

SECOND-ORDER VARIATIONAL ANALYSIS IN CONIC PROGRAMMING WITH APPLICATIONS TO OPTIMALITY AND STABILITY*

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Abstract. This paper is devoted to the study of a broad class of problems in conic programming modeled via parameter-dependent generalized equations. In this framework we develop a second-order generalized differential approach of variational analysis to calculate appropriate derivatives and coderivatives of the corresponding solution maps. These developments allow us to resolve some important issues related to conic programming. They include verifiable conditions for isolated calmness of the considered solution maps, sharp necessary optimality conditions for a class of mathematical programs with equilibrium constraints, and characterizations of tilt-stable local minimizers for cone-constrained problems. The main results obtained in the general conic programming setting are specified for and illustrated by the second-order cone programming.

Key words. variational analysis, second-order theory, conic programming, generalized differentiation, optimality conditions, isolated calmness, tilt stability

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1. Introduction. The major motivation for this paper comes from considering the following parametric problem of *conic programming* in finite-dimensional spaces:

$$(1.1) \quad \begin{array}{ll} \text{minimize} & \varphi(q, y) - \langle p, y \rangle \\ \text{subject to} & g(y) \in \Theta, \end{array}$$

where $y \in \mathbb{R}^m$ is the *decision* variable, $x = (q, p) \in \mathbb{R}^s \times \mathbb{R}^m$ is the two-component *perturbation parameter* (with q signifying the *basic* perturbations and p the *tilt* ones), $\Theta \subset \mathbb{R}^l$ is a *closed convex cone*, and $\varphi: \mathbb{R}^s \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g: \mathbb{R}^m \rightarrow \mathbb{R}^l$ are *twice continuously differentiable*, i.e., of class \mathcal{C}^2 . These are our *standing assumptions* in this paper unless otherwise stated.

The characteristic feature of the optimization problem (1.1) is the *cone constraint* given by $g(y) \in \Theta$, which unifies remarkable subclasses of conic programs when the cone Θ is given in a particular form. Among subclasses in conic programming well-recognized theoretically and most important for applications, we mention problems of

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second-order cone programming, semidefinite programming, and copositive programming; see, e.g., [1, 2, 4, 5, 6, 7, 8, 35, 43, 44] and the references therein. Note that the cone-constrained form (1.1) accommodates also the class of semi-infinite programs provided that Θ is a closed convex subcone of the corresponding infinite-dimensional space; see the recent paper [26] containing the study of nonsmooth conic programs in both finite and infinite dimensions.

It is well known from elementary variational analysis (see, e.g., [23, Proposition 5.1] and [40, Theorem 6.12]) that first-order necessary optimality conditions for problem (1.1) are described by the parameterized *generalized equation* (GE)

$$(1.2) \quad 0 \in f(x, y) + \widehat{N}_\Gamma(y) \quad \text{with } \Gamma := g^{-1}(\Theta)$$

in the sense of Robinson [38], where $\widehat{N}_\Gamma(y)$ stands for the *regular normal cone* to Γ at $y \in \Gamma$ (see section 2 below for the precise definition), and where $f(x, y) := \nabla_y \varphi(q, y) - p$ with the symbol ∇ used for the (partial) gradient of scalar functions as well as for the Jacobian matrix in the case of vector functions. Besides being associated with conic programs (1.1), model (1.2) is of independent interest and deserves study for its own sake. When the set Γ is convex, (1.2) encompasses classical variational inequalities, which have been widely studied in the literature together with similar models generated by other normal cone mappings replacing $\widehat{N}_\Gamma(y)$; see, e.g., [9, 21, 23, 34, 38, 39] and the references therein. In this paper we mainly concentrate on model (1.2) generated by the regular normal cone to Γ . Note that most of the results obtained below are new even when the set Γ is convex.

In what follows we consider the variational system (1.2), adding to our standing assumptions on g and Θ that $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is an arbitrary *continuously differentiable* mapping. This surely covers the original model (1.1) while having a much broader range of applications; some of them are given in section 6. Observe here another important case concerning perturbed conic programming that can be described by the GE (1.2). Let $x = (u, v) \in \mathbb{R}^s \times \mathbb{R}^l$, $y = (z, \lambda) \in \mathbb{R}^s \times \mathbb{R}^l$,

$$f(x, y) := \begin{bmatrix} \nabla_y \varphi(z) - u + \nabla g(z)^* \lambda \\ -g(z) + v \end{bmatrix},$$

and $\Gamma = \mathbb{R}^s \times \Xi^*$, where the notation A^* for a matrix A signifies the matrix *transposition/adjoint operator* while that of Ξ^* for the cone Ξ denotes the *dual/polar cone* of Ξ given by $\Xi^* := \{a \in \mathbb{R}^l \mid \langle a, b \rangle \leq 0 \text{ for all } b \in \Xi\}$. Then the corresponding GE (1.2) amounts (under some constraint qualification) to the *Karush–Kuhn–Tucker (KKT) system* associated with the following *canonically* perturbed conic program:

$$\begin{aligned} & \text{minimize} && \varphi(y) - \langle u, y \rangle \\ & \text{subject to} && g(y) - v \in \Xi. \end{aligned}$$

Consider the (generally set-valued) *solution map* $S: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ to the parametric variational system (1.2) defined by

$$(1.3) \quad S(x) := \{y \in \mathbb{R}^m \mid 0 \in f(x, y) + \widehat{N}_\Gamma(y)\} \quad \text{for all } x \in \mathbb{R}^n$$

with the *graph* $\text{gph } S := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in S(x)\}$, and let $(\bar{x}, \bar{y}) \in \text{gph } S$ be our *reference point*. Throughout the whole paper we impose the following assumptions on the set Θ and the mapping g in (1.2) standard in conic programming (see, e.g., [4]):

- (A1) The set Θ is \mathcal{C}^2 -reducible to a closed convex set Ξ at $\bar{z} := g(\bar{y})$, and the reduction is pointed. This means that there exist a neighborhood V of \bar{z} and a \mathcal{C}^2 mapping $h : V \rightarrow \mathbb{R}^k$ such that (i) for all $z \in V$ we have $z \in \Theta$ if and only if $h(z) \in \Xi$, where the cone $T_{\Xi}(h(\bar{z}))$ is pointed; (ii) $h(\bar{z}) = 0$ and the derivative mapping $\nabla h(\bar{z}) : \mathbb{R}^l \rightarrow \mathbb{R}^k$ is surjective/onto.
- (A2) The point $\bar{y} \in \mathbb{R}^m$ is *nondegenerate* for g with respect to Θ , i.e.,

$$(1.4) \quad \nabla g(\bar{y})\mathbb{R}^m + \text{lin}(T_{\Theta}(\bar{z})) = \mathbb{R}^l.$$

In (A1), (A2), $T_{\Omega}(a)$ stands for the classical tangent cone to Ω at $a \in \Omega$, and $\text{lin}(L)$ denotes the largest linear subspace of \mathbb{R}^l contained in $L \subset \mathbb{R}^l$.

Note that all the assumptions in (A1) are automatically fulfilled at any point $g(\bar{y}) \in \Theta$ in the following two important settings of cone programming: when Θ is either the *SDP cone*, i.e., the positive cone in semidefinite programming (cf. [3, Corollary 4.6]), or it is the *Lorentz cone*, known also as the *second-order cone* and as the *ice cream cone*; cf. [5, Lemma 15].

To proceed further, associate with (1.2) the *Lagrangian* function

$$(1.5) \quad \mathcal{L}(x, y, \lambda) := f(x, y) + \nabla g(y)^* \lambda \quad \text{with } \lambda \in \mathbb{R}^l.$$

As has been well recognized in models with conic constraints, under the assumptions imposed in (A2) there is a *unique* Lagrange multiplier λ satisfying the KKT conditions

$$(1.6) \quad \begin{aligned} 0 &= \mathcal{L}(x, y, \lambda), \\ 0 &\in -g(y) + N_{\Theta^*}(\lambda). \end{aligned}$$

The intention of this paper is to study two important stability properties of parametric equilibrium problems involving the solution map (1.3) and also to derive necessary optimality conditions for mathematical programs with equilibrium constraints (MPECs) generated by GE (1.2). It has been well understood in basic variational analysis that these three issues for general set-valued mappings are closely related to certain *generalized derivatives/coderivatives* of the solution maps in question. Observe to this end that the set-valued part of the variational system (1.2) is given by *normal cone* maps, i.e., by the corresponding *first-order subdifferential* of the indicator function of the underlying set $\Gamma = g^{-1}(\Theta)$. Therefore, applying yet another generalized differentiation, we enter *second-order variational analysis* of the initial cone-constrained systems. Our main tool will be *second-order* generalized derivatives; see section 2 for the precise definitions of these and related constructions in variational analysis and the subsequent sections for their calculations and applications in our setting. Note that the major results obtained in the paper in the *general conic programming* setting are specified for the case when Θ amounts to the Lorentz cone and illustrated by examples.

The rest of the paper is organized as follows. In section 2 we present *basic definitions* and discussions of the first-order and second-order generalized differential constructions used in the formulations and proofs of the main results below.

Section 3 is devoted to calculating the second-order generalized differential construction defined by *graphical derivative* of the *normal cone mapping* from (1.2) under the convexity assumption imposed on Γ . The principal result obtained here represents this generalized derivative in terms of the problem data involving the directional derivative of the metric *projection* onto the cone Θ .

Section 4 concerns calculating the *regular coderivative* of the *regular normal cone* mapping $\widehat{N}_\Gamma(y)$, which can be treated as the *regular second-order subdifferential* of the indicator function of Γ . The main result of this section gives in fact a new *second-order subdifferential chain rule* for the regular constructions under consideration in the case of *nondegeneracy* imposed in (A2).

In section 5 we present in a unified way formulas for the graphical derivatives and both regular and limiting coderivatives of the *solution map* (1.3) in terms of the initial data. They are based on the results from the preceding sections and on [35, Theorem 7]. The derivative/coderivative formulas derived in this way and combined with basic characterizations in general frameworks of variational analysis allow us to establish in section 6 new criteria for *isolated calmness* of the solution map (1.3) and sharp necessary optimality conditions for *MPECs* defined via (1.3) and to characterize *tilt stability* of local minimizers in conic programming.

Throughout the paper we use standard notation of variational analysis and optimization; see, e.g., the books [4, 23, 40]. Recall that the (Painlevé–Kuratowski) *outer limit* of the set-valued mapping/multifunction $F: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ as $z \rightarrow \bar{z}$ is defined by

$$(1.7) \quad \text{Lim sup}_{z \rightarrow \bar{z}} F(z) := \{v \in \mathbb{R}^s \mid \exists z_k \rightarrow \bar{z}, v_k \rightarrow v \text{ with } v_k \in F(z_k) \text{ as } k \in \mathbb{N}\}$$

and that, given a set $\Omega \subset \mathbb{R}^d$, the symbol $z \xrightarrow{\Omega} \bar{z}$ signifies that $z \rightarrow \bar{z}$ with $z \in \Omega$. As usual, the notation a^T indicates the vector transposition, I stands for the identity matrix, and \mathbb{B} denotes the closed unit ball of the space in question.

2. Tools of variational analysis. Generalized differentiation of nonsmooth and set-valued mappings, as well as generalized normals and tangents to sets, play a crucial role in modern variational analysis and optimization; see, e.g., the books [23, 40] and the references therein. In this section we briefly review some first-order and second-order generalized differential constructions employed in the paper, confining ourselves only to the settings that appear below. The reader can find more details and extended frameworks in the aforementioned books and in the papers we refer to.

