a suitable choice for normally admissible stratifications. In such a case one can enjoy special properties of faces of polyhedral sets and relations to tangent an critical cones.

In the following subsection, we revisit another previously developed representation of normal cones for the general case considered in Section 2.

3.2 Relation to a union of polyhedral sets

In [12], the authors studied the case of a union of general polyhedral sets. Apart from providing explicit formulas for values of limiting normal cone at a point, the authors in [12] also focused on several special cases of polyhedral sets, such as finite union of halfspaces and finite union of orthants. In this subsection, we briefly summarize their main result concerning the case of a union of R polyhedral sets, for details see [12, Section 6].

Consider Γ as in (2). For $x \in \Gamma$ denote the set of active components by

$$\mathbb{I}(x) = \{r \in \{1, \ldots, R\} |\, x \in \Omega_r\}.$$

Fix any $\bar{x} \in \Gamma$ and let us denote by Δ_r the polyhedral cones $\Delta_r := \mathrm{T}_{\Omega_r}(\bar{x})$. Then for $\Delta := \bigcup_{r \in \mathbb{I}(\bar{x})} \Delta_r$ one has

$$\mathrm{N}_\Gamma(\bar{x}) = \mathrm{N}_\Delta(0).$$

Now, for all $r \in \mathbb{I}(x)$, consider the explicit description of the polyhedral cones Δ_r

$$\Delta_r = \left\{x \,\middle|\, \left\langle c_t^r, x\right\rangle \le 0,\ t = 1, \ldots, T(r)\right\}.$$

Note that we will work with tangent and normal cones to Δ_r at 0 and that all constraints are active at this point. For $\mathbb{I} \subset \mathbb{I}(\bar{x})$ define the following index set

$$\mathcal{J}_\mathbb{I} = \begin{cases} \times_{r \in \mathbb{I}}\{1, \ldots, T(r)\} & \text{if } \mathbb{I} \ne \emptyset, \\ \{\emptyset\} & \text{if } \mathbb{I} = \emptyset, \end{cases}$$

which adopts the convention that $\mathcal{J}_\emptyset$ contains one element, an empty (zero-dimensional) vector.

For any integer vectors $\mathbb{I}^c = (i_{n_1}, \ldots, i_{n_L})$ and $J = (J_{n_1}, \ldots, J_{n_L}) \in \mathcal{J}_{\mathbb{I}^c}$ put

$$\Gamma_\mathbb{I}^J = \left\{x \,\middle|\, \begin{array}{l} \langle c_t^r, x\rangle \le 0,\ t = 1, \ldots, T(r),\ r \in \mathbb{I} \\ \left\langle c_{J_r}^r, x\right\rangle > 0,\ r \in \mathbb{I}^c \end{array} \right\}.$$

Then

$$\mathrm{N}_\Gamma(\bar{x}) = \bigcup_{\emptyset \neq \mathbb{I} \subset \mathbb{I}(\bar{x})} \bigcup_{J \in \mathcal{J}_{\mathbb{I}^c}} \bigcup_{x \in \Gamma_{\mathbb{I}}^J} \bigcap_{k \in \mathbb{I}} \hat{\mathrm{N}}_{\Delta_k}(x), \tag{25}$$

and for each $x \in \Gamma_{\mathbb{I}}^J$ and $r \in \mathbb{I}$ there exist exactly one subsets $\mathcal{J}_{x,r} \subset \{1, \ldots, T(r)\}$ such that

$$\begin{aligned} \langle c_t^r, x\rangle &= 0 \ \forall t \in \mathcal{J}_{x,r}, r \in \mathbb{I}, \\ \langle c_t^r, x\rangle &< 0 \ \forall t \in \{1, \ldots, T(r)\} \setminus \mathcal{J}_{x,r}, r \in \mathbb{I}, \\ \langle c_{J_r}^r, x\rangle &> 0 \ \forall r \in \mathbb{I}^c. \end{aligned} \tag{26}$$

For such x and fixed k we have $\hat{\mathrm{N}}_{\Delta_k}(x) = \mathrm{con}\{c_t^k \,|\, t \in \mathcal{J}_{x,k}\}$. For any subset $\mathcal{J} = \times_{r \in \mathbb{I}} \mathcal{J}_r \subset \mathcal{J}_{\mathbb{I}}$, put

$$\begin{aligned} R_{\mathbb{I}}^{\mathcal{J},J} &:= \mathrm{con}\{\{c_{J_r}^r \,|\, r \in \mathbb{I}^c\} \cup \{-c_t^r \,|\, r \in \mathbb{I}, t \in \{1, \ldots, n_r\} \setminus \mathcal{J}_r\}\}, \\ S_{\mathbb{I}}^{\mathcal{J}} &:= \mathrm{span}\{c_t^r \,|\, r \in \mathbb{I}, t \in \mathcal{J}_r\}, \end{aligned}$$

and

$$\mathcal{A}_{\mathbb{I}}^J := \{\mathcal{J} \subset \mathcal{J}_{\mathbb{I}} \,|\, R_{\mathbb{I}}^{\mathcal{J},J} \cap S_{\mathbb{I}}^{\mathcal{J}} = \{0\}\}. \tag{27}$$

Applying Motzkin's Theorem, solvability of systems of conditions (26) can be represented by elements of $\mathcal{A}_{\mathbb{I}}^J$.

