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9. Basic Solutions of Fuzzy Coalitional Games

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This chapter is concerned with basic concepts of solution for coalitional games with fuzzy coalitions in the case of finitely many players and transferable utility. The focus is on those solutions which preoccupy the main part of cooperative game theory (the core and the Shapley value). A detailed discussion or just the comprehensive overview of current trends in fuzzy games is beyond the reach of this chapter. Nevertheless, we mention current developments and briefly discuss other solution concepts.

The theory of cooperative games builds and analyses mathematical models of situations in which players can form coalitions and make binding agreements on how to share results achieved by these coalitions. One of the basic models of cooperative games is a *cooperative game in coalitional form* (briefly a *coalitional game* or a *game*). Following *Osborne* and *Rubinstein* [9.1] we assume that the data specifying a coalitional game are composed of:

- A nonempty set Ω (the set of players) and a nonempty set X (the set of consequences),
- A mapping V that assigns to every subset S of Ω a subset $V(S)$ of X, and
- A family $\{\succ_i\}_{i\in\Omega}$ of binary relations on X (players' preference relations).

The set Ω of all players is usually referred to as the *grand coalition*, subsets of Ω are called *coalitions*, and the mapping V is called the *characteristic function* (or *coalition function*) of the game.

This definition provides a rather general framework for analyzing many classes of coalitional games. The games of this type are usually called coalitional games *without side payments* or *without transferable payoff* (or *utility*). Obviously, for many purposes, this framework is too general because it neither specifies some useful structure of the set of consequences nor properties of preference relations. At the same time, this framework is also too restrictive because of requiring that the domain of the characteristic function must be the system of all subsets of the player set.

In this chapter, we are mainly concerned with coalitional games in which the number of players is finite. The number of players will be denoted by n and, without loss of generality, the players will be named by integers $1,2,\dots,n$. In other words, we set $\Omega = N$ where $N=\{1,2,\dots,n\}$. Moreover, we assume that the sets $V(S)$ of consequences are subsets of the n-dimensional real linear space $\mathbb{R}^n$, and that each player i prefers $(x_1,\dots,x_n)$ to $(y_1,\dots,y_n)$ if and only if $x_i > y_i$. Furthermore, we significantly restrict the generality by considering only the so-called coalitional games with transferable payoff or utility. This class of games is a subclass of games without transferable utility that is characterized by the property: for each coalition S, there exists a real number $v(S)$ such that

$$V(S)=\left\{x\in\mathbb{R}^n : \sum_{i\in S} x_i \le v(S) \text{ and } x_j = 0 \text{ if } j\notin S\right\}.$$

Evidently, each such game can be identified with the corresponding real-valued function v defined on the system of all subsets of N.

In coalitional games, whether with transferable or nontransferable utility, each player has only two alternatives of participation in a nonempty coalition: full participation or no participation. This assumption is too restrictive in many situations, and there has been a need for models that give players the possibility of participation in some or all intermediate levels between these two extreme involvements.

The first mathematical models in the form of coalitional games in which the players are permitted to participate in a coalition not only fully or not at all but also partially were proposed by *Butnariu* [9.2] and *Aubin* [9.3]. Aubin notices that the idea of partial participation in a coalition was used already in the *Shapley–Shubik* paper on market games [9.4]. In these models, the subsets of N no longer represent every possible coalition. Instead, a notion of a coalition has to be introduced that makes it possible to represent the partial membership degrees.

It has become customary to assume that a membership degree of player $i \in N$ is determined by a number a_i in the unit interval $I = [0,1]$, and to call the resulting vector $a = (a_1, \dots, a_n) \in I^n$ a *fuzzy coalition*. The n-dimensional cube I^n is thus identified with the set of all fuzzy coalitions. Every subset S of N, that is, every coalition S, can be viewed as an n-vector from $\{0,1\}^n$ whose ith components is 1 when $i \in S$ and 0 when $i \notin S$. These special fuzzy coalitions are often called *crisp* coalitions. Hence, we may think of the set of all fuzzy coalitions I^n as the convex closure of the set $\{0,1\}^n$ of all crisp coalitions. This leads to the notion of an n-player coalitional game with fuzzy coalitions and transferable utility (briefly a fuzzy game) as a bounded function $v\colon I^n \to \mathbb{R}$ satisfying $v(0) = 0$.

It turns out that most classes of coalitional games with transferable utility and most solution concepts have natural counterparts in the theory of fuzzy games with transferable utility. Therefore, in what follows, we start with the classical case (Sect. 9.1) and then deal with the fuzzy case (Sect. 9.2). Taking into account that, in comparison with the classical case, the theory of fuzzy games is relatively less developed, we focus attention on two well-established solution concepts of fuzzy games: the core and the Shapley value.

9.1 Coalitional Games with Transferable Utility

We know from the beginning of this chapter that from the mathematical point of view, every n-player coalitional game with transferable utility can be identified with a real-valued function v defined on the system of all subsets of the set $N = \{1, 2, \dots, n\}$. For convenience, we assume that always $v(\emptyset) = 0$.

It is customary to interpret the value $v(S)$ of the characteristic function v at coalition S as the worth of coalition S or the total payoff that coalition S will be able to distribute among its members, provided exactly the coalition S forms. However, equally well, the number $v(S)$ may represent the total cost of reaching some common goal of coalition S that must be shared by the members of S; or some other quantity, depending on the application field. In conformity with the players preferences stated previously, we usually assume that $v(S)$ represents the total payoff that S can distribute among its members.

Since the preferences are fixed, we denote the game given through N and v by (N, v), or simply v, and the collection of all games with fixed N by G_N. The sum $v + w$ of games from G_N defined by $(v + w)(S) = v(S) + w(S)$ for each coalition S is again a game from G_N. Moreover, if multiplication of $v \in G_N$ by a real number α is defined by $(\alpha v)(S) = \alpha v(S)$ for each coalition S, then αv also belongs to G_N. An important and well-known fact is that G_N endowed with these two algebraic operations is a real linear space.