Let us start with geometric objects. Given a set $\Omega \subset \mathbb{R}^d$ and a point $\bar{z} \in \Omega$, define the (Bouligand–Severi) *tangent/contingent cone* to Ω at \bar{z} via (1.7) by

$$(2.1) \quad T_\Omega(\bar{z}) := \text{Lim sup}_{t \downarrow 0} \frac{\Omega - \bar{z}}{t} = \left\{ u \in \mathbb{R}^d \mid \exists t_k \downarrow 0, u_k \rightarrow u \text{ with } \bar{z} + t_k u_k \in \Omega \right\}.$$

The (Fréchet) *regular normal cone* to Ω at $\bar{z} \in \Omega$ can be equivalently defined by

$$(2.2) \quad \widehat{N}_\Omega(\bar{z}) := \left\{ v \in \mathbb{R}^d \mid \limsup_{z \xrightarrow{\Omega} \bar{z}} \frac{\langle v, z - \bar{z} \rangle}{\|z - \bar{z}\|} \leq 0 \right\} = T_\Omega^*(\bar{z}).$$

The (Mordukhovich) *limiting normal cone* also can be defined in two equivalent ways: via the outer limit (1.7) of the regular normal cone (2.2) and via the (Euclidean) *metric projection* operator $P_\Omega(z) := \{y \in \Omega \text{ with } \|z - y\| = \text{dist}(z; \Omega)\}$ onto Ω by

$$(2.3) \quad N_\Omega(\bar{z}) := \text{Lim sup}_{z \xrightarrow{\Omega} \bar{z}} \widehat{N}_\Omega(z) = \text{Lim sup}_{z \rightarrow \bar{z}} \left\{ \text{cone}[z - P_\Omega(z)] \right\},$$

where Ω is assumed to be locally closed around \bar{z} in the second representation, and where the symbol “cone” signifies the (nonconvex) conic hull of a set. Note that both regular and limiting normal cones reduce to the classical normal cone of convex

analysis when the set Ω is convex and when the common notation $N_\Omega(\bar{z})$ is used. For general sets Ω we have the inclusion

$$\widehat{N}_\Omega(\bar{z}) \subset N_\Omega(\bar{z}) \quad \text{as } \bar{z} \in \Omega,$$

where the regular normal cone values are always convex, while it is not often the case for the limiting normal cone; see, e.g., $\Omega = \text{gph } |z| \subset \mathbb{R}^2$ at $\bar{z} = (0, 0)$. At the same time, in contrast to (2.2), we have $N_\Omega(\bar{z}) \neq \{0\}$ for boundary points. Furthermore, the limiting normal cone (2.3) and the corresponding generalized differential constructions for functions and multifunctions (in particular, those discussed below) possess *full calculi* based on *variational/extremal principles* of variational analysis; see [23, 40]. Although it is not generally the case for the regular normal cone, in this paper we derive new results in this direction for the *second-order* objects induced by (2.2).

Considering next set-valued (in particular, single-valued) mappings $F: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, we define for them the corresponding derivative and coderivative constructions generated by the tangent cone (2.1) and the normal cones (2.2) and (2.3), respectively. Given $(\bar{z}, \bar{w}) \in \text{gph } F$, the *graphical derivative* $DF(\bar{z}, \bar{w}): \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ of F at (\bar{z}, \bar{w}) is

$$(2.4) \quad DF(\bar{z}, \bar{w})(u) := \{q \in \mathbb{R}^s \mid (u, q) \in T_{\text{gph } F}(\bar{z}, \bar{w})\}, \quad u \in \mathbb{R}^d.$$

From the dual perspectives we define the *regular coderivative* $\widehat{D}^*F(\bar{z}, \bar{w}): \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ of F at $(\bar{z}, \bar{w}) \in \text{gph } F$ generated by the regular normal cone (2.2) as

$$(2.5) \quad \widehat{D}^*F(\bar{z}, \bar{w})(v) := \{p \in \mathbb{R}^d \mid (p, -v) \in \widehat{N}_{\text{gph } F}(\bar{z}, \bar{w})\}, \quad v \in \mathbb{R}^s,$$

and the corresponding *limiting coderivative* $D^*F(\bar{z}, \bar{w})$ generated by (2.3) as

$$(2.6) \quad D^*F(\bar{z}, \bar{w})(v) := \{p \in \mathbb{R}^d \mid (p, -v) \in N_{\text{gph } F}(\bar{z}, \bar{w})\}, \quad v \in \mathbb{R}^s.$$

If F is single-valued at \bar{z} , we drop \bar{w} in the notation of (2.4)–(2.6). Note that while the regular coderivative (2.5) is indeed dual to the graphical derivative (2.4) due to the duality correspondence in (2.2), the limiting coderivative (2.6) is *dual to none*, since the nonconvex normal cone (2.3) cannot be tangentially generated. In the case of smooth single-valued mappings, for all $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^s$ we have the representation

$$DF(\bar{z})(u) = \{\nabla F(\bar{z})u\} \quad \text{and} \quad \widehat{D}^*F(\bar{z})(v) = D^*F(\bar{z})(v) = \{\nabla F(\bar{z})^*v\}.$$

As mentioned in section 1, the main emphasis of this paper is on the *second-order* generalized differential constructions appropriate for the aforementioned applications. Among several approaches to second-order generalized differentiation in variational analysis (cf. [4, 23, 40]), we choose the set-valued version of the “derivative-of-derivative” approach initiated in [22], which is applied to arbitrary extended-real-valued functions and treats second-order differentiation of them as a certain generalized derivative of a (set-valued) first-order subdifferential mapping that reduces to the corresponding normal cone mapping in the case of the indicator functions of sets. In this way the (limiting) *second-order subdifferential/generalized Hessian*

$$(2.7) \quad \partial^2\varphi(\bar{z}, \bar{w})(v) := (D^*\partial\varphi)(\bar{z}, \bar{w})(v), \quad v \in \mathbb{R}^d,$$

of $\varphi: \mathbb{R}^d \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ at $\bar{w} \in \partial\varphi(\bar{z})$ has been introduced in [22] and then widely applied to various stability and optimality issues in mathematical programming; see, e.g., [8, 10, 13, 14, 19, 20, 22, 23, 26, 27, 29, 30, 31, 32, 33, 35, 36, 37, 45] and the

references therein. In (2.7), $\partial\varphi$ stands for the (first-order) *limiting subdifferential* mapping $\partial\varphi: \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ defined, via $\text{epi } \varphi := \{(z, \alpha) \in \mathbb{R}^{d+1} \mid \alpha \geq \varphi(z)\}$, by

$$(2.8) \quad \partial\varphi(z) := \{w \in \mathbb{R}^d \mid (w, -1) \in N_{\text{epi } \varphi}(z, \varphi(z))\}$$

if $\varphi(z) < \infty$ and $\partial\varphi(z) := \emptyset$ otherwise. In this paper we use it only in the case of the set *indicator function* $\varphi(z) = \delta_\Omega(z)$, equal to 0 when $z \in \Omega$ and to ∞ otherwise, when

$$\partial\varphi(z) = N_\Omega(z) \quad \text{and} \quad \partial^2\varphi(\bar{z}, \bar{w})(v) = D^*N_\Omega(\bar{z}, \bar{w})(v) \quad \text{for } \bar{z} \in \Omega \quad \text{and} \quad \bar{w} \in N_\Omega(\bar{z}).$$

Similarly to (2.7) we can define the *regular second-order subdifferential/generalized Hessian* $\widehat{\partial}^2\varphi(\bar{z}, \bar{w}) := (\widehat{D}^*\widehat{\partial}\varphi)(\bar{z}, \bar{w})$, where $\widehat{\partial}\varphi$ comes from (2.8) with the replacement of (2.3) by (2.2), and its indicator function specification $\widehat{D}^*\widehat{N}_\Omega(\bar{z}, \bar{w})$, which has already been studied and applied in [13, 14] in the settings different from this paper.

Yet another second-order construction for φ of our interest here has been considered in [40] under the name of the “subgradient graphical derivative” defined in scheme (2.7) as $(D\partial\varphi)(\bar{z}, \bar{w})(v)$ for $\bar{w} \in \partial\varphi(\bar{z})$, $v \in \mathbb{R}^d$. Note that for $\varphi \in \mathcal{C}^2$ we have

$$(D\partial\varphi)(\bar{z})(v) = \widehat{\partial}^2\varphi(\bar{z})(v) = \partial^2\varphi(\bar{z})(v) = \{\nabla^2\varphi(\bar{z})v\} \quad \text{for all } v \in \mathbb{R}^d$$

due to the symmetry of the classical Hessian. As for $\partial^2\varphi$ and $\widehat{\partial}^2\varphi$, we draw our attention to calculating $D\partial\varphi$ for $\varphi = \delta_\Gamma$, i.e., of the *graphical derivative of the normal cone mapping* $D\widehat{N}_\Gamma$ with $\Gamma = g^{-1}(\Theta)$ arising in conic programming.

3. Graphical derivatives of normal cone mappings. Throughout this section the set $\Gamma \subset \mathbb{R}^m$ from (1.2) is *convex*, i.e., $\widehat{N}_\Gamma(y)$ reduces to the normal cone $N_\Gamma(y)$ of convex analysis. The main result of the section provides a complete calculation of the graphical derivative of the normal cone mapping N_Γ . Apart from deriving the main result we provided a workable formula for the directional derivative of the metric projection onto Γ which seems to be of its independent interest.

The convexity of $\Gamma = g^{-1}(\Theta)$ is obviously ensured by the Θ -convexity of the mapping $g: \mathbb{R}^m \rightarrow \mathbb{R}^l$, in the sense that the set

$$\{(y, z) \in \mathbb{R}^m \times \mathbb{R}^l \mid g(y) - z \in \Theta\}$$

is convex. This definition corresponds to the “ $-\Theta$ -convexity” of g in the terminology of [4, Definition 2.103], while it is more convenient for us to use the term “ Θ -convexity” in the sense defined above. Since $g \in \mathcal{C}^2$, its Θ -convexity is equivalent to the condition

$$(3.1) \quad \langle \nabla^2 g(y)(h, h), \nu \rangle \geq 0 \quad \text{for all } \nu \in \Theta^* \quad \text{and } y, h \in \mathbb{R}^m,$$

which is assumed in the rest of this section.

Consider first the auxiliary *linear* GE

$$(3.2) \quad 0 \in y - u + N_\Gamma(y)$$

associated with the (unique) *metric projection* $P_\Gamma(u)$ of $u \in \mathbb{R}^m$ onto Γ . We clearly have that $y = P_\Gamma(u)$ if and only if

$$(3.3) \quad \begin{bmatrix} y \\ u - y \end{bmatrix} \in \text{gph } N_\Gamma.$$

Consider next the associated *perturbed* KKT system

$$(3.4) \quad \begin{aligned} 0 &= y - u + \nabla g(y)^* \nu, \\ -s &\in -g(y) + N_{\Theta^*}(\nu), \end{aligned}$$

with the Lagrange multiplier $\nu \in \mathbb{R}^l$, and denote by T the mapping $(u, s) \mapsto (y, \nu)$ defined by (3.4). Take a vector $\bar{u} \in \mathbb{R}^m$ with $P_{\Gamma}(\bar{u}) = \bar{y}$ and put $\bar{s} = 0$. It follows from (A2) that T is single-valued on a neighborhood of $(\bar{u}, 0)$ and $T(\bar{u}, 0) = \{(\bar{y}, \bar{\nu})\}$, where $\bar{\nu} \in \mathbb{R}^l$ is a unique Lagrange multiplier associated with the projection of \bar{u} onto Γ . Recall that a set-valued mapping $\Phi: \mathbb{R}^k \rightrightarrows \mathbb{R}^p$ has the *Aubin/Lipschitz-like property* around $(\bar{z}, \bar{w}) \in \text{gph } \Phi$ if there are neighborhoods \mathcal{U} of \bar{z} and \mathcal{V} of \bar{w} as well as a number $\kappa \geq 0$ such that

$$(3.5) \quad \Phi(z_1) \cap \mathcal{V} \subset \Phi(z_2) + \kappa \|z_2 - z_1\| \mathbb{B} \text{ for all } z_1, z_2 \in \mathcal{U}.$$

If Φ happens to be single-valued on \mathcal{U} , then the Aubin property around (\bar{z}, \bar{w}) reduces to the classical local Lipschitz continuity near \bar{z} .

The following lemma of its own interest gives a workable representation of the classical directional derivative of the (single-valued) metric projection operator onto the convex set Γ in terms of the initial data of (1.2) and plays a crucial role in deriving the main result of this section in what follows. It contains an assumption on the directional differentiability of the projection operator onto cone Θ , which is not restrictive and holds for the majority of conic programs important for optimization theory and applications (in particular, for semidefinite and second-order cone programming). On the other hand, the required directional differentiability of the projection operator is not always available for an arbitrary convex cone Θ in finite dimensions; see a rather involved counterexample [16] of a solid cone in \mathbb{R}^3 and also a much simpler, ingenious example in \mathbb{R}^2 constructed more recently in [41].