Proposition 2 *Under the notation above, the limiting normal cone to a finite union of polyhedral sets calculates as*

$$\mathrm{N}_\Gamma(\bar{x}) = \bigcup_{\emptyset \neq \mathbb{I} \subset \mathbb{I}(\bar{x})} \bigcup_{J \in \mathcal{J}_{\mathbb{I}^c}} \bigcup_{\mathcal{J} \in \mathcal{A}_{\mathbb{I}}^J} \bigcap_{k \in \mathbb{I}} \mathrm{con}\{c_j^k \,|\, j \in \mathcal{J}_k\}. \tag{28}$$

We will now compare the results of Proposition 2 to our results in Theorem 2. From direct comparison of sets defined by conditions (26) with sets $\Omega_{I,J}$ defined in (5), it follows that elements of $\mathcal{A}_{\mathbb{I}}^J$, which represent only the nonempty sets given by conditions (26), correspond to relatively open sets that form one particular normally admissible stratification of Γ. In fact, this is exactly the partition constructed in the proof of Lemma 1. Thus, it is not difficult to see that $\bigcap_{k\in\mathbb{I}} \mathrm{con}\{c_j^k \,|\, j \in \mathcal{J}_k\}$ in (28) corresponds to $\bigcap_{i \in \tilde{I}(s)} \hat{\mathrm{N}}_{\mathrm{cl}\,\Gamma_i}(\Gamma_s)$ in (12) via (8). Similarly $\bigcup_{\emptyset\neq\mathbb{I}\subset\mathbb{I}(\bar{x})} \bigcup_{J\in\mathcal{J}_{\mathbb{I}^c}} \bigcup_{\mathcal{J}\in\mathcal{A}_{\mathbb{I}}^J}$ in (28) corresponds to $\bigcup_{i\in I(\bar{s})}$ in (12).

Taking into account that there might exist other normally admissible stratifications of Γ with less components, we have managed to generalize the approach from [12] by considering a larger family of possible partitions instead of the particular one considered in [12]. On top of that, we are able to provide the corresponding result for the whole graph of N_Γ.

By means of the following example we show the differences in both approaches. These differences will become even clearer in Section 4 where we present an example in which a suitable choice of a normally admissible stratification plays a crucial role.

Example 3 Consider $\Gamma \subset \mathbb{R}^2$ to be a union of R different rays emanating from a common point $\bar{x} \in \mathbb{R}^2$. One can easily find a normally admissible stratification of Γ which consists of $R + 1$ sets. For such a normally admissible stratification, the application of Corollary 1 is straightforward and the number of elements in union (12) grows linearly in R. On the other hand, it is clear that direct application of Proposition 2 results in exponential growth of the number of elements in union (28).

4 Application to time dependent problems

In this section we will investigate a special structure of set Γ, which may arise during a discretization of time dependent problems [1, 4]. To give a short introduction, consider the following differential inclusion with given initial condition

$$\begin{aligned}&\dot{x}(t) \in \Lambda(t, x(t)),\ t \in [0, T] \text{ a.e.}\\&x(0) = x_0,\end{aligned} \tag{29}$$

where $[0, T]$ is time interval, $x : [0, T] \to \mathbb{R}^K$ is the state variable, $\Lambda : [0, T] \times \mathbb{R}^K \rightrightarrows \mathbb{R}^K$ is a multifunction and $x_0 \in \mathbb{R}^K$ is an initial point.

After performing a discretization of (29), we may obtain the following set of discretized feasible solutions to problem (29)

$$\Gamma := \left\{x \in \mathbb{R}^{KN} \,\middle|\, x_n \in \Lambda^n(x_{n-1}),\ n = 1, \dots, N\right\}. \tag{30}$$

Here, we consider $x = (x_1, \dots, x_N) \in \mathbb{R}^{KN}$ to be the discretization of the state variable $x(\cdot)$ and for notational simplicity, we identify the initial point x_0 from (29) with x_0 from (30). Moreover, $K \in \mathbb{N}$ is the dimension of the state variable x_n and $N \in \mathbb{N}$ denotes the number of time discretization steps. Finally, $\Lambda^n : \mathbb{R}^K \rightrightarrows \mathbb{R}^K$ for $n = 1, \dots, N$ are multifunctions.

The main goal of this section is to use particular structure of Γ defined by (30) and simplify the formula for $\operatorname{gph} \mathrm{N}_\Gamma$ from Theorem 2. To be able to do so, we will need the following assumption

$$\Lambda^n \text{ is a polyhedral multifunction for } n = 1, \dots, N, \tag{31}$$

where a polyhedral multifunction is a multifunction which graph is a finite union of polyhedral sets. We recall that there is a unique correspondence between multifunctions $S : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ and sets $A \subset \mathbb{R}^{p+q}$ via graph operator

$$A = \operatorname{gph} S := \left\{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \mid y \in S(x)\right\}.$$

Moreover, in this section, we will often work with a closure of multifunction $S : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$, which is denoted by $\operatorname{cl} S : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ and defined via its graph by $\operatorname{gph} \operatorname{cl} S = \operatorname{cl} \operatorname{gph} S$.

4.1 Theoretical background

In this subsection, we will provide a theoretical background for computation of $\operatorname{gph} \mathrm{N}_\Gamma$ where Γ is given by (30). In particular, we will express normally admissible stratification of Γ in terms of normally admissible stratifications of $\operatorname{gph} \Lambda^n$ and based on these partitions, we will provide a formula for computation of a normal cone to Γ based on normal cones to elements of partitions of $\operatorname{gph} \Lambda^n$.

Observe that under assumption (31), application of Lemma 1 yields a normally admissible stratification $\{A_i^n \subset \mathbb{R}^{2K} | \, i = 1, \dots, M(n)\}$ of $\operatorname{gph} \Lambda^n$ for all $n = 1, \dots, N$. Due to unique correspondence between multifunctions and their graphs, this is equivalent to existence of multifunctions $\Lambda_i^n : \mathbb{R}^K \rightrightarrows \mathbb{R}^K$ with $\operatorname{gph} \Lambda_i^n = A_i^n$ such that $\{\operatorname{gph} \Lambda_i^n | \, i = 1, \dots, M(n)\}$ is a normally admissible stratification of $\operatorname{gph} \Lambda^n$. Further, for $s \in \{1, \dots, M(n)\}$ we denote by $I^n(s) \subset \{1, \dots, M(n)\}$ and $\tilde{I}^n(s) \subset I^n(s)$ index sets (4) associated with this stratification.