Example 9.1 Simple games

If the range of a game v is the two-element set $\{0,1\}$ only, then the game can be viewed as a model of a voting system where each coalition $A \subseteq N$ is either *winning* ($v(A) = 1$) or *loosing* ($v(A) = 0$). Then it is natural to assume that the game also satisfies *monotonicity*; that is, if coalition A is winning and B is a coalition with $A \subseteq B$, then B is also winning. It is also natural to consider only games with at least one winning coalition. Thus, we define a *simple game* [9.5, Section 2.2.3] to be a $\{0,1\}$-valued coalitional game v such that the grand coalition is winning and $v(A) \le v(B)$, whenever $A \subseteq B$ for each $A, B \subseteq N$.

We say that a game v is *superadditive* if $v(A \cup B) \ge v(A) + v(B)$, for every disjoint pair of coalitions $A, B \subseteq N$. Consequently, in a superadditive game, it may be advantageous for members of disjoint coalitions A and B to form coalition $A \cup B$ because every pair of disjoint coalitions can obtain jointly at least as much as they could have obtained separately. Consequently, it is advantageous to form the largest possible coalitions, that is, the grand coalition.

The strengthening of the property of superadditivity is the assumption of nondecreasing marginal contribution of a player to each coalition with respect to coalition inclusion: a game v is said to be *convex* whenever

$$v(A \cup \{i\}) - v(A) \leq v(B \cup \{i\}) - v(B)$$

for each $i \in N$ and every $A \subseteq B \subseteq N \setminus \{i\}$. It can be directly checked that convexity of v is equivalent to

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B) \quad \text{for every } A, B \subseteq N\,.$$

Example 9.2
Let B be a nonempty coalition in a simple game (N, v). Then the game v_B given by

$$v_B(A) = \begin{cases} 1, & A \supseteq B\,, \\ 0, & \text{otherwise}\,, \end{cases} \qquad A \subseteq N\,,$$

is a convex simple game.

Example 9.3 Bankruptcy game [9.6]
Let $e > 0$ be the total value of assets held in a bankruptcy estate of a debtor and let N be the set of all creditors. Furthermore, let $d_i > 0$ be the debt to creditor $i \in N$. Assume that $e \leq \sum_{i \in N} d_i$. The *bankruptcy game* is then the game such that, for every $A \subseteq N$,

$$v(A) = \max\left(0, e - \sum_{i \in N \setminus A} d_i\right).$$

It can be shown that the bankruptcy game is convex.

There is a variety of solution concepts for coalitional games with n players. Some, like the core, stable set or bargaining set, may consist of sets of real n-vectors, while others offer as a solution of a game a single real n-vector.

9.1.1 The Core

Let v be an n-player coalitional game with transferable utility. The *core* of v is the set of all efficient payoff vectors $x \in \mathbb{R}^n$ upon which no coalition can improve, that is,

$$C(v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in A} x_i \geq v(A) \text{ for each } A \subseteq N \right\}. \quad (9.1)$$

The Bondareva–Shapley theorem [9.5, Theorem 3.1.4] gives a necessary and sufficient condition for the core nonemptiness in terms of the so-called balanced systems. It is easy to see that the core of every game is a (possibly empty) convex polytope. Moreover, the core of a convex game is always nonempty and its vertices can be explicitly characterized [9.7].

9.1.2 The Shapley Value

Let $f = (f_1, f_2, \ldots, f_n)$ be a mapping that assigns to every game v from some collection of games from G_N a real n-vector $f(v) = (f_1(v), f_2(v), \ldots, f_n(v))$. Following the basic interpretation of values of a characteristic functions as the total payoff, we can interpret the values of components of such a function as payoffs to individual players in game v.

Let $\mathcal{A}$ be a nonempty collection of games from G_N. A *solution function on* $\mathcal{A}$ is a mapping f from $\mathcal{A}$ into the n-dimensional real linear space $\mathbb{R}^n$. If the domain $\mathcal{A}$ of f is not explicitly specified, then it is assumed to be G_N. The collection of such mappings is too broad to contain only the mappings that lead to sensible solution concepts. Hence, to obtain reasonable solution concepts we have to require that the solution functions have some reasonable properties. One of the natural properties in many contexts is the following property of efficiency.

Property 9.1 Efficiency
A solution function f on a subset $\mathcal{A}$ of G_N is *efficient on* $\mathcal{A}$ if $f_1(v) + f_2(v) + \cdots + f_n(v) = v(N)$ for every game v from $\mathcal{A}$.

This property can be interpreted as a combination of the requirements of the *feasibility* defined by $f_1(v) + f_2(v) + \cdots + f_n(v) \leq v(N)$ and *collective rationality* defined by $f_1(v) + f_2(v) + \cdots + f_n(v) \geq v(N)$.

In addition to satisfying the efficiency condition, solution functions are required to satisfy a number of other desirable properties. To introduce some of them, we need further definitions.

Player i from N is a *null* player in game v if $v(S \cup \{i\}) = v(S)$ for every coalition S that does not contain player i; that is, participation of a null player in a coalition does not contribute anything to the coalition in question.

Player i from N is a *dummy* player in game v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every coalition S that does not contain player i; that is, a dummy player contributes to every coalition the same amount, his or her value of the characteristic function.

Players i and j from N are *interchangeable* in game v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S that contains neither player i nor player j. In other words, two players are interchangeable if they can replace each other in every coalition that contain one of them.

Property 9.2 Null player
A solution function f satisfies the *null player property* if $f_i(v) = 0$ whenever $v \in G_N$ and i is a null player in v.

Property 9.3 Dummy player
A solution function f satisfies the *dummy player property* if $f_i(v) = v(\{i\})$ whenever $v \in G_N$ and i is a dummy player in v.

Property 9.4 Equal treatment
A solution function f satisfies the *equal treatment property* if $f_i(v) = f_j(v)$ for every $v \in G_N$ and every pair of players i, j that are interchangeable in v.

These three properties are quite reasonable and attractive, especially from the point of fairness and impartiality: a player who contributes nothing should get nothing; a player who contributes the same amount to every coalition cannot expect to get anything else than he or she contributed; and two players who contribute the same to each coalition should be treated equally by the solution function.