LEMMA 3.1 (directional derivative of the projection operator). *In addition to the standing assumptions, suppose that the projection operator P_{Θ} onto Θ is directionally differentiable on \mathbb{R}^l . Then for any $\bar{u} \in \mathbb{R}^m$ such that $P_{\Gamma}(\bar{u}) = \bar{y}$ there is a neighborhood \mathcal{U} of \bar{u} such that the projection operator P_{Γ} onto Γ is directionally differentiable at each $u \in \mathcal{U}$ in every direction $h \in \mathbb{R}^m$. Furthermore, this directional derivative is calculated by $P'_{\Gamma}(u; h) = v_1$, where v_1 is the first component of the unique solution $v = (v_1, v_2) \in \mathbb{R}^m \times \mathbb{R}^l$ to the system of equations*

$$(3.6) \quad \begin{aligned} h &= \left(I + \sum_{i=1}^l \nu_i \nabla^2 g_i(y) \right) v_1 + \nabla g(y)^* v_2, \\ 0 &= \nabla g(y) v_1 - P'_{\Theta}(g(y) + \nu; \nabla g(y) v_1 + v_2) \end{aligned}$$

with $y = P_{\Gamma}(u)$ and $\nu = (\nu_1, \dots, \nu_l) \in \mathbb{R}^l$ being the unique Lagrange multiplier corresponding to the pair (u, y) in the KKT system (3.4) with $s = 0$.

Proof. It is easy to see that the mapping T can be represented as

$$T(u, s) = \{(y, \nu) \in \mathbb{R}^m \times \mathbb{R}^l \mid \Psi(u, s, y, \nu) \in \Lambda\},$$

where

$$\Psi(u, s, y, \nu) := \begin{bmatrix} y - u + (\nabla g(y))^* \nu \\ g(y) - s \\ \nu \end{bmatrix} \text{ and } \Lambda := \{0\}_{\mathbb{R}^m} \times \text{gph } N_{\Theta}.$$

We invoke now [23, Corollary 4.61] according to which the Aubin property of T around $(\bar{u}, \bar{s}, \bar{y}, \bar{\nu})$ is implied by the condition

$$\left. \begin{aligned} (\nabla_y \Psi(\bar{u}, \bar{s}, \bar{y}, \bar{\nu}))^* a &= 0 \\ (\nabla_\nu \Psi(\bar{u}, \bar{s}, \bar{y}, \bar{\nu}))^* a &= 0 \\ a &\in N_\Lambda(\Psi(\bar{u}, \bar{s}, \bar{y}, \bar{\nu})) \end{aligned} \right\} \implies a = 0,$$

which reduces in the case of Ψ under consideration to

$$(3.7) \quad \left. \begin{aligned} \left(I + \sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) a_1 + (\nabla g(\bar{y}))^* a_2 &= 0 \\ a_2 &\in D^* N_\Theta(g(\bar{y}), \bar{\nu})(\nabla g(\bar{y}) a_1) \end{aligned} \right\} \implies a_1 = 0, a_2 = 0$$

with $a_1 \in \mathbb{R}^m$ and $a_2 \in \mathbb{R}^l$. Multiplying scalarly the first equation of (3.7) from the left-hand side by a_1 and taking into account the inclusion for a_2 give us

$$\left\langle a_1, \left(I + \sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) a_1 \right\rangle + V = 0$$

with some $V \in \left\langle \nabla g(\bar{y}) a_1, D^* N_\Theta(g(\bar{y}), \bar{\nu})(\nabla g(\bar{y}) a_1) \right\rangle$.

From the assumed Θ -convexity of g and the monotonicity result in [37, Theorem 2.1] (implying $V \geq 0$), we immediately conclude that $a_1 = 0$. It remains to show that

$$\left. \begin{aligned} (\nabla g(\bar{y}))^* a_2 &= 0 \\ a_2 &\in D^* N_\Theta(g(\bar{y}), \bar{\nu})(0) \end{aligned} \right\} \implies a_2 = 0.$$

To proceed, we employ the reducibility of Θ at $g(\bar{y})$ and the nondegeneracy of \bar{y} for g with respect to Θ . The reducibility gives us a neighborhood \mathcal{V} of $\bar{z} := g(\bar{y})$, a \mathcal{C}^2 mapping $h : \mathcal{V} \rightarrow \mathbb{R}^k$, and a closed convex set $\Xi \subset \mathbb{R}^k$ such that $\Theta \cap \mathcal{V} = h^{-1}(\Xi) \cap \mathcal{V}$ and $\nabla h(\bar{z})$ is surjective. Thus we find a unique vector $\bar{\mu} \in N_\Xi(h(\bar{z}))$ satisfying

$$\bar{\nu} = \sum_{i=1}^k \bar{\mu}_i \nabla h_i(\bar{z}).$$

Furthermore, it follows from [29, Theorem 3.4] that

$$(3.8) \quad a_2 = (\nabla h(\bar{z}))^* b$$

with some vector $b \in D^* N_\Xi(h(\bar{z}), \bar{\mu})(0)$. Since $\nabla h(\bar{z}) \nabla g(\bar{y})$ is surjective due to the assumed nondegeneracy (see, e.g., [4, p. 315]), we get the implication

$$(\nabla g(\bar{y}))^* (\nabla h(\bar{z}))^* b = 0 \implies b = 0,$$

and so a_2 also vanishes by (3.8). Since T is locally single-valued, we conclude that it is Lipschitz continuous around $(\bar{u}, 0)$.

Define now the mapping $\Phi : \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$ by

$$(3.9) \quad \Phi(w, z, y, \nu) := \begin{bmatrix} w \\ z \\ y - w + (\nabla g(y))^* \nu \\ g(y) - z - P_\Theta(g(y) - z + \nu) \end{bmatrix}$$

and deduce from the convexity of Θ the equivalence

$$(3.10) \quad (y, \nu) = T(u, s) \iff \Phi(w, z, y, \nu) = \begin{bmatrix} u \\ s \\ 0 \\ 0 \end{bmatrix}.$$

We claim that Φ is a Lipschitzian homeomorphism near $(\bar{u}, 0, \bar{y}, \bar{\nu})$. Indeed, Φ is a Lipschitz function by virtue of our assumptions and there are a neighborhood \mathcal{O} of $(\bar{u}, 0, \bar{y}, \bar{\nu})$ and a single-valued Lipschitz mapping ϱ such that $\varrho(\bar{u}, 0, 0, 0) = (\bar{u}, 0, \bar{y}, \bar{\nu})$ and

$$\varrho(\cdot) = \Phi^{-1}(\cdot) \cap \mathcal{O}$$

on a neighborhood of $(\bar{u}, 0, 0, 0)$. This follows from the properties of T proved above and the equivalence (3.10). Since Φ is directionally differentiable by the posed assumptions, we can invoke the *inverse* mapping theorem by Kummer (see [17]) to conclude that ϱ is also directionally differentiable on a neighborhood of $(\bar{u}, 0, 0, 0)$. Its directional derivative on this neighborhood is related to the directional derivative of Φ by

$$\rho'(\Phi(w, z, y, \nu); (h, 0, 0, 0)) = \begin{bmatrix} h \\ 0 \\ v_1 \\ v_2 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} h \\ 0 \\ 0 \\ 0 \end{bmatrix} = \Phi'((w, z, y, \nu); (h, 0, v_1, v_2)).$$

This yields our conclusion on the directional differentiability of P_Γ with $P'_\Gamma(u; h) = v_1$ along the solution to (3.6) by using standard calculus rules in formula (3.9). \square

Remark 3.2 (discussions on projection directional differentiability).

(i) With some small changes of the argumentation in the above proof it can be shown that the assertion of Lemma 3.1 holds true if, instead of the Θ -convexity of g , we impose the *second-order condition*

$$\begin{aligned} &\langle a_1, (I + \Sigma \bar{\nu}_i \nabla^2 g_i(\bar{y})) a_1 \rangle + \langle \nabla g(\bar{y}) a_1, a_2 \rangle > 0 \\ &\text{for all } a_1 \in \mathbb{R}^m \setminus \{0\}, a_2 \in D^* N_\Theta(g(\bar{y}), \bar{\nu})(\nabla g(\bar{y}) a_1), \end{aligned}$$

which may well be fulfilled without assuming the Θ -convexity of g . However, the proof of Theorem 3.3, the main result of this section, requires the convexity of Γ anyway in order to employ the relationship between \widehat{N}_Γ and P_Γ .

(ii) The directional derivative of P_Γ has been calculated in [42] (based on the early results by Shapiro) in the case when g is an *affine* mapping and Θ is *second-order regular* in the sense of [4]. The obtained formula requires the calculation of the support function of the second-order tangent cone to Θ .

Lemma 3.1 established above is a crucial ingredient in deriving the main result of this section given in the following theorem.

THEOREM 3.3 (calculating the graphical derivative of the normal cone mapping). *Let the assumptions of Lemma 3.1 hold, $(\bar{u}, \bar{y}) \in \text{gph } P_\Gamma$, $\bar{w} := \bar{u} - \bar{y}$, and let $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_l)$ be a unique Lagrange multiplier associated with (\bar{u}, \bar{y}) via (3.4). Then the tangent cone (2.1) to the graph of N_Γ at (\bar{y}, \bar{w}) admits the representation*

$$(3.11) \quad T_{\text{gph } N_\Gamma}(\bar{y}, \bar{w}) = \left\{ (v, p) \mid \exists d \in \mathbb{R}^l \text{ with } p = \left(\sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) v + \nabla g(\bar{y})^* d, \right. \\ \left. \nabla g(\bar{y}) v = P'_\Theta(g(\bar{y}) + \bar{\nu}; \nabla g(\bar{y}) v + d) \right\}.$$

Consequently, for any direction $v \in \mathbb{R}^m$ the graphical derivative of N_Γ at (\bar{y}, \bar{w}) is

$$(3.12) \quad DN_\Gamma(\bar{y}, \bar{w})(v) = \left\{ p \in \mathbb{R}^l \mid \exists d \in \mathbb{R}^l \text{ with } p = \left(\sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) v + \nabla g(\bar{y})^* d, \right. \\ \left. \nabla g(\bar{y})v = P'_\Theta(g(\bar{y}) + \bar{\nu}; \nabla g(\bar{y})v + d) \right\}.$$

Proof. By [14, Proposition 3.1] we have the relationship

$$T_{\text{gph}P_\Gamma}(\bar{u}, \bar{y}) = \text{gph } P'_\Gamma(\bar{u}; \cdot).$$

Using this and Lemma 3.1 gives us the tangent cone representation

$$T_{\text{gph}P_\Gamma}(\bar{u}, \bar{y}) = \left\{ (h, v_1) \in \mathbb{R}^m \times \mathbb{R}^m \mid \exists v_2 \in \mathbb{R}^l \text{ such that} \right. \\ \left. h = \left(I + \sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) v_1 + \nabla g(\bar{y})^* v_2, \nabla g(\bar{y})v_1 = P'_\Theta(g(\bar{y}) + \bar{\nu}; \nabla g(\bar{y})v_1 + v_2) \right\}.$$

Since $(y, w) \in \text{gph } N_\Gamma$ if and only if $(y + w, y) \in \text{gph } P_\Gamma$, it follows by the elementary calculus rule from [40, Exercise 6.7] that

$$T_{\text{gph}N_\Gamma}(\bar{y}, \bar{w}) = \left\{ (v, p) \in \mathbb{R}^m \times \mathbb{R}^m \mid \begin{pmatrix} v + p \\ v \end{pmatrix} \in T_{\text{gph}P_\Gamma}(\bar{u}, \bar{y}) \right\},$$

which justifies the tangent cone formula (3.11). The graphical derivative result (3.12) follows immediately from (3.11) and definition (2.4). \square

We can see that formulas (3.11) and (3.12) for calculating the tangent cone to $\text{gph } N_\Gamma$ and the graphical derivative of N_Γ are valid for general conic programs while involving the directional derivative of the projection operator onto Θ . In particular, this directional derivative was calculated entirely in terms of the initial data for the following two remarkable subclasses of conic programming: *second-order cone programs* (SOCPs) defined by the *Lorentz cone*

$$(3.13) \quad \Theta = \mathcal{K}^l := \{(\theta_1, \dots, \theta_l) \in \mathbb{R}^l \mid \theta_1 \geq \|(\theta_2, \dots, \theta_l)\|\}$$

with the Euclidean norm $\|\cdot\|$ (see [36, Lemma 2]) and *semidefinite programs* (SDPs) with Θ being the cone of *symmetric positive semidefinite matrices*

$$(3.14) \quad \Theta := \mathcal{S}_+^l = \{A \in \mathbb{R}^{l \times l} \mid z^T A z \geq 0 \text{ for all } z \in \mathbb{R}^l\};$$

see [43, Theorem 4.7]. Let us present the calculation results for the case of $\Theta = \mathcal{K}^l$, which are used in what follows. Note that similar calculations can be done in the case when Θ is the *Cartesian product* of finitely many Lorentz cones.