Now, we consider the following sets

$$\Gamma_i := \Gamma_{i_1 \dots i_N} := \left\{x \in \mathbb{R}^{KN} \,\middle|\, x_n \in \Lambda_{i_n}^n(x_{n-1}),\ n = 1, \dots, N\right\} \tag{32}$$

for $i := (i_1, \dots, i_N)$ with $i_n \in \{1, \dots, M(n)\}$. Defining

$$\Theta := \left\{ i \in \bigtimes_{n=1}^{N} \{1, \dots, M(n)\} \,\middle|\, \Gamma_i \neq \emptyset \right\}, \tag{33}$$

we show that $\{\Gamma_i \,|\, i \in \Theta\}$ forms a normally admissible stratification of Γ. To this end we develop a series of lemmas which allow us to express properties of Γ in terms of properties of Λ^n.

Lemma 4 *For $i \in \Theta$ we have*

$$\operatorname{cl} \Gamma_i = \left\{ x \in \mathbb{R}^{KN} \,\middle|\, x_n \in (\operatorname{cl} \Lambda^n_{i_n})(x_{n-1}),\ n = 1, \dots, N \right\}. \tag{34}$$

Proof Denote the right–hand side of (34) by G. Directly from the definition of closure of a multifunction we have $\operatorname{cl} \Gamma_i \subset G$. To prove the opposite inclusion, consider some $x \in G$. Since $i \in \Theta$, there exists some $y \in \Gamma_i$, which means that $y_0 = x_0$ and $(y_{n-1}, y_n) \in \operatorname{gph} \Lambda^n_{i_n}$ for $n = 1, \dots, N$. Since $\operatorname{gph} \Lambda^n_{i_n}$ is convex and relatively open due to definition of normally admissible stratification, by virtue of Lemma A2 we obtain for $k \in \mathbb{N}$ and $n = 1 \dots, N$ the following formula

$$\left(\frac{1}{k} y_{n-1} + \left(1 - \frac{1}{k}\right) x_{n-1}, \frac{1}{k} y_n + \left(1 - \frac{1}{k}\right) x_n \right) \in \operatorname{gph} \Lambda^n_{i_n}.$$

Defining $z^k_n := \frac{1}{k} y_n + \left(1 - \frac{1}{k}\right) x_n$ and $z^k := (z^k_1, \dots, z^k_N)$ we have $z^k \in \Gamma_i$ and $z^k \to x$, which finishes the proof. □

Lemma 5 *For $s \in \Theta$ and index sets $I(s)$ and $\tilde{I}(s)$ defined by (4), it holds that*

$$I(s) = \left\{ i \in \Theta \,\middle|\, i_n \in I^n(s_n),\ n = 1, \dots, N \right\}, \tag{35a}$$

$$\tilde{I}(s) = \left\{ i \in I(s) \,\middle|\, \forall j \in I(s) : j_n \in I^n(i_n),\ n = 1, \dots, N \implies i = j \right\}, \tag{35b}$$

where index sets $I^n(s_n)$ are associated to a normally admissible stratifications of $\operatorname{gph} \Lambda^n$ *for $n = 1, \dots, N$. Moreover, for any $i \in I(s)$ condition (3) holds true.*

Proof First, take any $i \in I(s)$. From the definition of $I(s)$ this is equivalent to $\Gamma_s \cap \operatorname{cl} \Gamma_i \neq \emptyset$, which implies $i \in \Theta$. For contradiction assume that there is some n such that $i_n \notin I^n(s_n)$. This means that $\operatorname{gph} \Lambda^n_{s_n} \cap \operatorname{gph} \operatorname{cl} \Lambda^n_{i_n} = \emptyset$. Using Lemma 4 this further implies that $\Gamma_s \cap \operatorname{cl} \Gamma_i = \emptyset$, which concludes the contradiction.

Now, take any $i \in \Theta$ such that $i_n \in I^n(s_n)$ for all $n = 1, \dots, N$. Due to definition of $I^n(s)$ this implies $\operatorname{gph} \Lambda^n_{s_n} \cap \operatorname{gph} \operatorname{cl} \Lambda^n_{i_n} \neq \emptyset$ for all n. By condition (3) for stratification of $\operatorname{gph} \Lambda^n$ this implies $\operatorname{gph} \Lambda^n_{s_n} \subset \operatorname{gph} \operatorname{cl} \Lambda^n_{i_n}$ for all n. Invoking Lemma 4, we have $\Gamma_s \subset \operatorname{cl} \Gamma_i$. Firstly, this implies that $\Gamma_s \cap \operatorname{cl} \Gamma_i = \Gamma_s \neq \emptyset$ proving (35a), and secondly it also means that property (3) holds true as well.

Formula (35b) then follows directly from (35a) and (4d). □

Lemma 6 $\{\Gamma_i \,|\, i \in \Theta\}$ *forms a normally admissible stratification of* Γ.

Proof Observe first that due to definition of Θ we have $\Gamma = \cup_{i \in \Theta} \Gamma_i$ and that all Γ_i are nonempty. Since $\{\operatorname{gph} \Lambda^n_j \,|\, j \in \{1, \dots, M(n)\}\}$ is a normally admissible stratification of $\operatorname{gph} \Lambda^n$, it follows that Γ_i are pairwise disjoint. Hence we have shown that $\{\Gamma_i \,|\, i \in \Theta\}$ is indeed a partition of Γ.