The next property reflects the natural requirement that the solution function should be independent of the players' names. Let v be a game from G_N and $\pi: N \to N$ be a permutation of N, and let the image of coalition S under π be denoted by $\pi(S)$. It is obvious that, for every $v \in G_N$, the function πv defined on G_N by $(\pi v)(S) = v(\pi(S))$ is again a game from G_N. Apparently, the game πv differs from game v only in players' names; they are interchanged by the permutation π.

Property 9.5 Anonymity
A solution function f is said to be *anonymous* if, for every permutation π of N, we have $f_i(\pi v) = f_{\pi(i)}(v)$ for every game $v \in G_N$ and every player $i \in N$.

When a game consists of two independent games played separately by the same players or if a game is split into a sum of games, then it is natural to require the following property of additivity.

Property 9.6 Additivity
A solution function f on G_N is said to be *additive* if $f(u+v) = f(u) + f(v)$ for every pair of games u and v from G_N.

The requirement of additivity differs from the previous conditions in one important aspect. It involves two different games that may or may not be mutually dependent. In contrast, the dummy player and equal treatment properties involve only one game, and the anonymity property involves only those games which are completely determined by a single game.

Remark 9.1
The terminology introduced in the literature for various properties of players and solution functions is not completely standardized. For example, some authors use the term *dummy player* and *symmetric players* (or *substitutes*), for what we call null player and interchangeable players, respectively. Moreover, the term *symmetry* is sometimes used for our equal treatment and sometimes for our anonymity.

One of the most studied and most influential single-valued solution concept for coalitional games with transferable utility is the Shapley solution function or briefly the Shapley value, proposed by *Shapley* in 1953 [9.8]. The simplest way of introducing the Shapley value is to define it explicitly by the following well-known formula for calculation of its components.

Definition 9.1
The Shapley value on a subset $\mathcal{A}$ of G_N is a solution function φ on $\mathcal{A}$ whose components $\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v)$ at game $v \in \mathcal{A}$ are defined by

$$\varphi_i(v) = \sum_{S:\, i \in S} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S \setminus \{i\})]\,, \tag{9.2}$$

where the sum is meant over all coalitions S containing player i, and s generically stands for the number of players in coalition S.

To clarify the basic idea behind this definition, we first recall the notion of players' marginal contributions to coalitions.

Definition 9.2
For each player i and each coalition S, a *marginal contribution* of player i to coalition S in game v from G_N is the number $m_i^v(S)$ defined by

$$m_i^v(S) = \begin{cases} v(S) - v(S \setminus \{i\}) & \text{if} \quad i \in S \\ v(S \cup \{i\}) - v(S) & \text{if} \quad i \notin S \end{cases} .$$

Now imagine a procedure for dividing the total payoff $v(N)$ among the members of N in which the players enter a room in some prescribed order and each player receives his or her marginal contribution as payoff to the coalition of players already being in the room. Suppose that the prescribed order is $(\pi(1), \pi(2), \dots, \pi(n))$ where $\pi: N \to N$ is a fixed permutation of N. Then the procedure under consideration determines the payoffs to individual players as follows: before the first player $\pi(1)$ entered the room, there was the empty coalition waiting in the room. After player $\pi(1)$ enters, the coalition in the room becomes $\{\pi(1)\}$ and the player receives $v(\{\pi(1)\}) - v(\emptyset)$. Similarly, before the second player $\pi(2)$ entered, there was coalition $\{\pi(1)\}$ waiting in the room. After player $\pi(2)$ enters, coalition $\{\pi(1), \pi(2)\}$ is formed in the room and player $\pi(2)$ receives $v(\{\pi(1), \pi(2)\}) - v(\{\pi(1)\})$. This continues till the last player $\pi(n)$ enters and receives $v(N) - v(N \setminus \{\pi(n)\})$.

Let S_i^π denote the coalition of players preceding player i in the order given by $(\pi(1), \pi(2), \dots, \pi(n))$; that is, $S_i^\pi = \{\pi(1), \pi(2), \dots, \pi(j-1)\}$ where j is the uniquely determined member of N such that $i = \pi(j)$. Because there are $n!$ possible orders, the arithmetical average of the marginal contributions of player i taken over all possible orderings is equal to the number $(1/n!) \sum m_i^v(S_i^\pi)$ where the sum is understood over all permutations π of N. This number is exactly the i-th component of the Shapley value. Therefore, in addition to the equality (9.2) we also have the equality

$$\varphi_i(v) = \frac{1}{n!} \sum_\pi m_i^v(S_i^\pi) \tag{9.3}$$

for computing the components of the Shapley value.

In addition to satisfying the condition of efficiency, the Shapley value has a number of other useful properties. In particular, it satisfies all properties 9.2–9.6. Remarkably, no other solution function on G_N satisfies the properties of null player, equal treatment, and additivity at the same time.

Theorem 9.1 Shapley
For each N, there exists a unique solution function on G_N satisfying the properties of efficiency, null player, equal treatment, and additivity; this solution function is the Shapley value introduced by Definition 9.1.

The standard proof of this basic result follows from the following facts:

- The collection $\{u_T : T \neq \emptyset, T \subseteq N\}$ of *unanimity games* defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} , \tag{9.4}$$

  form a base of the linear space G_N.
- The null player and equal treatment properties guarantee that φ is determined uniquely on multiples of unanimity games.
- The property of additivity (combined with the fact that the unanimity games form a basis) makes it possible to extend φ in a unique way to the whole space G_N.

In addition to the original axiomatization by Shapley, there exist several equally beautiful alternative axiomatizations of the Shapley value that do not use the property of additivity [9.9, 10].