Prior to presenting the corresponding results from [36] for $\Theta = \mathcal{K}^l$, let us recall the relevant notation and facts from the theory of symmetric cones needed below; see, e.g., [12]. Given any vector $u = (u_1, \tilde{u}) \in \mathbb{R} \times \mathbb{R}^{l-1}$, we have its *spectral decomposition*

$$(3.15) \quad u = \lambda_1(u)c_1(u) + \lambda_2(u)c_2(u),$$

where $\lambda_1(u), \lambda_2(u)$ and $c_1(u), c_2(u)$ are the *spectral values* and *vectors* of u given by

$$(3.16) \quad \lambda_i(u) = u_1 + (-1)^i \|\tilde{u}\| \text{ and}$$

$$(3.17) \quad c_i(u) = \begin{cases} \frac{1}{2} \left(1, (-1)^i \frac{\tilde{u}}{\|\tilde{u}\|} \right) & \text{if } \tilde{u} \neq 0, \\ \frac{1}{2} \left(1, (-1)^i v \right) & \text{if } \tilde{u} = 0 \end{cases}$$

for $i = 1, 2$, with v being any unit vector in \mathbb{R}^{l-1} . The following proposition taken from [36, Lemma 2] describes the directional derivative of the metric projection onto the Lorentz cone $\Theta = \mathcal{K}^l$. Note that in this case we have $\Theta^* = -\mathcal{K}^l$.

PROPOSITION 3.4 (directional derivative of the projection onto the Lorentz cone). *Let $\Theta = \mathcal{K}^l$. Then the projection operator P_Θ is directionally differentiable at u and for any direction $h \in \mathbb{R}^l$ we have the following relationships:*

(i) *If $u \in \text{int } \Theta \cup \text{int } \Theta^*$, then $P'_\Theta(u; h) = \nabla P_\Theta(u)h$ with*

$$\nabla P_\Theta(u) = 2 \sum_{i=1}^2 [\beta^{[1]}(\lambda(u))]_{ii} c_i(u) c_i(u)^T + [\beta^{[1]}(\lambda(u))]_{12} A(u),$$

where we have $A(u) := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - (c_2(u) - c_1(u))(c_2(u) - c_1(u))^T$,
 $\lambda(u) := (\lambda_1(u), \lambda_2(u))$, and

$$[\beta^{[1]}(\lambda)]_{ij} := \begin{cases} \frac{\beta(\lambda_i) - \beta(\lambda_j)}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ \beta'(\lambda_i) & \text{if } \lambda_i = \lambda_j, \end{cases} \quad i, j = 1, 2,$$

for the first divided difference matrix of the real-valued function $\beta(x) = (x)_+ := \max\{0, x\}$ at λ , with $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\lambda_1 \lambda_2 \neq 0$.

(ii) *If $u \in \text{bd } \Theta^* \setminus \{0\}$, then $P'_\Theta(u; h) = 2 \langle (c_2(u), h) \rangle_+ c_2(u)$.*

(iii) *If $u \in \text{bd } (\Theta) \setminus \{0\}$, then $P'_\Theta(u; h) = h - 2 \langle (c_1(u), h) \rangle_- c_1(u)$ with the notation $(x)_- := \min\{0, x\}$.*

(iv) *If $u = 0$, then $P'_\Theta(u; h) = P_\Theta(h)$.*

4. Regular coderivatives of normal cone mappings. This section is devoted to deriving a formula for calculating the regular coderivative (2.5) of the regular normal cone mapping (2.2) to the set $\Gamma = g^{-1}(\Theta)$ from (1.2) via the initial data g and Θ under the standing assumptions formulated in section 1. Note that due to the duality relationship between the regular normal and tangent cones in (2.2) we have, for an arbitrary set-valued mapping $F: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, that

$$(4.1) \quad \widehat{D}^* F(\bar{y}, \bar{v})(w) = \{p \in \mathbb{R}^d \mid \langle p, u \rangle \leq \langle w, q \rangle \text{ for all } (u, q) \in \text{gph } DF(\bar{y}, \bar{v})\}$$

whenever $(\bar{y}, \bar{v}) \in \text{gph } F$ and $w \in \mathbb{R}^s$. This implies that formula (3.12) of Theorem 3.3 induces, under the assumptions of section 3, the corresponding representation of the regular coderivative of N_Γ by the duality relation (4.1).

However, in this section we explore another route to deal with the regular normal and coderivative constructions, which treats them directly, with no appeal to the tangential counterparts and duality correspondences. In this way we arrive at a new *second-order chain rule* to calculate the regular coderivative of the regular normal cone mapping \widehat{N}_Γ in the setting under consideration *without* imposing the convexity assumption on the set Γ as well as assuming the directional differentiability of the projection operator P_Θ . In the case of the limiting second-order subdifferential (2.7) such a calculus approach has been developed in a series of previous publications (see,

e.g., [23, 29, 30, 32, 35]), with the result for D^*N_Γ closest to our setting appearing in [35, Theorem 7] under the nondegeneracy assumption. We are not familiar with any previous developments in this direction for the case of $\widehat{D}^*\widehat{N}_\Gamma$.

To proceed, consider an $[l \times k]$ -matrix $A(\cdot)$ whose entries are continuously differentiable functions of $x \in \mathbb{R}^n$, and let the components of $b(\cdot) \in \mathbb{R}^p$ be continuously differentiable functions of x . Then we have by the classical product rule that

$$(4.2) \quad \nabla(A(\cdot)b(\cdot))|_{x=\bar{x}} = \nabla(A(\cdot)b(\bar{x}))|_{x=\bar{x}} + \nabla(A(\bar{x})b(\cdot))|_{x=\bar{x}}.$$

THEOREM 4.1 (second-order chain rule for the regular coderivative of the regular normal cone mapping). *Let $\bar{v} \in \widehat{N}_\Gamma(\bar{y})$ under the standing assumptions from section 1. Then for all $w \in \mathbb{R}^m$ we have the representation*

$$(4.3) \quad \widehat{D}^*\widehat{N}_\Gamma(\bar{y}, \bar{v})(w) = \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w + \nabla g(\bar{y})^* \widehat{D}^* N_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w),$$

where $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \mathbb{R}^l$ is a unique solution to the system

$$(4.4) \quad \nabla g(\bar{y})^* \lambda = \bar{v}, \quad \lambda \in N_\Theta(g(\bar{y})).$$

Proof. First we justify the chain rule (4.3) in the case when the derivative operator $\nabla g(\bar{y}): \mathbb{R}^m \rightarrow \mathbb{R}^l$ is *surjective*, i.e., the associated Jacobian matrix $\nabla g(\bar{y})$ with the same notation is of *full rank*. Consider the set

$$Q := \{(y, \lambda, v) \in \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^m \mid v = \nabla g(y)^* \lambda, \lambda \in N_\Theta(g(y))\}$$

and observe that $Q = \Phi^{-1}(\{0\} \times \text{gph } N_\Theta)$ with

$$\Phi(y, \lambda, v) := \begin{bmatrix} v - \nabla g(y)^* \lambda \\ g(y) \\ \lambda \end{bmatrix}.$$

It easily follows from the surjectivity of $\nabla g(\bar{y})$ that the operator $\nabla \Phi(\bar{y}, \bar{\lambda}, \bar{v})$ is also surjective, and then invoking [23, Corollary 1.15] gives us

$$\widehat{N}_Q(\bar{y}, \bar{\lambda}, \bar{v}) = \left\{ \left[\begin{array}{c} \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})p + \nabla g(\bar{y})^* q \\ \nabla g(\bar{y})p + r \\ -p \end{array} \right] \mid p \in \mathbb{R}^m, (q, r) \in \widehat{N}_{\text{gph } N_\Theta}(g(\bar{y}), \bar{\lambda}) \right\}.$$

Since $\text{gph } \widehat{N}_\Gamma$ is the canonical projection of the set Q onto the space generated by the first and the third components, we get from [40, Theorem 6.43] that

$$(4.5) \quad \widehat{N}_{\text{gph } \widehat{N}_\Gamma}(\bar{y}, \bar{v}) \subset \left\{ (z, u) \in \mathbb{R}^l \times \mathbb{R}^m \mid \begin{array}{l} z = \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})(-u) + \nabla g(\bar{y})^* q, \\ (q, \nabla g(\bar{y})u) \in \widehat{N}_{\text{gph } N_\Theta}(g(\bar{y}), \bar{\lambda}) \end{array} \right\}.$$

By (2.5) this implies in turn with putting $w := -u$ that

$$(4.6) \quad \widehat{D}^*\widehat{N}_\Gamma(\bar{y}, \bar{v})(w) \subset \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w + \nabla g(\bar{y})^* \widehat{D}^* N_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w).$$

To justify the opposite inclusion in (4.3), we express according to [23] the set \widehat{N}_Γ locally around \bar{y} as follows:

$$\widehat{N}_\Gamma(y) = \psi(y, N_\Theta(g(y))) = \bigcup \{ \psi(y, u) \mid u \in N_\Theta(g(y)) \}$$

with the function $\psi: \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^m$ defined by

$$\psi(y, u) := \nabla g(y)^* u \text{ for all } y \in \mathbb{R}^m \text{ and } u \in \mathbb{R}^l.$$

We pick now an arbitrary pair (z, w) satisfying

$$(z, -\nabla g(\bar{y})w) \in \widehat{N}_{\text{gph}N_\Theta}(g(\bar{y}), \bar{\lambda})$$

and observe that

$$\text{gph}(N_{\Theta \circ g}) = \{ (a, b) \in \mathbb{R}^m \times \mathbb{R}^l \mid (g(a), b) \in \text{gph} N_\Theta \}.$$

It ensures by [23, Corollary 1.15] and the surjectivity of $\nabla g(\bar{y})$ that

$$\widehat{N}_{\text{gph}(N_{\Theta \circ g})}(\bar{y}, \bar{\lambda}) = \begin{bmatrix} \nabla g(\bar{y})^* & 0 \\ 0 & I \end{bmatrix} \widehat{N}_{\text{gph}N_\Theta}(g(\bar{y}), \bar{\lambda}),$$

and so the inclusion

$$(\nabla g(\bar{y})^* z, -\nabla g(\bar{y})w) \in \widehat{N}_{\text{gph}(N_{\Theta \circ g})}(\bar{y}, \bar{\lambda})$$

holds true. Invoking now the arguments used in the proof of [23, Lemma 1.126] gives us

$$\left(\sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w + \nabla g(\bar{y})^* z, -w \right) \in \widehat{N}_{\text{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{v}),$$

which justifies together with (4.6) the validity of (4.3) in the case of surjectivity.

It remains to replace the surjectivity of $\nabla g(\bar{y})$ by the weaker *nondegeneracy* assumption from (A2). To proceed, we employ the local representation of Θ provided by its *reducibility* at $g(\bar{y})$; see the assumptions in (A1) with the notation used therein.