To prove that this partition is a normally admissible stratification of Γ, it remains to show that Γ_i are relatively open and convex, $\operatorname{cl}\Gamma_i$ are polyhedral and that property (3) holds. Since Γ_i can be written as an intersection of N relatively open convex sets, it is relatively open and convex as well. Similarly, as $\operatorname{cl}\Gamma_i$ is an intersection of N polyhedral sets due to Lemma 4, it is polyhedral. Finally, condition (3) follows directly from Lemma 5 and so the proof has been finished. □

The following theorem proposes a convenient formula for computation of $\hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_s)$. This formula is presented purely in terms of individual Λ^n and not the original Γ. The consequences of this theorem will be later seen in Section 4.2.

Theorem 3 *Assume that Γ is defined via (30) and that assumption (31) is satisfied. Assume moreover that $\{\operatorname{gph}\Lambda_i^n \mid i = 1,\dots,M(n)\}$ forms a normally admissible stratification of $\operatorname{gph}\Lambda^n$ for all $n = 1,\dots,N$. Then for any $s \in \Theta$ and $i \in I(s)$ we have*

$$\hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\Gamma_s) = \left\{ \begin{pmatrix} p_1 + q_1 \\ \vdots \\ p_N + q_N \end{pmatrix} \in \mathbb{R}^{KN} \;\middle|\; \begin{array}{c} \begin{pmatrix} p_{n-1} \\ q_n \end{pmatrix} \in \hat{\mathrm{N}}_{\operatorname{cl}\operatorname{gph}\Lambda^n_{i_n}}(\operatorname{gph}\Lambda^n_{s_n}),\ n = 1,\dots,N \\ p_N = 0 \end{array} \right\}.$$

Proof The set $\operatorname{cl}\Gamma_i$ can be by virtue of Lemma 4 written as multivalued inverse $F^{-1}(\Omega_i)$, where

$$F(x) := \begin{pmatrix} x_0 \\ x_1 \\ \hline x_1 \\ x_2 \\ \hline \vdots \\ \hline x_{N-1} \\ x_N \end{pmatrix}, \quad \Omega_i := \begin{pmatrix} \operatorname{cl}\operatorname{gph}\Lambda^1_{i_1} \\ \hline \operatorname{cl}\operatorname{gph}\Lambda^2_{i_2} \\ \hline \vdots \\ \hline \operatorname{cl}\operatorname{gph}\Lambda^N_{i_N} \end{pmatrix}.$$

Now, consider some $\bar{x} \in \Gamma_s \subset \operatorname{cl}\Gamma_i$ and define $\bar{x}_0 = x_0$. Since F is affine linear function and Ω_i is a polyhedral set, multifunction $S_i(p) := \{x \mid p + F(x) \in \Omega_i\}$ is calm at $(0, \bar{x})$. Then [14, Proposition 3.4] implies that $\mathrm{N}_{\operatorname{cl}\Gamma_i}(\bar{x}) \subset (\nabla F(\bar{x}))^\top \mathrm{N}_{\Omega_i}(F(\bar{x}))$. But since Ω_i is convex, it is regular, and thus [25, Theorem 6.14] implies that

$$\hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\bar{x}) = (\nabla F(\bar{x}))^\top \hat{\mathrm{N}}_{\Omega_i}(F(\bar{x})), \tag{36}$$

Plugging in the original data, we observe that $x^* \in \hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}(\bar{x})$ if and only if for every $n = 1,\dots,N$ there exist some multipliers $p_{n-1}, q_n \in \mathbb{R}^K$ with

$$\begin{pmatrix} p_{n-1} \\ q_n \end{pmatrix} \in \hat{\mathrm{N}}_{\operatorname{cl}\operatorname{gph}\Lambda^n_{i_n}}(\bar{x}_{n-1}, \bar{x}_n),\ n = 1,\dots,N,$$

such that equations $x_n^* = p_n + q_n$ hold for $n = 1,\dots,N$ with $p_N := 0$. But this is equivalent to the stated result by virtue of Lemma 5, Lemma 6 and Theorem 1. □

The previous result may be used directly to calculate $\operatorname{gph}\hat{\mathrm{N}}_\Gamma$ and $\operatorname{gph}\mathrm{N}_\Gamma$, and $\hat{\mathrm{N}}_\Gamma(\bar{x})$ for $\bar{x} \in \Gamma$, using Theorem 2 and Corollary 1, respectively. We note that $I(s)$ can be computed in a convenient way due to Lemma 5.

Remark 3 Even though we were able to express $I(s)$ in terms of $I^n(s_n)$ in Lemma 5 and similarly $\hat{\mathrm{N}}_{\operatorname{cl}\Gamma_i}$ in terms of $\hat{\mathrm{N}}_{\operatorname{cl}\operatorname{gph}\Lambda^n_{i_n}}$ in Theorem 3, we are convinced that it is not possible to derive a similar formula for $\hat{\mathrm{N}}_\Gamma$. In this remark we show that the following intuitive formula

$$\hat{\mathrm{N}}_\Gamma(\Gamma_s) = \left\{ \begin{pmatrix} p_1 + q_1 \\ \vdots \\ p_N + q_N \end{pmatrix} \in \mathbb{R}^{KN} \;\middle|\; \begin{array}{c} \begin{pmatrix} p_{n-1} \\ q_n \end{pmatrix} \in \hat{\mathrm{N}}_{\mathrm{gph}\,\Lambda^n}(\mathrm{gph}\,\Lambda^n_{s_n}),\ n = 1, \dots, N \\ p_N = 0 \end{array} \right\} \tag{37}$$

does not hold true. This is closely connected with violation of the so–called intersection property [10, Definition 9] for (36), which says that

$$\bigcap_{i \in I(s)} (\nabla F(\bar{x}))^\top \hat{\mathrm{N}}_{\Omega_i}(F(\bar{x})) = (\nabla F(\bar{x}))^\top \bigcap_{i \in I(s)} \hat{\mathrm{N}}_{\Omega_i}(F(\bar{x})).$$