9.1.3 Probabilistic Values

Let us fix some player i and, for every coalition S that does not contain player i, denote by $\alpha_i(S)$ the number $s!(n-s-1)!/n!$. The family $\{\alpha_i(S) : S \subseteq N \setminus \{i\}\}$ is a probability distribution over the set of coalitions not containing player i. Because the i-th component of the Shapley value can be computed by

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \alpha_i(S)[v(S \cup \{i\}) - v(S)] ,$$

we see that the i-th component of the Shapley value is the expected marginal contribution of player i with respect to the probability measure $\{\alpha_i(S) : S \subseteq N \setminus \{i\}\}$ and that the Shapley value belongs to the following class of solution functions:

Definition 9.3
A solution function f on a subset $\mathcal{A}$ of G_N is called *probabilistic on* $\mathcal{A}$ if, for each player i, there ex-

ists a probability distribution $\{p_i(S) : S \subseteq N \setminus \{i\}\}$ on the collection of coalitions not containing i such that

$$f_i(v) = \sum_{S \subseteq N \setminus \{i\}} p_i(S)[v(S \cup \{i\}) - v(S)] \tag{9.5}$$

for every $v \in \mathcal{A}$.

The family of probabilistic solution functions embraces an enormous number of functions [9.11]. The efficient probabilistic solution functions are often called *quasi-values*, and the anonymous probabilistic solution functions are called *semivalues*. Since the Shapley value is anonymous and efficient on G_N, we know that it is both a quasivalue and a semivalue on G_N. Moreover, the Shapley value is the only probabilistic solution function with these properties.

Theorem 9.2 Weber
If N has at least three elements, then the Shapley value is the unique probabilistic solution function on G_N, that is, anonymous and efficient.

Another widely known probabilistic solution function is the function proposed originally only for voting games by *Banzhaf* [9.12].

Definition 9.4
The *Banzhaf value* on G_N is a solution function ψ on G_N whose components at game v are defined by

$$\psi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}}[v(S \cup \{i\}) - v(S)] \,. \tag{9.6}$$

Again, by simple computation, we can verify that the Banzhaf solution is a probabilistic solution function. Consequently, the i-th component of the Banzhaf solution is the expected marginal contribution of player i with respect to the probability measure $\{\beta_i(S) : S \subseteq N \setminus \{i\}\}$, where $\beta_i(S) = 1/2^{n-1}$ for each subset S of $N \setminus \{i\}$. From the probabilistic point of view, the Banzhaf solution is based on the assumption that each player i is equally likely to join any subcoalition of $N \setminus \{i\}$. On the other hand, the Shapley value is based on the assumption that the coalition the player i enters is equally likely to be of any size s, $0 \leq s \leq n-1$, and that all coalitions of this size are equally likely.

9.2 Coalitional Games with Fuzzy Coalitions

Since the publication of *Aubin*'s seminal paper [9.3], cooperative scenarios allowing for players' fractional membership degrees in coalitions have been studied. In such situations, the subsets of N no longer model every possible coalition. Instead, a notion of coalitions has to be introduced that makes it possible to represent the partial membership degrees. It has become customary to assume that a membership degree of player $i \in N$ is determined by a number a_i in the unit interval $I = [0, 1]$, and to call the resulting vector $a = (a_1, \dots, a_n) \in I^n$ a *fuzzy coalition*. (The choice of I^n is not the only possible choice, see [9.13] or the discussion in [9.14].) The n-dimensional cube I^n is thus identified with the set of all fuzzy coalitions. Every subset A of N, that is, every classical coalition, can be viewed as a vector $\mathbb{1}_A \in \{0, 1\}^n$ with coordinates

$$(\mathbb{1}_A)_i = \begin{cases} 1 & \text{if } i \in A \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

These special fuzzy coalitions are also called *crisp* coalitions. When $A = \{i\}$ is a singleton, we write simply $\mathbb{1}_i$ in place of $\mathbb{1}_{\{i\}}$. Hence, we may think of the set of all fuzzy coalitions I^n as the convex closure of the set $\{0, 1\}^n$ of all crisp coalitions; see [9.14] for further explanation of this convexification process.

Several definitions of fuzzy games appear in the literature [9.3, 15]. We adopt the one used by *Azrieli* and *Lehrer* [9.13]. However, note that the authors of [9.13] use a slightly more general definition, since they consider a fuzzy coalition a to be any nonnegative real vector such that $a \leq q$, where $q \in \mathbb{R}^n$ is a given nonnegative vector.

Definition 9.5
An n-player *game (with fuzzy coalitions and transferable utility)* is a bounded function $v \colon I^n \to \mathbb{R}$ satisfying $v(\mathbb{1}_\emptyset) = 0$.

If we want to emphasize the dependence of Definition 9.5 on the number n of players, then we write (I^n, v) in place of v. Further, by $\bar{v}$ we denote the restriction of v to all crisp coalitions

$$\bar{v}(A) = v(\mathbb{1}_A) \,, \quad A \subseteq N \,. \tag{9.7}$$

Hence, every game with fuzzy coalitions v induces a classical coalition game $\bar{v}$ with transferable utility.

Most solution concepts of the cooperative game theory have been generalized to games with fuzzy coalitions. A *payoff vector* is any vector x with n real coordinates, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. In a particular game with fuzzy coalitions (I^n, v), each player $i \in N$ obtains the amount of utility x_i as a result of his cooperative activity. Consequently, a fuzzy coalition $a \in I^n$ gains the amount

$$\langle a, x \rangle = \sum_{i=1}^{n} a_i x_i ,$$

which is just the weighted average of the players' payoffs x with respect to their participation levels in the fuzzy coalition a. By a *feasible payoff* in game (I^n, v), we understand a payoff vector x with $\langle \mathbb{1}_N, x \rangle \leq v(\mathbb{1}_N)$.

The following general definition captures most solution concepts for games with fuzzy coalitions.

Definition 9.6
Let Γ_N be a class of all games with fuzzy coalitions (I^n, v) and let Λ_N be its nonempty subclass. A *solution* on Λ_N is a function σ that associates with each game (I^n, v) in Λ_N a subset $\sigma(I^n, v)$ of the set

$$\{x \in \mathbb{R}^n | \langle \mathbb{1}_N, x \rangle \leq v(\mathbb{1}_N)\}$$

of all feasible payoffs in game (I^n, v).

The choice of σ is governed by all thinkable rules of economic rationality. Every solution σ is thus determined by a system of restrictions on the set of all feasible payoff vectors in the game. For example, we may formulate a set of axioms for σ to satisfy or single out inequalities making the payoffs in $\sigma(I^n, v)$ stable, in some sense.