Denote $f := h \circ g$ and observe that its derivative $\nabla f(\bar{y})$ is surjective due to the nondegeneracy assumption. Applying (4.3) to the mapping f allows us to find a unique multiplier $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_k) \in N_\Xi(f(\bar{x}))$ satisfying the relationships $\nabla f(\bar{x})^* \bar{\mu} = \bar{v}$ and

$$(4.7) \quad \widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{v})(w) = \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 f_i(\bar{y}) \right) w + \nabla f(\bar{y})^* \widehat{D}^* N_\Xi(f(\bar{y}), \bar{\mu})(\nabla f(\bar{y})w).$$

By the classical chain rule we have the equalities

$$(4.8) \quad \begin{aligned} \nabla f(\bar{y}) &= \nabla(h \circ g)(\bar{y}) = \nabla h(g(\bar{y})) \nabla g(\bar{y}), \\ \nabla f(\bar{y})^* &= \nabla g(\bar{y})^* \nabla h(g(\bar{y}))^*. \end{aligned}$$

Furthermore, it follows from the surjectivity of $\nabla h(g(\bar{y}))$ and the application of (4.3) to h that

$$\widehat{D}^* N_\Theta(g(\bar{y}), \bar{\lambda})(\nu) = \left(\sum_{i=1}^k \bar{\vartheta}_i \nabla^2 h_i(g(\bar{y})) \right) \nu + \nabla h(g(\bar{y}))^* \widehat{D}^* N_\Xi(h(g(\bar{y})), \bar{\vartheta})(\nabla h(g(\bar{y}))\nu)$$

for all $\nu \in \mathbb{R}^k$, where $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_k)$ is a unique vector from \mathbb{R}^k such that

$$\bar{\nu} \in N_{\Xi}(h(g(\bar{y}))) \text{ and } \nabla h(g(\bar{y}))^* \bar{\nu} = \bar{\lambda}.$$

Since $h(g(\bar{y})) = f(\bar{y})$, it follows from the uniqueness of multiplier $\bar{\mu}$ in (4.7) that $\bar{\nu} = \bar{\mu}$. Indeed, both multipliers $\bar{\nu}$ and $\bar{\mu}$ belong to $N_{\Xi}(f(\bar{y}))$, and by (4.8) we have

$$\nabla f(\bar{y})^* \bar{\nu} = \nabla g(\bar{y})^* \nabla h(g(\bar{y}))^* \bar{\nu} = \nabla g(\bar{y})^* \bar{\lambda} = \bar{v}.$$

Taking this into account, observe the equalities

$$\begin{aligned} & \nabla f(\bar{y})^* \widehat{D}^* N_{\Xi}(f(\bar{y}), \bar{\mu})(\nabla f(\bar{y})w) \\ &= \nabla g(\bar{y})^* \nabla h(g(\bar{y}))^* \widehat{D}^* N_{\Xi}(h(g(\bar{y})), \bar{\nu})(\nabla h(g(\bar{y}))\nabla g(\bar{y})w) \\ &= \nabla g(\bar{y})^* \left[\widehat{D}^* N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) - \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right) \nabla g(\bar{y})w \right]. \end{aligned}$$

To complete the proof of the theorem, let us show that

$$\left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 f_i(\bar{y}) \right) w = \left(\sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right) w + \nabla g(\bar{y})^* \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right) \nabla g(\bar{y})w.$$

To proceed, we invoke the product rule (4.2) and get the equality

$$\begin{aligned} \nabla(\nabla f_i(\bar{y})) &= \nabla(\nabla g(\cdot)^* \nabla h_i(g(\cdot)))|_{y=\bar{y}} \\ &= \nabla(\nabla g(\cdot)^* \nabla h_i(g(\bar{y})))|_{y=\bar{y}} + \nabla g(\bar{y})^* \nabla(\nabla h_i(g(\cdot)))|_{y=\bar{y}}. \end{aligned}$$

This allows us to conclude that

$$(4.9) \quad \begin{aligned} \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 f_i(\bar{y}) \right) w &= \left[\nabla \left(\nabla g(\cdot)^* \left(\sum_{i=1}^k \bar{\mu}_i \nabla h_i(g(\bar{y})) \right) \right) \right]_{y=\bar{y}} w \\ &\quad + \nabla g(\bar{y})^* \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right) \nabla g(\bar{y})w. \end{aligned}$$

Since $\sum_{i=1}^k \bar{\mu}_i \nabla h_i(g(\bar{y})) = \bar{\lambda}$, the first term on the right-hand side of (4.9) amounts to

$$(\nabla([\nabla g_1(\cdot), \dots, g_l(\cdot)] \bar{\lambda}))|_{y=\bar{y}} w = \left(\sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right) w,$$

which justifies (4.3) and thus completes the proof of the theorem. \square

To facilitate the usage of formula (4.3), e.g., in the case of $\Theta = \mathcal{K}^l$, we restate it now in terms of the regular coderivative of the metric projection mapping P_{Θ} .

COROLLARY 4.2 (regular coderivative of the regular normal cone and projection mappings). *In the setting of Theorem 4.1 we have*

$$\begin{aligned} \widehat{D}^* \widehat{N}_{\Gamma}(\bar{y}, \bar{v})(w) &= \left\{ \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w + (\nabla g(\bar{y}))^* p \right. \\ &\quad \left. - \nabla g(\bar{y})w \in \widehat{D}^* P_{\Theta}(g(\bar{y}) + \bar{\lambda})(-\nabla g(\bar{y})w - p) \right\}. \end{aligned}$$

Proof. It follows from the well-known relationship

$$P_{\Theta} = (I + N_{\Theta})^{-1}$$

between the projection and normal cone operators for convex sets (see, e.g., [40, Proposition 6.17]) that $(y, w) \in \text{gph } N_{\Theta}$ if and only if $(y + w, y) \in \text{gph } P_{\Theta}$. Hence we get from [40, Exercise 6.7] that

$$\widehat{N}_{\text{gph } N_{\Theta}}(g(\bar{y}), \bar{\lambda}) = \{(p, r) \mid p = u + w, r = u, (u, w) \in \widehat{N}_{\text{gph } P_{\Theta}}(g(\bar{y}) + \bar{\lambda}, g(\bar{y}))\},$$

and it suffices to apply the definition of the regular coderivative. \square

Note that a similar relationship to that in Corollary 4.2 holds for the *limiting coderivatives* of N_{Γ} and P_{Θ} . In the Lorentz cone case $\Theta = \mathcal{K}^l$ the corresponding formulas for the regular and limiting coderivatives of the projection operator P_{Θ} can be found [36, Theorem 1] and [36, Theorems 2 and 3], respectively.

To conclude this section, let us discuss further specifications and extensions of the second-order chain rule obtained in Theorem 4.1.

Remark 4.3 (calculating coderivatives of the normal cone mapping defined by the SDP cone). Besides the Lorentz cone case discussed above, the second-order calculus rule in Theorem 4.1 as well as its limiting counterpart from [35, Theorem 7] allow us to fully calculate the corresponding coderivatives $\widehat{D}^* \widehat{N}_{\Gamma}(\bar{y}, \bar{v})$ and $D^* N_{\Gamma}(\bar{y}, \bar{v})$ entirely via the problem data for the SDP cone $\Theta = \mathcal{S}_+^l$ from (3.14). This is based on the calculations of the corresponding coderivative constructions for $P_{\mathcal{S}_+^l}$ given recently in [8, Proposition 3.2 and Theorem 3.1]. It is worth mentioning that the calculation of the limiting coderivative $D^* N_{\Gamma}(\bar{y}, \bar{v})$ has been effectively applied in [28] to characterize tilt and full stability in SDP problems.

Remark 4.4 (infinite-dimensional extensions). Observe that Theorem 4.1 holds as formulated in arbitrary *Banach spaces*, where the notion of nondegeneracy is taken from [4, Definition 4.70] without assuming the finite-dimensionality of the spaces in question. Indeed, the only change in the proof given above is to replace the application of the finite-dimensional result from [40, Theorem 6.43] by its Banach space counterpart from [25, Theorem 4.2] ensuring the *equality* in (4.5) and hence in (4.1) under the surjectivity assumption imposed on $\nabla g(\bar{y})$. In this way we can also derive the Banach space version of second-order chain rule from [35, Theorem 7] for the limiting constructions. Furthermore, the developed approach based on [23, Lemma 1.126] and its proof in the case of surjectivity and on the nondegeneracy reduction to the surjectivity case employed in [35, Theorem 7] and in Theorem 4.1 above allows us to establish—in the general case of *nondegeneracy*—the *exact/equality* type second-order chain rules for *various combinations* of coderivatives and first-order subdifferentials defined via the dual “derivative-of-derivative” scheme of (2.7) (see, e.g., [23, 27, 29]) in arbitrary Banach spaces.

Remark 4.5 (relations to tangential constructions). Having in hand the calculations of Theorem 4.1, it is appealing to employ them in deriving workable formulas for the corresponding representations of the graphical derivative of \widehat{N}_{Γ} by *reversing* the duality scheme (4.1) between DF and $\widehat{D}^* F$. However, the realization of this scheme requires the *graphical regularity* of F (cf. [40, Corollary 6.29]), which in fact reduces to a certain *smoothness* of F in the case of *graphically Lipschitzian* (in the sense of [40, Definition 9.66] and [23, Definition 1.45]) mappings; see [23, Theorem 1.46] with the references and discussions therein. These observations show that the reversed duality scheme *cannot* be applied to the normal cone mappings under consideration.

5. Generalized derivatives of solution maps. The results obtained in sections 3 and 4 together with [35, Theorem 7] and standard calculus rules of generalized differentiation allow us to establish workable formulas for calculating and estimating graphical derivatives and both regular and limiting coderivatives of the solution map S from (1.3). They are crucial for the subsequent applications in section 6. We unify these results in the next theorem.

THEOREM 5.1 (derivatives and coderivatives of solution maps). *Considering the solution map S in (1.3), fix $(\bar{x}, \bar{y}) \in \text{gph } S$. Let $\bar{\lambda} \in \mathbb{R}^l$ be a unique Lagrange multiplier satisfying the KKT system*

$$(5.1) \quad \mathcal{L}(\bar{x}, \bar{y}, \lambda) = 0, \quad \lambda \in N_{\Theta}(g(\bar{y})).$$

Then the following assertions hold:

(i) Under the assumptions of Theorem 3.3 we have the inclusion

$$(5.2) \quad \begin{aligned} & DS(\bar{x}, \bar{y})(v) \\ & \subset \{u \in \mathbb{R}^m \mid \exists d \in \mathbb{R}^l \text{ such that } 0 = \nabla_x f(\bar{x}, \bar{y})v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})u + \nabla g(\bar{y})^*d, \\ & \quad \nabla g(\bar{y})u = P'_{\Theta}(g(\bar{y}) + \bar{\lambda}; \nabla g(\bar{y})u + d)\}, \quad v \in \mathbb{R}^n, \end{aligned}$$

which becomes an equality if in addition $\nabla_x f(\bar{x}, \bar{y})$ is surjective.

(ii) In addition to the standing assumptions of section 1, suppose that $\nabla_x f(\bar{x}, \bar{y})$ is surjective. Then for any $v \in \mathbb{R}^m$ we have the regular coderivative representation

$$(5.3) \quad \begin{aligned} \widehat{D}^*S(\bar{x}, \bar{y})(v) = & \left\{ \nabla_x f(\bar{x}, \bar{y})^*w \mid 0 \in v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^*w \right. \\ & \left. + \nabla g(\bar{y})^* \widehat{D}^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \right\}. \end{aligned}$$

(iii) In the setting of (ii) we have the limiting coderivative representation

$$(5.4) \quad \begin{aligned} D^*S(\bar{x}, \bar{y})(v) = & \left\{ \nabla_x f(\bar{x}, \bar{y})^*w \mid 0 \in v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^*w \right. \\ & \left. + \nabla g(\bar{y})^*D^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \right\}. \end{aligned}$$

Proof. To verify (5.2), observe that

$$(5.5) \quad \text{gph } S = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid h(x, y) := \begin{bmatrix} y \\ -f(x, y) \end{bmatrix} \in \text{gph } \widehat{N}_{\Gamma} \right\}.$$

Thus we can deduce from [40, Theorem 6.31] that

$$\begin{aligned} T_{\text{gph } S}(\bar{x}, \bar{y}) \subset & \left\{ (v, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid \begin{bmatrix} u \\ -\nabla_x f(\bar{x}, \bar{y})v - \nabla_y f(\bar{x}, \bar{y})u \end{bmatrix} \right. \\ & \left. \in T_{\text{gph } \widehat{N}_{\Gamma}}(\bar{y}, -f(\bar{x}, \bar{y})) \right\}, \end{aligned}$$

which holds as equality provided that $\nabla_x f(\bar{x}, \bar{y})$ is surjective; see [40, Exercise 6.7]. Employing now formula (3.11) for calculating the tangent cone $T_{\text{gph } \widehat{N}_{\Gamma}}$ under the assumptions made in section 3 and then recalling the definitions of Lagrangian (1.5) and graphical derivative (2.4), we ensure the validity of inclusion (5.2) and then the equality therein under the additional surjectivity assumption on $\nabla_x f(\bar{x}, \bar{y})$.