Indeed, consider the following example with $N = 2$, $K = 2$,

$$\mathrm{gph}\,\Lambda^1 = \left[\mathbb{R} \times \mathbb{R} \times \{0\} \times \mathbb{R}_{--}\right] \bigcup\ [\mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\}] \bigcup \left\{(a, b, c, d) \in \mathbb{R}^4 \mid c \in \mathbb{R}_{--}, d = -c\right\},$$
$$\mathrm{gph}\,\Lambda^2 = \left[\mathbb{R}_{--} \times \{0\} \times \mathbb{R} \times \mathbb{R}\right] \bigcup\ [\{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}] \bigcup \left\{(a, b, c, d) \in \mathbb{R}^4 \mid a \in \mathbb{R}_{++}, b = -a\right\}$$

and initial point $x_0 = (0, 0)$. Then one observes that $\Gamma = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$ and thus for any $\bar{x} \in \Gamma$ we have $\mathrm{N}_\Gamma(\bar{x}) = \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\}$. On the other hand, the right–hand side of formula (37) results in $\mathbb{R}_+ \times \mathbb{R}_+ \times \{0\} \times \{0\}$ and thus (37) does not hold true.

4.2 Example

Consider set

$$\Gamma := \left\{(y, z) \in \mathbb{R}^N \times \mathbb{R}^N \mid z_n \in \mathrm{N}_{[0, y_{n-1}]}(y_n),\ n = 1, \dots, N\right\} \tag{38}$$

with $y_0 = 1$. Such set arises in delamination modeling [26] where variable $y_n \in [0, 1]$ signifies the delamination level of an adhesive. Specifically, $y_n = 1$ corresponds to a situation where the adhesive is not damaged while $y_n = 0$ corresponds to a complete delamination. Due to the definition of normal cone, we see that (38) contains a hidden constraint $0 \le y_n \le y_{n-1}$, meaning that a glue cannot heal back to its original state y_0. When considering optimal control or parameter identification in such model, it is advantageous to compute $\mathrm{gph}\,\mathrm{N}_\Gamma$, see [3].

We are not able to use the standard results of variational analysis to compute $\mathrm{N}_\Gamma(\bar{y}, \bar{z})$. Since the set $[0, y_{n-1}]$ depends on y, we would have to introduce first additional variables. For example, it is possible to rewrite

$$z_n \in \mathrm{N}_{[0, y_{n-1}]}(y_n)$$

into the following system

$$\begin{aligned} z_n &= z_n^+ + z_n^-, \\ z_n^+ &\in \mathrm{N}_{(-\infty, 0]}(y_n - y_{n-1}), \\ z_n^- &\in \mathrm{N}_{[0, \infty)}(y_n). \end{aligned}$$

However, Mangasarian–Fromovitz constraint qualification is not satisfied for this case if $\bar{y}_{n-1} = \bar{y}_n = 0$, and thus results such [25, Theorem 6.14] or [20] cannot be used. Considering this reformulation, it would be possible to use calculus rules with calmness constraint qualification [14] leading only to an inclusion instead of equality.

For these reasons, we will compute $\mathrm{gph}\,\mathrm{N}_\Gamma$ with Γ defined in (38) using Theorem 2 and Theorem 3. We consider $x = (y, z)$ and rewrite $z_n \in \mathrm{N}_{[0, y_{n-1}]}(y_n)$ equivalently as

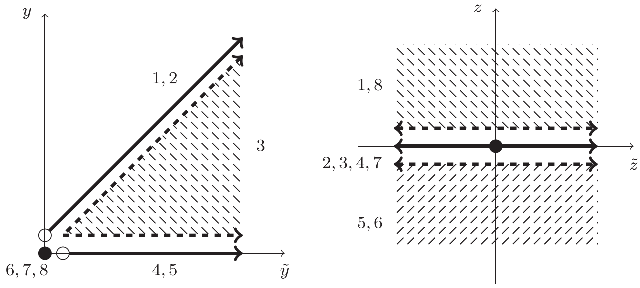

Fig. 3 Partition of gph Λ^n from (39)

$(y_n, z_n) \in \Lambda^n(y_{n-1}, z_{n-1}) = \bigcup_{j=1}^{8} \Lambda_j^n(y_{n-1}, z_{n-1})$ with initial condition $(y_0, z_0) = (1, 0)$ and Λ_i^n, $i = 1, \ldots, 8$, being defined via respective graphs as follows

$$\begin{aligned}
\operatorname{gph} \Lambda_1^n &= \left\{(\tilde{y}, \tilde{z}, y, z) \in \mathbb{R}^4 \mid \tilde{y} \in \mathbb{R}_{++}, \tilde{z} \in \mathbb{R}, y = \tilde{y}, z \in \mathbb{R}_{++}\right\}, \\
\operatorname{gph} \Lambda_2^n &= \left\{(\tilde{y}, \tilde{z}, y, 0) \in \mathbb{R}^4 \mid \tilde{y} \in \mathbb{R}_{++}, \tilde{z} \in \mathbb{R}, y = \tilde{y}\right\}, \\
\operatorname{gph} \Lambda_3^n &= \left\{(\tilde{y}, \tilde{z}, y, 0) \in \mathbb{R}^4 \mid \tilde{y} \in \mathbb{R}_{++}, \tilde{z} \in \mathbb{R}, y \in (0, \tilde{y})\right\}, \\
\operatorname{gph} \Lambda_4^n &= \mathbb{R}_{++} \times \mathbb{R} \times \{0\} \times \{0\}, \\
\operatorname{gph} \Lambda_5^n &= \mathbb{R}_{++} \times \mathbb{R} \times \{0\} \times \mathbb{R}_{--}, \\
\operatorname{gph} \Lambda_6^n &= \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}_{--}, \\
\operatorname{gph} \Lambda_7^n &= \{0\} \times \mathbb{R} \times \{0\} \times \{0\}, \\
\operatorname{gph} \Lambda_8^n &= \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}_{++}.
\end{aligned} \tag{39}$$

Then, $\{\operatorname{gph} \Lambda_j^n \mid j = 1, \ldots, 8\}$ forms a normally admissible stratification of gph Λ^n for all $n = 1, \ldots, N$, see Fig. 3.