9.2.1 Multivalued Solutions

Core

The core is a solution concept σ defined on the whole class of games with fuzzy coalitions Γ_N. We present the definition that appeared in [9.3].

Definition 9.7
Let $N = \{1, \ldots, n\}$ be the set of all players and $v \in \Gamma_N$. The *core* of v is a set

$$C(v) = \{x \in \mathbb{R}^n | \langle \mathbb{1}_N, x \rangle = v(\mathbb{1}_N), \langle a, x \rangle \geq v(a) , \text{ for every } a \in I^n\} . \quad (9.8)$$

In words, the core of v is the set of all payoff vectors x such that no coalition $a \in I^n$ is better off when accepting any other payoff vector $y \notin C(v)$. This is a consequence of the two conditions in (9.8): *Pareto efficiency* $\langle \mathbb{1}_N, x \rangle = v(\mathbb{1}_N)$ requires that the profit of the grand coalition is distributed among all the players in N and *coalitional rationality* $\langle a, x \rangle \geq v(a)$ means that no coalition $a \in I^n$ accepts less than is its profit $v(a)$.

Observe that the core $C(v)$ of a game with fuzzy coalitions v is an intersection of uncountably many halfspaces $\langle a, x \rangle \geq v(a)$ with the affine hyperplane $\langle \mathbb{1}_N, x \rangle = v(\mathbb{1}_N)$. This implies that the core is a possibly empty compact convex subset of $\mathbb{R}^n$, since $C(v)$ is included in the core (9.1) of a classical coalition game $\bar{v}$ given by (9.7). In this way, we may think of the Aubin's core $C(v)$ as a refinement of the classical core (9.1).

A payoff x in the core $C(v)$ must meet uncountably many restrictions represented by all coalitions I^n. This raises several questions:

1. When is $C(v)$ nonempty/empty?
2. When is $C(v)$ reducible to the intersection of finitely many sets only?
3. For every $a \in I^n$, is there a core element $x \in C(v)$ giving coalition a exactly its worth $v(a)$?
4. Is there an allocation rule for assigning payoffs in $C(v)$ to fuzzy coalitions?

Azrieli and *Lehrer* formulated a necessary and a sufficient condition for the core nonemptiness [9.13], thus generalizing the well-known Bondareva–Shapley theorem for classical coalition games. We will need an additional notion in order to state their result. The *strong superadditive cover* of a game $v \in \Gamma_N$ is a game $\hat{v} \in \Gamma_N$ such that, for every $a \in I^n$,

$$\hat{v}(a) = \sup \left\{ \sum_{k=1}^{\ell} \lambda_k v(a^k) \middle| \ell \in \mathbb{N},\ a^i \leq a,\ \lambda_i \geq 0, \right. \\ \left. \sum_{k=1}^{\ell} \lambda_k a^k = a, \quad i = 1, \ldots, \ell \right\} .$$

The nonemptiness of $C(v)$ depends on value of $\hat{v}$ at one point only.

Theorem 9.3 Azrieli and Lehrer [9.13]
Let $v \in \Gamma_N$. The core $C(v)$ is nonempty if and only if $v(\mathbb{1}_N) = \hat{v}(\mathbb{1}_N)$.

The above theorem answers Question 1. Nevertheless, it may be difficult to check the condition $v(\mathbb{1}_N) =$

$\hat{v}(\mathbb{1}_N)$. Can we simplify this task for some classes of games? In particular, can we show that the shape of the core is simpler on some class of games? This leads naturally to Question 2. *Branzei* et al. [9.16] showed that the class of games for which this holds true is the class of convex games. We say that a game $v \in \Gamma_N$ is *convex*, whenever the inequality

$$v(a+c) - v(a) \leq v(b+c) - v(b) \qquad (9.9)$$

is satisfied for every $a, b, c \in I^n$ such that $b+c \in I^n$ and $a \leq b$. A word of caution is in order here: in general, as shown in [9.13], the convexity of the game $v \in \Gamma_N$ does not imply and is not implied by the convexity of v as an n-place real function. The game-theoretic convexity captures the economic principle of nondecreasing marginal utility. Interestingly, this property makes it possible to simplify the structure of $C(v)$.

Theorem 9.4 Branzei et al. [9.12]
Let $v \in \Gamma_N$ be a convex game. Then $C(v) \neq \emptyset$ and, moreover, $C(v)$ coincides with the core $C(\bar{v})$ of the classical coalition game $\bar{v}$.

The previous theorem, which solves Question 2, provides in fact the complete characterization of core on the class of convex games with fuzzy coalitions. Indeed, since the game $\bar{v}$ is convex, we can use the result of *Shapley* [9.7] to describe the shape of $C(v) = C(\bar{v})$.

The point 3 motivates the following definition. A game $v \in \Gamma_N$ is said to be *exact* whenever for every $a \in I^n$, there exists $x \in C(v)$ such that $\langle a, x\rangle = v(a)$. The class of exact games can be explicitly described [9.13].

Theorem 9.5
Let $v \in \Gamma_N$. Then the following properties are equivalent:

i) v is exact;
ii) $v(a) = \min\{\langle a, x\rangle | x \in C(v)\}$;
iii) v is simultaneously
  a) a concave, positively homogeneous function on I^n, and
  b) $v(\lambda a + (1-\lambda)\mathbb{1}_N) = \lambda v(a) + (1-\lambda)v(\mathbb{1}_N)$, for every $a \in I^n$ and every $0 \leq \lambda \leq 1$.

The second equivalent property enables us to generate many examples of exact games – it is enough to take the minimum of a family of linear functions, each of which coincides at point $\mathbb{1}_N$.