Let us next verify (ii). Taking into account the surjectivity of $\nabla h(\bar{x}, \bar{y})$ from (5.5) due to the surjectivity of $\nabla_x f(\bar{x}, \bar{y})$ and using the formula for the normal cone of the inverse image of sets under smooth mappings (see, e.g., [23, Corollary 1.15] and [40, Exercise 6.7]) as well as definition (2.5), we arrive at

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ u \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^m \text{ with } u = \nabla_x f(\bar{x}, \bar{y})^* w, \right. \\ \left. 0 \in v + \nabla_y f(\bar{x}, \bar{y})^* w + \widehat{D}^* \widehat{N}_\Gamma(\bar{y}, -f(\bar{x}, \bar{y}))(w) \right\}.$$

Then formula (5.3) is directly implied by Theorem 4.1.

To justify finally assertion (iii), we proceed similarly to the proof of (ii) under the same assumptions, replacing the second-order chain rule from Theorem 4.1 by the one obtained in [35, Theorem 7]. \square

Employing the complete calculations from Proposition 3.4 of the directional derivative of the projection operator P_Θ for the case of the Lorentz cone $\Theta = \mathcal{K}^l$ allows us to express the results of Theorem 5.1(i) entirely via the initial data of the corresponding SOCP. This is illustrated by the following example.

Example 5.2 (calculating graphical derivative of solutions maps for SOCPs). Consider the following GE of type (1.2):

$$x \in y + \widehat{N}_{g^{-1}(\Theta)}(y),$$

where $g(y) := (1 - \frac{1}{2}y_1^2, y_2, y_3 + 1)$, $\Theta = \mathcal{K}^3$, and $(\bar{x}, \bar{y}) = (0, 0)$. Then $g(\bar{y}) = (1, 0, 1)$, $\bar{\lambda} = (0, 0, 0)$, $\text{Im} \nabla g(\bar{y}) = \{0\} \times \mathbb{R} \times \mathbb{R}$, and $\text{lin } T_{\mathcal{K}^3}(g(\bar{y})) = \{(\alpha, \beta, \alpha) : \alpha, \beta \in \mathbb{R}\}$. This ensures that \bar{y} is a nondegenerate point of g with respect to \mathcal{K}^3 . It is also easy to check (3.1), which ensures that g is Θ -convex. Moreover, on the basis of Proposition 3.4(iii), we have for $u := g(\bar{y}) + \bar{\lambda} = (1, 0, 1)$ and for any direction h that

$$P'_\Theta(u; h) = h - 2(\langle c_1(u), h \rangle)_- c_1(u) = h - \frac{1}{2}(h_1 - h_3)_- \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

with $c_1(u) = \frac{1}{2}(1, 0, -1)$ and the notation $(x)_- := \min\{0, x\}$. Thus, the surjectivity of $\nabla_x f(\bar{x}, \bar{y}) = -I$ and the equality in (5.2) lead us to the precise formula

$$DS(\bar{x}, \bar{y})(v) = \left\{ u \mid u_1 = v_1, u_2 = v_2, u_3 = \begin{cases} v_3 & \text{if } v_3 \leq 0 \\ 0 & \text{if } v_3 > 0 \end{cases} \right\}$$

for calculating the graphical derivative of the solution map in question.

On the basis of Corollary 4.2, formula (5.3) can be reformulated in terms of the regular coderivative of the *projection operator* as follows:

$$(5.6) \quad \widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ \nabla_x f(\bar{x}, \bar{y})^* w \mid \text{there is } p \in \mathbb{R}^l \text{ such that} \right. \\ \left. 0 = v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* w + (\nabla g(\bar{y}))^* p \right. \\ \left. - \nabla g(\bar{y}) w \in \widehat{D}^*P_\Theta(g(\bar{y}) + \bar{\lambda})(-p - \nabla g(\bar{y})w) \right\}.$$

This variant will be used in Example 5.3 for $\Theta = \mathcal{K}^3$ given below.

Note that the corresponding limiting counterpart of (5.6) attains the form

$$(5.7) \quad D^*S(\bar{x}, \bar{y})(v) = \left\{ \nabla_x f(\bar{x}, \bar{y})^* w \mid \text{there is } p \in \mathbb{R}^l \text{ such that} \right. \\ \left. 0 = v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* w + (\nabla g(\bar{y}))^* p \right. \\ \left. - \nabla g(\bar{y}) w \in D^*P_\Theta(g(\bar{y}) + \bar{\lambda})(-p - \nabla g(\bar{y})w) \right\}.$$

Let us next illustrate the statements of Theorem 5.1(ii) and Theorem 5.1(iii) by the following example from second-order cone programming.

Example 5.3 (calculating coderivatives of solution maps in SOCPs). Consider the GE as in Example 5.2 with the same $\Theta = \mathcal{K}^3$, $(\bar{x}, \bar{y}) = (0, 0)$ but with the different $g(y) := (1 + y_3, y_2^2, 1 + y_2)$. Then $g(\bar{y}) = (1, 0, 1)$, $\bar{\lambda} = (0, 0, 0)$, $\text{Im}\nabla g(\bar{y}) = \mathbb{R} \times \{0\} \times \mathbb{R}$, and $\text{lin} T_{\mathcal{K}^3}(g(\bar{y})) = \{(\alpha, \beta, \alpha) \mid \alpha, \beta \in \mathbb{R}\}$. This ensures that \bar{y} is a nondegenerate point of g with respect to \mathcal{K}^3 . Employing further [36, Theorem 1(ii)], we calculate the regular coderivative $\widehat{D}^*P_{\Theta}(g(\bar{y}) + \bar{\lambda})$ by

$$(5.8) \quad \widehat{D}^*P_{\Theta}(g(\bar{y}) + \bar{\lambda})(u) = \{z \in \mathbb{R}^3 \mid u - z \in \mathbb{R}_+c_1, \langle z, c_1 \rangle \geq 0\},$$

where the spectral vector $c_1 := c_1(g(\bar{y}) + \bar{\lambda})$ in this case is

$$c_1 = \frac{1}{2}(1, 0, -1).$$

Using (5.6) gives us the regular coderivative expression in the two equivalent forms:

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \begin{cases} \left\{ v + \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \mid \alpha \in \left[\frac{1}{2}(v_2 - v_3), 0 \right] \right\} & \text{if } v_3 \geq v_2, \\ \emptyset & \text{otherwise;} \end{cases}$$

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ -w \begin{cases} w_1 = -v_1, \\ \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \in \begin{cases} \text{co} \left\{ -\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, -\begin{bmatrix} \frac{v_2 + v_3}{2} \\ \frac{v_2 + v_3}{2} \end{bmatrix} \right\} & \text{if } v_3 \geq v_2 \\ \emptyset & \text{otherwise} \end{cases} \end{cases} \right\}.$$

Likewise, on the basis of [36, Theorem 3(i)] we obtain that

$$D^*P_{\Theta}(z)(u) = \begin{cases} \text{co} \{u, A(z)u\} & \text{if } \langle u, c_1 \rangle \geq 0, \\ \{u, A(z)u\} & \text{otherwise,} \end{cases}$$

where $z := g(\bar{y}) + \bar{\lambda}$, and where

$$A(z) := P_{(c_1)^\perp}(z) = I + \frac{1}{2} \begin{bmatrix} -1 & \frac{\tilde{z}^T}{\|\tilde{z}\|_2} \\ \frac{\tilde{z}}{\|\tilde{z}\|_2} & -\frac{\tilde{z}(\tilde{z})^T}{\|\tilde{z}\|_2^2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

via the spectral decomposition of the vector $z = g(\bar{y}) + \bar{\lambda}$; cf. (3.15) and (3.16) above. It follows now from (5.7) that

$$D^*S(\bar{x}, \bar{y})(v) = \left\{ -w \left| \begin{array}{l} w_1 = -v_1, \\ \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \in \begin{cases} \text{co} \left\{ -\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, -\begin{bmatrix} \frac{v_2 + v_3}{2} \\ \frac{v_2 + v_3}{2} \end{bmatrix} \right\} & \text{if } v_3 \geq v_2 \\ \left\{ -\begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, -\begin{bmatrix} \frac{v_2 + v_3}{2} \\ \frac{v_2 + v_3}{2} \end{bmatrix} \right\} & \text{otherwise} \end{cases} \end{array} \right\}.$$

6. Applications. This section contains some applications of the results obtained above on computing the generalized derivative/coderivative constructions for solution maps to the following three important issues in variational analysis: *isolated calmness*, *optimality conditions for MPECs*, and *tilt stability* in conic programming. Accordingly, we split this section into three subsections.

6.1. Isolated calmness of solution maps. Given $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, recall that it has the *isolated calmness* property at $(\bar{x}, \bar{y}) \in \text{gph } F$ if there are neighborhoods U of \bar{x} and V of \bar{y} and a constant $\kappa \geq 0$ such that

$$(6.1) \quad F(x) \cap V \subset \{\bar{y}\} + \kappa \|x - \bar{x}\| \mathbb{B} \quad \text{for all } x \in U.$$

This well-posedness/stability property and its equivalent description as *strong metric subregularity* of the inverse F^{-1} play a significant role in variational analysis and optimization; see, e.g., [9] and the references therein.

It is known from [18] (cf. also [9, Theorem 4C.1]) that isolated calmness of an arbitrary closed-graph multifunction F between finite-dimensional spaces at any point $(\bar{x}, \bar{y}) \in \text{gph } F$ can be fully characterized via the graphical derivative (2.4) as follows:

$$(6.2) \quad DF(\bar{x}, \bar{y})(0) = \{0\}.$$

Following this line and utilizing the graphical derivative calculations for the solution map (1.3) in Theorem 5.1(i) allow us to derive, on the basis of (6.2), verifiable characterizations of the isolated calmness property of S for the general convex cone Θ in (1.3) and provide its implementation for $\Theta = \mathcal{K}^l$.

THEOREM 6.1 (isolated calmness of solution maps in perturbed conic programming). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ for the solution map (1.3) under the assumptions of Theorem 3.3, and let $\bar{\lambda} \in \mathbb{R}^l$ be a unique Lagrange multiplier satisfying (5.1). Then S enjoys the isolated calmness property at (\bar{x}, \bar{y}) provided that $u = 0$ for any solution $(u, d) \in \mathbb{R}^m \times \mathbb{R}^l$ of the system*

$$(6.3) \quad \begin{aligned} 0 &= \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})u + \nabla g(\bar{y})^* d, \\ \nabla g(\bar{y})u &= P'_{\Theta}(g(\bar{y}) + \bar{\lambda}; \nabla g(\bar{y})u + d). \end{aligned}$$

If in addition the partial Jacobian $\nabla_x f(\bar{x}, \bar{y})$ is surjective, then the above condition is also necessary for the isolated calmness of S at (\bar{x}, \bar{y}) .

Proof. The proof follows directly by substituting the graphical derivative calculations from Theorem 5.1(i) into the isolated calmness criterion (6.2). \square

In the case of the Lorentz cone $\Theta = \mathcal{K}^l$ the complete calculation of the directional derivative P'_Θ provided in Proposition 3.4 leads us to the isolated calmness characterization of Theorem 6.1 entirely via the initial data of SOCPs. We illustrate this with the following example.

Example 6.2 (verifying isolated calmness for SOCPs with no Aubin property). Consider the solution map $S: \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the GE (1.2) with the following initial data:

$$f(x, y) = (y_2, -x + y_1), \Gamma = \{y \in \mathbb{R}^2 \mid (y_1 - \frac{1}{2}y_2^2, y_2) \in \mathcal{K}^2\}, \bar{x} = 0 \text{ and } \bar{y} = (0, 0) \in \mathbb{R}^2.$$

In this case $\nabla g(\bar{y})$ is surjective and so all assumptions of Theorem 3.3, used in Theorems 5.1(i) and 6.1, are satisfied. It is easy to see that the only solution to the corresponding KKT system (5.1) is $\bar{\lambda} = 0 \in \mathbb{R}^2$. Observe that Γ is convex (because g is Θ -convex), $(\bar{x}, \bar{y}) \in \text{gph } S$, and S does not possess the Aubin property around (\bar{x}, \bar{y}) because $S(x) = \emptyset$ for all $x < 0$.

Moreover, the criterion from Theorem 6.1 attains the form

$$(6.4) \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = P'_\Theta \left(0; \begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} \right) = P_{\mathcal{K}^2} \left(\begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} \right) \Rightarrow u = 0.$$

The classical characterization of projections onto convex cones implies that for any u satisfying the condition on the left-hand side of (6.4) one has

$$0 = \left\langle \begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle = -2u_1u_2.$$

So, there are only two possible cases: either $u_1 = 0$ or $u_2 = 0$. In both cases implication (6.4) holds true and so the isolated calmness property of S at (\bar{x}, \bar{y}) has been established.