Next, directly from (4), we obtain for all $n = 1, \ldots, N$

$$\begin{aligned}
&I^n(1) = \{1\} && I^n(5) = \{5\}, \\
,\ &I^n(2) = \{1, 2, 3\}, && I^n(6) = \{5, 6\}, \\
&I^n(3) = \{3\}, && I^n(7) = \{1, \ldots, 8\}, \\
&I^n(4) = \{3, 4, 5\}, && I^n(8) = \{1, 8\}.
\end{aligned} \tag{40}$$

To construct normally admissible stratification of Γ, we need to characterize Θ given by (33).

Lemma 7 *Setting $i_0 = 1$, it holds that*

$$\Theta = \left\{ (i_1, \ldots, i_N) \in \{1, \ldots, 8\}^N \;\middle|\; \begin{array}{r} i_{n-1} \in \{1, 2, 3\} \implies i_n \in \{1, 2, 3, 4, 5\} \\ i_{n-1} \in \{4, 5, 6, 7, 8\} \implies i_n \in \{6, 7, 8\} \end{array} \right\}. \tag{41}$$

Proof Denote the right–hand side of (41) by A. If $i \in \Theta$, then there exists some $(y, z) \in \Gamma_i$. If $i_{n-1} \in \{1, 2, 3\}$, then we have $y_{n-1} > 0$, which immediately implies $i_n \in \{1, 2, 3, 4, 5\}$. If $i_{n-1} \in \{4, 5, 6, 7, 8\}$, then $y_n = 0$ and thus $i_n \in \{6, 7, 8\}$. Hence $\Theta \subset A$.

To finish the proof, consider now any $i \in A$ and define y coordinatewise as follows

$$y_n = \begin{cases} y_{n-1} & \text{if } i_n \in \{1, 2\}, \\ \frac{1}{2} y_{n-1} & \text{if } i_n = 3, \\ 0 & \text{if } i_n \in \{4, 5, 6, 7, 8\}, \end{cases}$$

with $y_0 = 1$. Then it is not difficult to find z such that $(y, z) \in \Gamma_i$, and thus $i \in \Theta$, which completes the proof. $\square$

Now we have enough information to compute $\operatorname{gph} \mathrm{N}_\Gamma$ using Theorem 3. For simplicity, we will compute $\mathrm{N}_\Gamma(\bar{y}, \bar{z})$ for two given points $(\bar{y}, \bar{z})$. The first one is rather simple and will be computed thoroughly, while for the second one we show only the first stage of the computation.

Example 4 Consider Γ defined in (38) with $N = 5$, $\bar{y} = (1, 0.5, 0, 0, 0)$ and $\bar{z} = (1, 0, 0, 1, -1)$. First, we realize that $\bar{s} = (1, 3, 4, 8, 6)$, where $\bar{s} \in \Theta$ is the unique index such that $(\bar{y}, \bar{z}) \in \Gamma_{\bar{s}}$. Employing (40), we realize that

$$I^1(\bar{s}_1) = \{1\},\ I^2(\bar{s}_2) = \{3\},\ I^3(\bar{s}_3) = \{3, 4, 5\},\ I^4(\bar{s}_4) = \{1, 8\} \text{ and } I^5(\bar{s}_5) = \{5, 6\}.$$

Then, denoting $i = (1, 3, 3, 1, 5)$, $j = (1, 3, 4, 8, 6)$ and $k = (1, 3, 5, 8, 6)$, Lemma 5 together with formula (41) yields

$$\begin{aligned} I(\bar{s}) &= \{i, j, k\}, \\ I(i) &= \tilde{I}(i) = \{i\}, \\ I(j) &= \{i, j, k\}, \quad \tilde{I}(j) = \{i, k\}, \\ I(k) &= \tilde{I}(k) = \{k\}. \end{aligned}$$

Thus, invoking formula (12) we have

$$\mathrm{N}_\Gamma(\bar{y}, \bar{z}) = \hat{\mathrm{N}}_{\operatorname{cl} \Gamma_i}(\Gamma_i) \cup \left[\hat{\mathrm{N}}_{\operatorname{cl} \Gamma_i}(\Gamma_j) \cap \hat{\mathrm{N}}_{\operatorname{cl} \Gamma_k}(\Gamma_j) \right] \cup \hat{\mathrm{N}}_{\operatorname{cl} \Gamma_k}(\Gamma_k).$$

Each of the regular normal cones in this formula can be computed via application of Theorem 3 with the use of the following regular normal cones, $n = 1, \dots, 5$,

$$\begin{aligned} \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_1^n}(\operatorname{gph} \Lambda_1^n) &= \left\{ (\tilde{\alpha}, 0, \alpha, 0) \in \mathbb{R}^4 \,|\, \tilde{\alpha} \in \mathbb{R}, \alpha = -\tilde{\alpha} \right\}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_3^n}(\operatorname{gph} \Lambda_3^n) &= \{0\} \times \{0\} \times \{0\} \times \mathbb{R}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_3^n}(\operatorname{gph} \Lambda_4^n) &= \{0\} \times \{0\} \times \mathbb{R}_- \times \mathbb{R}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_5^n}(\operatorname{gph} \Lambda_4^n) &= \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}_+, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_5^n}(\operatorname{gph} \Lambda_5^n) &= \{0\} \times \{0\} \times \mathbb{R} \times \{0\}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_5^n}(\operatorname{gph} \Lambda_6^n) &= \mathbb{R}_- \times \{0\} \times \mathbb{R} \times \{0\}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_6^n}(\operatorname{gph} \Lambda_6^n) &= \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_1^n}(\operatorname{gph} \Lambda_8^n) &= \left\{ (\tilde{\alpha}, 0, \alpha, 0) \in \mathbb{R}^4 \,|\, \tilde{\alpha} \in \mathbb{R}, \alpha \le -\tilde{\alpha} \right\}, \\ \hat{\mathrm{N}}_{\operatorname{cl} \operatorname{gph} \Lambda_8^n}(\operatorname{gph} \Lambda_8^n) &= \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}. \end{aligned}$$