Question 4 amounts to asking for the existence of allocation rules in the sense of *Lehrer* [9.17] or dynamic procedures for approximating the core elements by *Wu* [9.18]. A bargaining procedure for recovering the elements of the Aubin's core $C(v)$ is discussed in [9.19], where the authors present the so-called Cimmino-style bargaining scheme. For a game $v \in \Gamma_N$ and some *initial payoff* $x^0 \in \mathbb{R}^n$, the goal is to recover a sequence of payoffs converging to a core element, provided that $C(v) \neq \emptyset$. We consider a probability measure that captures the bargaining power of coalitions $a \in I^n$: a *coalitional assessment* is any complete probability measure ν on I^n. In what follows, we will require that ν is such that, for every Lebesgue measurable set $A \subseteq I^n$,

$$\nu(A) > 0\,, \quad \text{whenever } A \text{ is open or } \mathbb{1}_N \in A\,. \qquad (9.10)$$

Let $x \in \mathbb{R}^n$ be an arbitrary payoff and $a \in I^n$. We denote

$$C_a(v) = \begin{cases} \{y \in \mathbb{R}^n | \langle a, y\rangle \geq v(a)\} & a \in I^n \setminus \{\mathbb{1}_N\}\,, \\ \{y \in \mathbb{R}^n | \langle \mathbb{1}_N, y\rangle = v(\mathbb{1}_N)\} & a = \mathbb{1}_N\,. \end{cases}$$

What happens when payoff x is accepted by a, that is, $x \in C_a(v)$? Then coalition a has no incentive to bargain for another payoff. On the contrary, if $x \notin C_a(v)$, then a may seek the payoff $P_a x \in C_a(v)$ such that $P_a x$ is the closest to x in some sense. Specifically, we will assume that $P_a x$ minimizes the Euclidean distance of x from set $C_a(v)$. This yields the formula

$$\begin{aligned} &P_a x \\ &= \arg\min_{y \in C_a(v)} \|y - x\| \\ &= \begin{cases} x + \dfrac{\max\{0, v(a) - \langle a, x\rangle\}}{\|a\|^2} a & a \in I^n \setminus \{\mathbb{1}_\emptyset, \mathbb{1}_N\}\,, \\ x + \dfrac{v(\mathbb{1}_N) - \langle \mathbb{1}_N, x\rangle}{n} \mathbb{1}_N & a = \mathbb{1}_N\,, \\ x & a = \mathbb{1}_\emptyset\,, \end{cases} \end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm. After all coalitions $a \in I^n$ have raised their requests on the new payoff $P_a x$, we will average their demands with respect to the coalitional assessment ν in order to obtain a new proposal payoff vector Px. Hence, Px is computed as

$$Px = \int_{I^n} P_a x \, \mathrm{d}\nu(a)\,.$$

The integral on the right-hand side is well defined, whenever ν is Lebesgue measurable. The amalgamated projection operator P is the main tool in the *Cimmino-style bargaining procedure*: an initial payoff x^0 is arbitrary, and we put $x^k = Px^{k-1}$, for each $k = 1, 2, \ldots$

Theorem 9.6
Let $v \in \Gamma_N$ be a continuous game with fuzzy coalitions and let ν be a coalitional assessment satisfying (9.10):

1. If the sequence $(x^k)_{k\in\mathbb{N}}$ generated by the Cimmino procedure is bounded and
$$\lim_{k\to\infty} \int_{[0,1]^n} \left\|x^k - P_a x^k\right\| \, d\nu(a) = 0 , \qquad (9.11)$$
then $C(v) \neq \emptyset$ and $\lim_{k\to\infty} x^k \in C(v)$.
2. If the sequence $(x^k)_{k\in\mathbb{N}}$ is unbounded or (9.11) does not hold, then $C(v) = \emptyset$.

The interested reader is invited to consult [9.19] for further details and numerical experiments.

9.2.2 Single-Valued Solutions

Shapley value

Aubin defined Shapley value on spaces of games with fuzzy coalitions possessing nice analytical properties [9.3, 14, Chapter 13.4]. Specificically, let a function $v : \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous and Lipschitz in the neighborhood of $\mathbb{1}_N$. Such functions are termed *generalized sharing games with side payments* by *Aubin* [9.14, Chap. 13.4]. The restriction of v onto the cube I^n is clearly a game with fuzzy coalitions and therefore we would not make any distinction between v and its restriction to I^n. In addition, assume that function v is continuously differentiable at $\mathbb{1}_N$ and denote by $\mathcal{G}_N^1$ the class of all such games with fuzzy coalitions. Hence, we may put

$$\sigma(v) = \nabla v(\mathbb{1}_N) , \quad v \in \mathcal{G}_N^1 . \qquad (9.12)$$

Each coordinate $\sigma_i(v)$ of the gradient vector $\sigma(v)$ captures the marginal contribution of player $i \in N$ to the grand coalition $\mathbb{1}_N$. As pointed out by Aubin, the gradient measures the roles of the players as pivots in game v. Moreover, the operator σ given by (9.12) can be considered as a generalized Shapley value on the class of games $\mathcal{G}_N^1$ (cf. Theorem 9.1): *Aubin* proved [9.14, Chapter 13.4] that the operator defined by (9.12) satisfies

$$\langle \mathbb{1}_N, \sigma(v)\rangle = v(\mathbb{1}_N) ,$$

for every game $v \in \mathcal{G}_N^1$, and

$$\sigma_i(\pi v) = \sigma_{\pi(i)}(v) ,$$

for every player $i \in N$ and every permutation π of N. Moreover, σ fulfills a certain variant of the Dummy Property.

When defining a value on games with fuzzy coalitions, many other authors [9.15, 20] proceed in the following way: a classical cooperative game is extended from the set of all crisp coalitions to the set of all fuzzy coalitions. The main issue is to decide on the nature of this extension procedure and to check that the extended game with fuzzy coalitions inherits all or at least some properties of the function that is extended (such as superadditivity or convexity). Clearly, there are as many choices for the extension as there are possible interpolations of a real function on $\{0, 1\}^n$ to the cube $[0, 1]^n$.

Tsurumi et al. [9.20] used the Choquet integral as an extension. Specifically, for every $a \in I^n$, let $V_a = \{a_i | a_i > 0, i \in N\}$ and let $n_a = |V_a|$. Without loss of generality, we may assume that the elements of V_a are ordered and write them as $b_1 < \cdots < b_{n_a}$. Further, put $[a]_y = \{i \in N | a_i \geq y\}$, for each $a \in I^n$ and for each $y \in [0, 1]$.