6.2. MPECs. The main concern of this subsection is the following general optimization problem, which belongs to the class of MPECs:

$$(6.5) \quad \begin{array}{ll} \text{minimize} & \varphi(x, y) \\ \text{subject to} & 0 \in f(x, y) + \widehat{N}_\Gamma(y) \end{array}$$

with the cost function $\varphi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and the constraints described by the GE (1.2) under the standing assumptions formulated in section 1. Optimization problems of this type have drawn much attention in the literature, mainly in the case of convex sets Γ when the GE in (6.5) reduces to the classical parameterized variational inequality; see the books [21, 23, 34] and the references therein. When deriving optimality conditions, one typically works with the limiting coderivative of S . A systematic study of MPEC models with the regular coderivative of S was started recently in [14] and [13]. These papers are devoted to the study of the parameterized solution maps (1.3) and applications to necessary optimality conditions in MPECs (6.5) in the case of Γ being a set of feasible solutions to problems of nonlinear programming with \mathcal{C}^2 inequality constraints. The next theorem seems to be the first result in this direction for MPECs described via the *regular coderivative* of the normal cone mapping to feasible sets in *conic programming*.

THEOREM 6.3 (necessary optimality conditions for MPECs governed by regular normal cone mappings in conic programming). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ be a local optimal solution to MPEC (6.5), and let $\bar{\lambda} \in \mathbb{R}^l$ be a unique Lagrange multiplier satisfying (5.1). In addition to the standing assumptions on f and Γ formulated in section 1,*

suppose that φ is (Fréchet) differentiable at (\bar{x}, \bar{y}) and that the operator $\nabla_x f(\bar{x}, \bar{y})$ is surjective. Then there exists an MPEC multiplier $\bar{\mu} \in \mathbb{R}^m$ such that

$$(6.6) \quad \begin{aligned} 0 &= \nabla_x \varphi(\bar{x}, \bar{y}) + \nabla_x f(\bar{x}, \bar{y})^* \bar{\mu}, \\ 0 &\in \nabla_y \varphi(\bar{x}, \bar{y}) + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* \bar{\mu} + \nabla g(\bar{y})^* \widehat{D}^* \widehat{N}_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y}) \bar{\mu}). \end{aligned}$$

Proof. MPEC (6.5) obviously can be rewritten in the unconstrained form

$$(6.7) \quad \text{minimize } \phi(x, y) := \varphi(x, y) + \delta_{\text{gph}S}(x, y) \text{ over all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$$

via the indicator function $\delta_{\text{gph}S}$ of the graph of the solution map (1.3). Applying to the optimal solution (\bar{x}, \bar{y}) of (6.7) the generalized Fermat rule $0 \in \widehat{\partial}\phi(\bar{x}, \bar{y})$ via the regular subdifferential and then the regular subdifferential sum rule for $\varphi + \delta_{\text{gph}S}$ with the differentiable function φ (see, e.g., [23, Propositions 1.107 and 1.114]), we get

$$0 \in \nabla \varphi(\bar{x}, \bar{y}) + \widehat{N}_{\text{gph}S}(\bar{x}, \bar{y}),$$

which amounts by virtue of (2.5) to the regular coderivative condition

$$(6.8) \quad 0 \in \nabla_x \varphi(\bar{x}, \bar{y}) + \widehat{D}^* S(\bar{x}, \bar{y})(\nabla_y \varphi(\bar{x}, \bar{y})).$$

To complete the proof of the theorem, it remains to employ Theorem 5.1(ii) on the calculation of the regular coderivative of the solution map in (6.8). \square

Conditions (6.6) correspond to the notion of S-stationarity developed in connection with mathematical programs with complementarity constraints. They are substantially sharper (more selective) than their M-stationarity counterpart, where the regular coderivative of \widehat{N}_Θ is replaced by the limiting one.

Remark 6.4 (necessary optimality conditions for nonsmooth MPECs associated with conic programming). Following the proof of Theorem 6.3 and using more involved results from nonsmooth optimization together with Theorem 5.1(ii), we can consider MPEC (6.5) even for nonsmooth cost functions and derive for it both *upper subdifferential* necessary optimality conditions via the upper version $\widehat{\partial}^+ \varphi(\bar{z}) := -\widehat{\partial}(-\varphi)(\bar{z})$ of the regular subdifferential and *lower subdifferential* necessary optimality conditions via the limiting subdifferential (2.8); cf. [23, Propositions 5.2 and 5.3] in the general constrained framework.

For some applications it is again more convenient to present necessary optimality conditions of Theorem 6.3 via the regular coderivative of the metric *projection operator* P_Θ onto the underlying cone Θ .

COROLLARY 6.5 (necessary optimality conditions for MPECs via the regular coderivative of the projection operator). *In the setting of Theorem 6.3 there are multipliers $\bar{\mu} \in \mathbb{R}^m$ and $\bar{v} \in \mathbb{R}^l$ such that we have the inclusion*

$$(6.9) \quad -\nabla g(\bar{y}) \bar{\mu} \in \widehat{D}^* P_\Theta(g(\bar{y}) + \bar{\lambda})(-\nabla g(\bar{y}) \bar{\mu} - \bar{v})$$

along with the equality system

$$(6.10) \quad \begin{aligned} 0 &= \nabla_x \varphi(\bar{x}, \bar{y}) + \nabla_x f(\bar{x}, \bar{y})^* \bar{\mu}, \\ 0 &= \nabla_y \varphi(\bar{x}, \bar{y}) + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* \bar{\mu} + \nabla g(\bar{y})^* \bar{v}. \end{aligned}$$

Proof. The proof follows immediately from the relationships between the projection and normal cone operators for convex sets used in the proof of Corollary 4.2. \square

As discussed in Remark 4.3, the results obtained in [36] and [8] allow us to calculate the regular coderivative of the projection mapping in (6.9) entirely via the initial data for the cases of the Lorentz and SDP cones Θ , respectively, and hence effectively implement the MPEC necessary optimality conditions of Corollary 6.5 in these settings. Let us illustrate it by the following example of an MPEC generated by the Lorentz cone $\Theta = \mathcal{K}^3$ in (6.5).

Example 6.6 (illustrating the MPEC optimality conditions for the Lorentz cone). Consider MPEC (6.5) with $x \in \mathbb{R}^3$, $y \in \mathbb{R}^3$, and

$$\varphi(x, y) := y_1 - y_2 + \frac{1}{2}y_3^2,$$

in which the equilibrium is governed by the GE from Example 5.3. It is easy to see that the pair $(\bar{x}, \bar{y}) = (0, 0)$ is a local minimizer of this MPEC. Since all assumptions of Theorem 6.3 are fulfilled, we can invoke Corollary 6.5 and conclude from (6.10) that $\bar{\mu} = 0$, $\bar{\nu}_1 = -1$, and $\bar{\nu}_2 = 1$. Thus it remains to find $\bar{\nu}_3 \in \mathbb{R}$ such that the vector $\bar{\nu} = (-1, 1, \bar{\nu}_3)$ satisfies relation (6.9), which reads in this case as

$$0 \in \widehat{D}^* P_{\Theta}(g(\bar{y}) + \bar{\lambda})(-\bar{\nu}).$$

It holds by (5.8) for $\bar{\nu}_3 = 0$, which confirms therefore that the solution (\bar{x}, \bar{y}) indeed fulfills the optimality conditions of Corollary 6.5.

6.3. Tilt stability in conic programming. The limiting coderivative of S from (5.4) can be used in testing the Aubin property via the so-called Mordukhovich criterion [40, Theorem 9.40].¹ Formula (5.4) can be, however, employed also in the characterization of another important stability property related to local minimizers.

DEFINITION 6.7 (tilt stability of local minimizers for extended-real-valued functions). *Let $\phi: \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ be an extended-real-valued function finite at \bar{y} . We say that \bar{y} is a tilt-stable local minimizer of ϕ if there is $\gamma > 0$ such that the mapping*

$$M: p \mapsto \operatorname{argmin} \{ \phi(y) - \phi(\bar{y}) - \langle p, y - \bar{y} \rangle \mid \|y - \bar{y}\| \leq \gamma \}$$

is single-valued and Lipschitzian on some neighborhood of $p = 0$ with $M(0) = \{\bar{y}\}$.

This notion has been introduced by Poliquin and Rockafellar in [37] and characterized therein in terms of the second-order subdifferential of ϕ . A combination of the results from [37] and Theorem 5.1(iii) enables us to characterize the tilt stability of the conic program

$$(6.11) \quad \text{minimize } \varphi(y) \quad \text{subject to } y \in \Gamma = g^{-1}(\Theta)$$

with a twice continuously differentiable objective $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}$ under the standing assumptions on the initial data imposed in section 1.

Recent years have witnessed strong interest in tilt stability and its applications from several viewpoints of new developments in variational analysis and generalized differentiation; see, e.g., [10, 11, 20, 27, 31, 32, 33]. The closest to our developments in this paper are those presented in [27, 31, 32, 33], where a number of necessary conditions, sufficient conditions, and complete characterizations of tilt stability were obtained on the basis of second-order generalized differential calculus for various classes of constrained optimization problems including classical nonlinear programs

¹For a different approach to the Aubin property in conic programs see [15].

with equality and inequality constraints, mathematical programs with polyhedral constraints, and the so-called extended nonlinear programs with C^2 data. We are not familiar with any results in this direction for general or special classes of problems of conic programming of type (6.11).

Note that the tilt stability of a local optimal solution $\bar{y} \in \Gamma$ to (6.11) amounts to the single-valuedness and Lipschitz continuity of the mapping

$$(6.12) \quad \mathcal{M}: p \mapsto \operatorname{argmin} \{ \varphi(y) - \varphi(\bar{y}) - \langle p, y - \bar{y} \rangle \mid y \in \Gamma, \|y - \bar{y}\| \leq \gamma \}$$

around the nominal parameter value $p = 0$ with some $\gamma > 0$ and $\mathcal{M}(0) = \{\bar{y}\}$. Prior to deriving the respective characterizations, let us give an example showing that isolated local minimizers in the standard sense may not be tilt-stable for simple cone programs described by the Lorentz cone $\Theta = \mathcal{K}^3$ in (6.11).

Example 6.8 (not tilt-stable isolated local minimizers in second-order cone programming). Consider the following SOCP:

$$\text{minimize } \varphi(y) := y_1^3 + y_1 - y_3 \quad \text{subject to } y = (y_1, y_2, y_3) \in \mathcal{K}^3.$$

Thus in this case $\Theta = \Gamma = \mathcal{K}^3$ and g is the identity map. It is easy to check that $\bar{y} = (0, 0, 0)$ is the only optimal solution of the problem, which is an isolated minimizer. Fix any number $\gamma > 0$ and choose the sequence $p_k = (0, 0, \frac{3}{k}) \in \mathbb{R}^3$ as $k \in \mathbb{N}$. Then for any $y \in \mathcal{K}^3$ with $\|y - \bar{y}\| \leq \gamma$ and for any $k \in \mathbb{N}$ we have the inequalities

$$\varphi(y) - \langle p_k, y \rangle = y_1^3 + y_1 - \left(1 + \frac{3}{k}\right)y_3 \geq y_1^3 - \frac{3}{k}y_1 \geq -\frac{2}{k\sqrt{k}},$$

which become equalities for $y_k = (\frac{1}{\sqrt{k}}, 0, \frac{1}{\sqrt{k}})$. Note furthermore that

$$\|y_k - \bar{y}\| = \frac{\sqrt{2}}{\sqrt{k}} > \frac{\sqrt{k}}{3} \|p_k - 0\| \quad \text{for all } k \in \mathbb{N},$$

which shows that the argminimum mapping (6.12) is not Lipschitz continuous. Thus the minimizer \bar{y} is not tilt-stable for this cone-constrained program.

The next theorem provides a characterization of tilt-stable minimizers for general conic programs defined in (6.11).