This results in

$$
\begin{aligned}
\mathrm{N}_\Gamma(\bar{y},\bar{z}) &= \cup_{t\in\mathbb{R}} \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ \{t\}\times\mathbb{R} \\ \{-t\}\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix} \bigcup \left[\cup_{s\in\mathbb{R}} \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ (-\infty,s]\times\mathbb{R} \\ (-\infty,-s]\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix} \bigcap \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ \mathbb{R}\times\mathbb{R}_+ \\ \mathbb{R}\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix} \right] \bigcup \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ \mathbb{R}\times\{0\} \\ \mathbb{R}\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix} \\
&= \cup_{t\in\mathbb{R}} \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ \{t\}\times\mathbb{R} \\ \{-t\}\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix} \bigcup \cup_{s\in\mathbb{R}} \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ (-\infty,s]\times\mathbb{R}_+ \\ (-\infty,-s]\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix} \bigcup \begin{bmatrix} \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R} \\ \mathbb{R}\times\{0\} \\ \mathbb{R}\times\{0\} \\ \mathbb{R}\times\{0\} \end{bmatrix}. \qquad (42)
\end{aligned}
$$

Example 5 In the setting of Example 4 we consider $\bar{y} = (1, 0.5, 0, 0, 0)$ and $\bar{z} = (1, 0, 0, 0, 1)$. Then we have $\bar{s} = (1, 3, 4, 7, 8)$ and

$$
I(\bar{s}) = \left\{ i \in \{1,\dots,8\}^N \;\middle|\; \begin{array}{l} i_1 = 1,\; i_2 = 3,\; i_3 \in \{3,4,5\} \\ i_3 = 3 \implies i_4 \in \{1,2,3,4,5\} \\ i_3 \in \{4,5\} \implies i_4 \in \{4,5,6,7,8\} \\ i_4 \in \{1,2,3\} \implies i_5 = 1 \\ i_4 \in \{4,5,6,7,8\} \implies i_5 = 8 \end{array} \right\}.
$$

It is not difficult to verify that $I(\bar{s})$ contains 15 elements and hence we will have to consider a union with respect to 15 elements in (12). Then it would be necessary to compute $\tilde{I}(s)$ for every $s \in I(\bar{s})$ using Lemma 5, which would, however, in most cases amount to only one or two elements.

Finally, in the light of Example 4 and especially Example 5 we present another comparison of our approach with the theory developed in [12]; a comparison which was already slightly touched in Example 3.

Remark 4 Consider set Γ defined in (38) and let us show that even though the approach developed in this paper is not simple, it could be more applicable than the approach developed in [12]. There it is necessary to compute $\mathrm{T}_\Gamma(\bar{y},\bar{z})$ first, which, due to our best knowledge, cannot be tackled by standard calculus rules because of the same reasons as described earlier in this subsection. Even though it is possible to derive formula for $\mathrm{T}_\Gamma(\bar{y},\bar{z})$ directly from the definition, it is not a simple task.

Consider now the same point $(\bar{y},\bar{z})$ as in Example 4. With the notation of Section 3.2 it is possible to show that $|\mathbb{I}(\bar{x},\bar{y})| = 2$ with

$$
\Delta_1 = \bigcup_{t\in\mathbb{R}_+} \begin{bmatrix} \{0\}\times\mathbb{R} \\ \mathbb{R}\times\{0\} \\ \{t\}\times\{0\} \\ \{t\}\times\mathbb{R} \\ \{0\}\times\mathbb{R} \end{bmatrix}, \qquad \Delta_2 = \begin{bmatrix} \{0\}\times\mathbb{R} \\ \mathbb{R}\times\{0\} \\ \{0\}\times\mathbb{R}_- \\ \{0\}\times\mathbb{R} \\ \{0\}\times\mathbb{R} \end{bmatrix}.
$$

Now, we show that a direct application of Proposition 2 can be rather cumbersome. It is clear that the first union in (28) will be performed with respect to three elements. Since each Δ_i can be described as an intersection of 11 halfspaces, the any fixed $\mathbb{I}$ for expressing the second and third union in (28), one has to check 121 combinations of sets $R_{\mathbb{I}}^{\mathcal{J},J}$ and $S_{\mathbb{I}}^{\mathcal{J}}$, leading together to necessity of solving 363 systems of linear (27). The number is so high because the majority of this systems will have some solution apart from 0 and thus the set $A_{\mathbb{I}}^{J}$ will contain lesser number of elements. Note that in Example 4 we need to compute only union of 3 elements. The situation would become more difficult, or possibly intractable should we consider $(\bar{y}, \bar{z})$ as in Example 5.

Another approach to compute the desired normal cone is to realize that Δ_1 and Δ_2 differ only at components y_3, z_3 and y_4, and so we obtain from [12, Proposition 3.1] that

$$\mathrm{N}_\Gamma(\bar{x}, \bar{y}) = \mathbb{R} \times \{0\} \times \{0\} \times \mathbb{R} \times \Omega \times \{0\} \times \mathbb{R} \times \{0\},$$

where

$$\begin{aligned} \Omega &= \operatorname{bd} \Theta_1^* \bigcup (\Theta_1^* \cap \Theta_2^*) \bigcup \operatorname{bd} \Theta_2^*, \\ \Theta_1 &= \{(y_3, 0, y_4) |\, y_3 \in \mathbb{R}_+,\ y_4 = y_3\}, \\ \Theta_2 &= \{0\} \times \mathbb{R}_- \times \{0\}. \end{aligned} \tag{43}$$

After computing the polars to Θ_1 and Θ_2, it becomes clear that the three elements of unions in (42) and (43) do correspond.