Definition 9.8
A game with fuzzy coalitions (I^n, v) is a *game with Choquet integral form* whenever

$$v(a) = \sum_{i=1}^{n_a} \bar{v}([a]_{b_i})(b_i - b_{i-1}) , \quad a \in I^n ,$$

where $b_0 = 0$. Let Γ_N^C be the class of all games with Choquet integral form.

In the above definition the function v is the so-called Choquet integral [9.21] of a with respect to the restriction $\bar{v}$ of v to all crisp coalitions. It was shown that every game $v \in \Gamma_N^C$ is monotone whenever $\bar{v}$ is monotone [9.20, Lemma 2] and that v is a continuous function on I^n [9.20, Theorem 2]. The authors define a mapping

$$f : \Gamma_N^C \to ([0, \infty)^n)^{I^n} ,$$

which is called a *Shapley function*, by the following assignment

$$f_i(v)(a) = \sum_{i=1}^{n_a} f_i^0(\bar{v})([a]_{b_i})(b_i - b_{i-1}) ,$$
$$i \in N,\ v \in \Gamma_N^C,\ a \in I^n ,$$

where

$$f_i^0(\bar{v})(B) = \sum_{\substack{A \subseteq B \\ i \in A}} \frac{(|A|-1)!(|B|-|A|)!)}{|B|!}(\bar{v}(A) - \bar{v}(A \setminus \{i\}) ,$$

$$B \subseteq N ,$$

whenever $i \in B$, and $f_i^0(\bar{v})(B) = 0$, otherwise. Observe that $f_i(v)(a)$ is the Choquet integral of a with respect to $f_i^0(v)$ and that $f_i^0(\bar{v})(B)$ is the Shapley value of $\bar{v}$ with the grand coalition N replaced with the coalition B. Before we show that the Shapley function has some expected properties, we prepare the following definitions. Let $a \in I^n$ and $i, j \in N$. For each $b \in I^n$ with $b \leq a$, define a vector b_{ij}^a whose coordinates are

$$(b_{ij}^a)_k = \begin{cases} b_i \wedge a_j & k = i , \\ b_j \wedge a_i & k = j , \\ b_k & \text{otherwise} , \end{cases} \qquad k \in N .$$

For an arbitrary $b \in I^n$, put

$$(b'_{ij}[a])_k = \begin{cases} b_j & k = i , \\ b_i & k = j , \\ b_k & \text{otherwise} , \end{cases} \qquad k \in N .$$

Clearly, we have both $b_{ij}^a \leq a$ and $b'_{ij}[a] \leq a$. The following theorem is proved in [9.20].

Theorem 9.7
The operator $f : \Gamma_N^C \to ([0, \infty)^n)^{I^n}$ has the following properties:

1. If $v \in \Gamma_N^C$ and $a \in I^n$, then
$$\sum_{i \in N} f_i(v)(a) = v(a) \quad \text{and} \quad f_j(v)(a) = 0 ,$$
for every $j \in N$ such that $a_j = 0$.
2. If $v \in \Gamma_N^C$, $a \in I^n$, and $b \in I^n$ such that $v(b \wedge c) = v(b)$, for every $c \in I^n$ with $c \leq a$, then
$$f_i(v)(a) = f_i(v)(b) \quad \text{for every } i \in N .$$
3. If $v \in \Gamma_N^C$, $a \in I^n$, a_{ij}^a is such that $v(a_{ij}^a \wedge c) = v(b)$, for every $c \in I^n$ with $c \leq a$, and $v(b) = v(b'_{ij})$ for every $b \in I^n$ with $b \leq a_{ij}^a$, then
$$f_i(v)(a) = f_j(v)(a) .$$
4. If $v, w \in \Gamma_N^C$, then $v + w \in \Gamma_N^C$, and
$$f_i(v + w)(a) = f_i(v)(a) + f_i(w)(a) ,$$
for every $i \in N$ and every $a \in I^n$.

The previous theorem thus says that the Shapley function f on the class of games Γ_N^C has the properties analogous to the Shapley value: efficiency, the carrier property, symmetry, and additivity.

Butnariu and *Kroupa* [9.15] studied a value operator on the class of fuzzy games (I^n, v) satisfying

$$v(a) = \sum_{t \in [0,1]} \psi(t) v(a^t), \quad a \in I^n ,$$

where $\psi : [0, 1] \to \mathbb{R}$ fulfills

$$(\psi(t) = 0 \text{ iff } t = 0) \text{ and } \psi(1) = 1$$

and

$$a^t = \{i \in N | a_i = t\} .$$

The class of such fuzzy games is denoted by Γ_N^ψ. The so-called Shapley mapping function can be axiomatized on Γ_N^ψ [9.15, Axioms 1–3]: it turns out that there is only one Shapley mapping $\Phi : \Gamma_N^\psi \to (\mathbb{R}^n)^{I^n}$ [9.15, Theorem 1].

Theorem 9.8
There exists a unique Shapley mapping $\Phi : \Gamma_N^\psi \to (\mathbb{R}^n)^{I^n}$ and it is given by the following formula:

$$\Phi_i(v)(a) = \begin{cases} \psi(r) \sum\limits_{S \in \mathcal{P}_i(a^r)} \dfrac{(|S|-1)!(|a^r|-|S|)!}{|a^r|!}(v(S) - v(S \setminus \{i\})) , \\ \qquad \text{if } a_i = r > 0 , \\ 0, \quad \text{otherwise} , \end{cases}$$

where

$$\mathcal{P}_i(a^r) = \{R \subseteq N | i \in R \text{ and } R \subseteq a^r\} .$$

The expected total allocation of player $i \in N$ is then obtained as

$$\hat{\Phi}_i(v) = \int_{I^n} \Phi_i(v)(a) \, da .$$

provided that the above Lebesgue integral exists. The operator $\hat{\Phi} = (\hat{\Phi}_1, \dots, \hat{\Phi}_n)$ is called the *cumulative value* of v. If the weight function ψ is bounded and Lebesgue integrable, then [9.15, Theorem 2] shows that the cumulative value is well-defined and its coordinates are

$$\hat{\Phi}_i(v) = v(\mathbb{1}_i) \int_0^1 \psi(t)\, \mathrm{d}t$$

for each $i \in N$.