THEOREM 6.9 (characterization of tilt-stable local minimizers for general conic programs). *Let $\bar{y} \in g^{-1}(\Theta)$ under our standing assumptions, and let $\bar{\lambda} \in \Theta^* \subset \mathbb{R}^l$ be a unique Lagrange multiplier satisfying the KKT system*

$$(6.13) \quad \nabla_y L(\bar{y}, \lambda) = 0, \quad \langle \lambda, g(\bar{y}) \rangle = 0$$

with the Lagrangian $L(y, \lambda) := \varphi(y) + \langle \lambda, g(y) \rangle$. Then \bar{y} is a tilt-stable local minimizer of (6.11) if and only if for all $w \in \mathbb{R}^m \setminus \{0\}$ we have the inequality

$$(6.14) \quad \langle w, \nabla_{yy}^2 L(\bar{y}, \bar{\lambda})w \rangle + \langle \nabla g(\bar{y})w, D^* N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \rangle > 0,$$

where the inner product in the second term is understood pointwisely as for any element from the coderivative set therein.

Proof. Observe first that thanks to the posed assumptions Γ is prox-regular at \bar{y} and, consequently, [37, Theorem 4.2] can be applied. The condition from that statement characterizing that \bar{y} gives a tilt-stable local minimum in (6.11) attains the form

$$(6.15) \quad \langle w, \nabla^2 \varphi(\bar{y})w \rangle > -\langle u, w \rangle \quad \text{whenever } u \in D^* N_{\Gamma}(\bar{y}, -\nabla \varphi(\bar{y}))(w) \text{ and } w \neq 0.$$

This condition amounts to the inequality

$$(6.16) \quad \langle w, z \rangle > 0 \text{ whenever } z \in D^*S^{-1}(\bar{y}, \bar{x})(w), w \neq 0,$$

where S is the solution map associated with the GE

$$(6.17) \quad x \in \nabla\varphi(y) + N_\Gamma(y)$$

and $\bar{x} = 0$. Using the relationship

$$z \in D^*S^{-1}(\bar{y}, \bar{x})(w) \text{ if and only if } -w \in D^*S(\bar{x}, \bar{y})(-z),$$

the result follows directly from (5.4). \square

If the constraint mapping g in the conic program (6.11) is Θ -convex as defined at the beginning of section 3, we get the following sufficient condition for tilt stability.

COROLLARY 6.10 (sufficient condition for tilt-stable minimizers of conic programs with Θ -convex constraints). *In addition to the assumptions of Theorem 6.9, suppose that the mapping g is Θ -convex. Then the condition*

$$(6.18) \quad \langle w, \nabla^2\varphi(\bar{y})w \rangle > 0 \text{ whenever } \nabla g(\bar{y})w \in \text{dom } D^*N_\Theta(g(\bar{y}), \bar{\lambda}) \text{ and } w \neq 0$$

is sufficient for \bar{y} to be a tilt-stable local minimizer of the conic program (6.11).

Proof. It follows from the *monotonicity* result of [37, Theorem 2.1] and the maximal monotonicity of the normal cone mapping in convex analysis that

$$\langle \nabla g(\bar{y})w, D^*N_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \rangle \geq 0 \text{ for all } w \in \mathbb{R}^m.$$

Furthermore, it follows from the inclusion $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \Theta^*$ and the Θ -convexity description (3.1) for \mathcal{C}^2 mappings that

$$\langle w, \nabla_{yy}^2 L(\bar{y}, \bar{\lambda})w \rangle = \langle w, \nabla^2\varphi(\bar{y})w \rangle + \sum_{i=1}^l \bar{\lambda}_i \langle w, \nabla^2 g_i(\bar{y})w \rangle \geq \langle w, \nabla^2\varphi(\bar{y})w \rangle.$$

Thus condition (6.18) implies (6.14), and we complete the proof of the corollary. \square

7. Conclusion. This paper presents calculations of the major derivative and coderivative constructions of variational analysis for solution maps to parameterized GEs/KKT systems associated with conic constraints. The results established in this direction are based on new second-order calculus rules of generalized differentiation derived in the paper, which are of their own interest. The obtained derivative and coderivative formulas are applied to deriving sharp necessary optimality conditions for a class of MPECs with conic constraints and to characterizing two important stability properties, namely, isolated calmness of solution maps and tilt stability of local optimal solutions. These general results are specified for an important class of equilibria with second-order cone constraints and illustrated by examples. Moreover, they open a possibility to deal efficiently also with parameterized SDPs.

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REFERENCES

- [1] F. ALIZADEH AND D. GOLDFARB, *Second-order cone programming*, Math. Program., 95 (2003), pp. 3–51.
- [2] F. ALVAREZ, J. LPEZ, AND H. RAMÍREZ C., *Interior proximal algorithm with variable metric for second-order cone programming: Applications to structural optimization and support vector machines*, Optim. Methods Softw., 25 (2010), pp. 859–881.
- [3] J. F. BONNANS, R. COMINETTI, AND A. SHAPIRO, *Second-order optimality conditions based on parabolic second-order tangent sets*, SIAM J. Optim., 9 (1999), pp. 466–492.
- [4] J. F. BONNANS AND A. SHAPIRO, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.
- [5] J. F. BONNANS AND H. RAMÍREZ C., *Perturbation analysis of second-order cone programming problems*, Math. Program., 104 (2005), pp. 205–227.
- [6] J. F. BONNANS AND H. RAMÍREZ C., *Strong Regularity of Semidefinite Programs*, Technical report DIM-CMM 137, Universidad de Chile, 2005.
- [7] S. BUNDFUSS AND M. DÜR, *An adaptive linear approximation algorithm for copositive programs*, SIAM J. Optim., 20 (2009), pp. 30–53.
- [8] C. DING, D. SUN, AND J. J. YE, *First-order optimality conditions for mathematical programs with semidefinite cone complementarity constraints*, Math. Program., 147 (2014), pp. 539–579.
- [9] A. L. DONTCHEV AND R. T. ROCKAFELLAR, *Implicit Functions and Solution Mappings*, Springer, New York, 2009.
- [10] D. DRUSVYATSKIY AND A. S. LEWIS, *Tilt stability, uniform quadratic growth, and strong metric regularity of the subdifferential*, SIAM J. Optim., 23 (2013), pp. 256–267.
- [11] A. C. EBERHARD AND R. WENCZEL, *A study of tilt-stable optimality and sufficient conditions*, Nonlinear Anal., 75 (2012), pp. 1260–1281.
- [12] U. FARAUT AND A. KORÁNYI, *Analysis on Symmetric Cones*, Oxford University Press, New York, 1994.
- [13] R. HENRION, A. Y. KRUGER, AND J. V. OUTRATA, *Some remarks on stability of generalized equations*, J. Optim. Theory Appl., 159 (2013), pp. 681–697.
- [14] R. HENRION, J. V. OUTRATA, AND T. SUROWIEC, *On regular coderivatives in parametric equilibria with non-unique multipliers*, Math. Program., 136 (2012), pp. 111–131.
- [15] D. KLATTE AND B. KUMMER, *Aubin property and uniqueness of solutions in cone constrained optimization*, Math. Methods Oper. Res., 77 (2013), pp. 291–304.
- [16] J. B. KRUSKAL, *Two convex counterexamples: A discontinuous envelope function and a non-differentiable nearest-point mapping*, Proc. Amer. Math. Soc., 23 (1969), pp. 697–703.
- [17] B. KUMMER, *Newton's method based on generalized derivatives for nonsmooth functions: Convergence analysis*, in Advances in Optimization, W. Oettli and D. Pallaschke, eds., Lecture Note in Econom. and Math. Systems, 382, Springer, Berlin, 1992, pp. 171–194.
- [18] A. B. LEVY, *Implicit multifunction theorems for the sensitivity analysis of variational conditions*, Math. Program., 74 (1996), pp. 333–350.
- [19] A. B. LEVY, R. A. POLIQUIN, AND R. T. ROCKAFELLAR, *Stability of locally optimal solutions*, SIAM J. Optim., 10 (2000), pp. 580–604.
- [20] A. S. LEWIS AND S. ZHANG, *Partial smoothness, tilt stability, and generalized Hessians*, SIAM J. Optim., 23 (2013), pp. 74–94.
- [21] T. LUO, J.-S. PANG, AND D. RALPH, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, UK, 1996.
- [22] B. S. MORDUKHOVICH, *Sensitivity analysis in nonsmooth optimization*, in Theoretical Aspects of Industrial Design, D. A. Field, and V. Komkov, eds., SIAM, Philadelphia, 1992, pp. 32–46.
- [23] B. S. MORDUKHOVICH, *Variational Analysis and Generalized Differentiation, I: Basic Theory*, Springer, Berlin, 2006.
- [24] B. S. MORDUKHOVICH, *Variational Analysis and Generalized Differentiation, II: Applications*, Springer, Berlin, 2006.
- [25] B. S. MORDUKHOVICH, N. M. NAM, AND B. WANG, *Metric regularity of mappings and generalized normals to set images*, Set-Valued Var. Anal., 17 (2009), pp. 359–387.
- [26] B. S. MORDUKHOVICH AND T. T. A. NGHIA, *Nonsmooth cone-constrained optimization with applications to semi-infinite programming*, Math. Oper. Res., 39 (2014), pp. 301–337.
- [27] B. S. MORDUKHOVICH AND T. T. A. NGHIA, *Second-order characterizations of tilt stability with applications to nonlinear programming*, Math. Program., DOI:10.1007/s10107-013-0739-8.
- [28] B. S. MORDUKHOVICH, T. T. A. NGHIA, AND R. T. ROCKAFELLAR, *Full stability in finite-dimensional optimization*, Math. Oper. Res., DOI:10.1287/moor.2014.0669.

- [29] B. S. MORDUKHOVICH AND J. V. OUTFATA, *On second-order subdifferentials and their applications*, SIAM J. Optim., 12 (2001), pp. 139–169.
- [30] B. S. MORDUKHOVICH AND J. V. OUTFATA, *Coderivative analysis of quasi-variational inequalities with applications to stability in optimization*, SIAM J. Optim., 18 (2007), pp. 389–412.
- [31] B. S. MORDUKHOVICH AND J. V. OUTFATA, *Tilt stability in nonlinear programming under Mangasarian-Fromovitz constraint qualification*, Kybernetika, 49 (2013), pp. 446–464.
- [32] B. S. MORDUKHOVICH AND R. T. ROCKAFELLAR, *Second-order subdifferential calculus with applications to tilt stability in optimization*, SIAM J. Optim., 22 (2012), pp. 953–986.
- [33] B. S. MORDUKHOVICH, R. T. ROCKAFELLAR, AND M. E. SARABI, *Characterizations of full stability in constrained optimization*, SIAM J. Optim., 23 (2013), pp. 1810–1849.
- [34] J. V. OUTFATA, M. KOČVARA AND J. ZOWE, *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*, Kluwer, Dordrecht, Netherlands, 1998.
- [35] J. V. OUTFATA AND H. RAMÍREZ C., *On the Aubin property of critical points to perturbed second-order cone programs*, SIAM J. Optim., 21 (2011), pp. 798–823.
- [36] J. V. OUTFATA AND D. SUN, *On the coderivative of the projection operator onto the second-order cone*, Set-Valued Anal., 16 (2008), pp. 999–1014.
- [37] R. A. POLIQUIN AND R. T. ROCKAFELLAR, *Tilt stability of a local minimum*, SIAM J. Optim., 8 (1998), pp. 287–299.
- [38] S. M. ROBINSON, *Generalized equations and their solutions, I: Basic theory*, Math. Program. Study, 10 (1979), pp. 128–141.
- [39] S. M. ROBINSON, *Strongly regular generalized equations*, Math. Oper. Res., 5 (1980), pp. 43–62.
- [40] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational Analysis*, Springer, Berlin, 1998.
- [41] A. SHAPIRO, *Directionally nondifferentiable metric projection*, J. Optim. Theory Appl., 81 (1994), pp. 203–204.
- [42] A. SHAPIRO, *Differentiability Properties of Metric Projections Onto Convex Sets*, http://www.optimization-online.org/DB_HTML/2013/11/4119.html.
- [43] D. SUN AND J. SUN, *Semismooth matrix valued functions*, Math. Oper. Res., 27 (2002), pp. 150–169.
- [44] H. WOLKOWICZ, R. SAIDAL, AND L. VANDENBERGHE, EDS., *Handbook of Semidefinite Programming: Theory, Algorithms and Applications*, Kluwer, Dordrecht, Netherlands, 2000.
- [45] J. J. YE, *Constraint qualifications and necessary optimality conditions for optimizations problems with variational inequality constraints*, SIAM J. Optim., 10 (2000), pp. 943–962.