Finally, we would like to point out that non-regular points (such as $\bar{y}_n = \bar{y}_{n-1} > 0$ and $\bar{z}_n > 0$) fit well into our approach, while in [12] these points considerably increase the number of halfplanes defining Δ_i.

5 Conclusion

In this paper, we have proposed a new approach for computation of Fréchet and limiting normal cones to a set which can be expressed as a finite union of convex polyhedra. Moreover, we have compared our results to several selected known results, and applied the proposed approach to the case of time dependent problems.

We believe that, based on Remark 2, our approach can be used to derive stability conditions for general bilevel programs where the constraints on the lower level amount to a polyhedral set. In this way, results of [5] dealing with MPCCs might be generalized. This, however, goes beyond the scope of this paper.

Appendix A: Auxiliary lemmas

Lemma A1 *Consider continuous functions $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \dots, I$ and affine linear $h_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \dots, J$ and define the following set*

$$A = \{x |\, g_i(x) < 0,\ h_j(x) = 0,\ i = 1, \dots, I,\ j = 1, \dots, J\}.$$

Then A is relatively open. Moreover, if g_i are convex for all $i = 1, \dots, I$ and A is nonempty, then

$$\operatorname{cl} A = \{x |\, g_i(x) \le 0,\ h_j(x) = 0,\ i = 1, \dots, I,\ j = 1, \dots, J\}. \tag{44}$$

Proof Since g_i are continuous, $A_1 := \{x |\, g_i(x) < 0,\ i = 1, \dots, I\}$ is an open set. As h_j are affine linear, we know that $A_2 := \{x |\, h_j(x) = 0,\ j = 1, \dots, J\}$ is an affine subspace. Thus, $A = A_1 \cap A_2$ is relatively open.

To prove the second result, denote the right–hand side of (44) by B. Clearly, we have $\operatorname{cl} A \subset B$ without any additional assumptions. To show the opposite inclusion, consider any $x \in B$. Since A is nonempty, there exists $\bar{x}$ such that $g_i(\bar{x}) < 0$ and $h_j(\bar{x}) = 0$. Due to the assumptions, we know that $x_n := (1 - \frac{1}{n})x + \frac{1}{n}\bar{x} \in A$ and $x_n \to x$, which finishes the proof. □

Lemma A2 *Assume that $A \subset \mathbb{R}^n$ is convex and relatively open and consider some $x \in A$ and $y \in \operatorname{cl} A$. Then for all $\lambda \in (0, 1)$ we have $\lambda x + (1 - \lambda)y \in A$.*

Proof The statement is a direct consequence of [24, Theorem 6.1]. $\square$

Lemma A3 *Consider a normally admissible stratification $\{\Gamma_s \,|\, s = 1, \ldots, S\}$ of Γ and some $\mathcal{S} \subset \{1, \ldots, S\}$. Then*

$$\bigcap_{s \in \mathcal{S}} \operatorname{cl} \Gamma_s = \bigcup_{\{t \,|\, \mathcal{S} \subset I(t)\}} \Gamma_t. \tag{45}$$

Proof Assume that $x \in \operatorname{cl} \Gamma_s$ for all $s \in \mathcal{S}$. Then there exists some t such that $x \in \Gamma_t$. But this means that $x \in \Gamma_t \cap \operatorname{cl} \Gamma_s$ for all $s \in \mathcal{S}$ and thus $s \in I(t)$ for all $s \in \mathcal{S}$, meaning that $\mathcal{S} \subset I(t)$.

On the other hand, consider any t such that $\mathcal{S} \subset I(t)$. Then for any $s \in \mathcal{S}$, we have $s \in \mathcal{S} \subset I(t)$, and thus $\Gamma_t \subset \operatorname{cl} \Gamma_s$, which finishes the proof. $\square$

Lemma A4 *For a polyhedral set C consider its all nonempty relatively open faces C_s with $s = 1, \ldots, S$. Then $\{C_s \,|\, s = 1, \ldots, S\}$ forms a normally admissible stratification of C.*

Proof Since all properties of Definition 2 apart from formula (3) obviously hold, it remains to verify this formula. Consider thus some C_s and C_i such that $C_s \cap \operatorname{cl} C_i \neq \emptyset$. Since we can write

$$\begin{aligned}
C &= \{x \,|\, \langle c_t, x\rangle \le b_t,\ t = 1, \ldots, T\},\\
C_s &= \{x \,|\, \langle c_t, x\rangle < b_t,\ t \in \mathcal{T}_{11},\ \langle c_t, x\rangle = b_t,\ t \in \mathcal{T}_{12}\},\\
\operatorname{cl} C_i &= \{x \,|\, \langle c_t, x\rangle \le b_t,\ t \in \mathcal{T}_{21},\ \langle c_t, x\rangle = b_t,\ t \in \mathcal{T}_{22}\},
\end{aligned}$$

where $\mathcal{T}_{j1} \cap \mathcal{T}_{j2} = \emptyset$ and $\mathcal{T}_{j1} \cup \mathcal{T}_{j2} = \{1, \ldots, T\}$ for $j = 1, 2$ and since there is some $x \in C_s \cap \operatorname{cl} C_i$, we have $\mathcal{T}_{11} \subset \mathcal{T}_{21}$ and thus $C_s \subset \operatorname{cl} C_i$, which finishes the proof. $\square$

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