Owen's approach to classical Shapley value [9.22] cannot be, strictly speaking, classified as an attempt to define a Shapley-style value on some class of games with fuzzy coalitions, but we mention his construction for the sake of completeness. The idea is to extend a game $v \in G_N$ with crisp coalitions from its domain $\{\mathbb{1}_A | A \subseteq N\}$ to the whole unit cube I^n by way of the multilinear interpolation. The resulting *multilinear extension* $\tilde{v}$ can be described explicitly as the function

$$\tilde{v}(a) = \sum_{A \subseteq N} \left[\prod_{i \in A} a_i \prod_{i \notin A} (1 - a_i) \right] v(A) ,$$
$$a = (a_1, \dots, a_n) \in I^n . \tag{9.13}$$

Function $\tilde{v}$ is linear in each of its variables separately and $v(A) = \tilde{v}(\mathbb{1}_A)$, for each $A \subseteq N$. The usual formula (9.13) for the Shapley value $\phi(v)$ of v now takes the following *diagonal form* [9.22]

$$\phi_i(v) = \int_0^1 \frac{\partial \tilde{v}}{\partial x_i}(t, \dots, t)\, \mathrm{d}t . \tag{9.14}$$

Hence, $\phi_i(v)$ is completely determined by the behavior of the function $\tilde{v}$ in the neighborhood of the diagonal in I^n. The formula (9.14) is important from the computational point of view: its use in connection with statistical techniques can enhance computations with the Shapley value – see [9.23, Chap. XII.4] for further details.

Since the space of games with crisp coalitions is finite dimensional unlike the space of games with fuzzy coalitions, there is no general approach to the Shapley value of fuzzy games. Even a direct comparison of the cumulative value introduced above with the Shapley function on the space of games Γ_N^C of *Tsurumi* et al. [9.20] is hardly possible since the domains of Shapley operators are essentially different. The selection of the right space of games and an appropriate solution thus vary from one application to another.

9.3 Final Remarks

We presented results concerning basic concepts of solution for coalitional games with fuzzy coalitions and finitely many players in the case of transferable utility. We concentrated on those solutions which preoccupy the main part of cooperative game theory (the core and the Shapley value). A detailed discussion or just the comprehensive overview of the current trends in fuzzy games is beyond the reach of this chapter. Nevertheless, in this section we mention current developments and briefly discuss other solution concepts. The reader should always consult the relevant reference for the specification of the concepts used by the cited authors; for example, we can find at least two definitions of a convex fuzzy game:

1. *Azrieli* and *Lehrer* [9.13] and [9.16] use the definition (9.9) employed herein;
2. *Tsurumi* et al. [9.20] call a game with fuzzy coalitions v *convex* whenever
$$v(a \vee b) + v(a \wedge b) \geq v(a) + v(b)$$
holds true for every $a, b \in I^n$.

Shellshear [9.24] employs the concavification of the fuzzy game – the strong supperadditive cover – in order to show [9.24, Theorem 4.4] that the strong superadditive cover has a stable core if and only if the original game has a stable core. Further, he investigates important properties of the concavification and its superdifferential; new necessary and sufficient conditions for core stability are given in [9.23, Chap. XII.4].

Yang et al. [9.25] introduced the concept of bargaining sets for games with fuzzy coalitions; they prove that the bargaining set coincides with the Aubin core whenever a game is continuous and convex. *Liu* and *Liu* [9.26] extended the results from [9.25] in order to overcome some weakness of the previously used fuzzy bargaining sets. The concept of the classical Mas-Colells bargaining set was also generalized and the authors proved existence theorems for such fuzzy bargaining sets. Moreover, both Aumann and Maschler and Mas-Colell fuzzy bargaining sets of a continuous convex cooperative fuzzy game coincide with its Aubin core.

A fuzzy game is represented as a convex program in [9.27]. It is shown that the optimum of the program determines the optimal coalitions as well as the optimal rewards for the players. Further, this framework seems to unify a number of existing representations of solutions: the core, the least core, and the nucleolus.

Wu [9.28] investigates various types of cores based on the dominance among payoff vectors and the concepts of the true payoff and quasi-payoff of a fuzzy coalition.

Interpretational difficulties related to fuzzy games are pointed out by *Mareš* and *Vlach* in [9.29]. The authors propose an alternative model for a fuzzy coalition – a collection of crisp coalitions – and discuss some of its consequences.

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81.5 Conclusions

The analysis of the experimental results of the bio-inspired methods considered in this paper, the FPSO+FGA (FPSO: fuzzy particle swarm optimization; FGA: fuzzy generic algorithm), lead us to the conclusion that for the optimization of these benchmark mathematical functions execution on the GPU is a good alternative, because it is easier and very fast to optimize and achieve good results than to try it with PSO, GA, SA, and genetic pattern search (GPS) on the CPU [81.14], especially when the number of dimensions is increased. This is because processing on GPUs is faster than processing on CPUs. Also, the experimental results obtained with the use of GPUs in this research were compared with another similar approach [81.20, 21] and achieved good results quickly.

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Acknowledgements

A.4 Aggregation Functions on [0,1]

by Radko Mesiar, Anna Kolesárová, Magda Komorníková

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A.5 Monotone Measures-Based Integrals

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A.6 The Origin of Fuzzy Extensions

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A.7 *F*-Transform

by Irina Perfilieva

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A.8 Fuzzy Linear Programming and Duality

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A.9 Basic Solutions of Fuzzy Coalitional Games

by Tomáš Kroupa, Milan Vlach

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B.10 Basics of Fuzzy Sets

by János Fodor, Imre Rudas

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B.12 Fuzzy Implications: Past, Present, and Future

by Michał Baczyński, Balasubramaniam Jayaram, Sebastià Massanet, Joan Torrens

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B.16 An Algebraic Model of Reasoning to Support Zadeh's CWW

by Enric Trillas

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B.20 Application of Fuzzy Techniques to Autonomous Robots

by Ismael Rodríguez Fdez, Manuel Mucientes, Alberto Bugarín Diz

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