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Limiting Normal Operator in Quasiconvex Analysis

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Abstract Inspired by similar definition in subdifferential theory, we define limiting sublevel set and limiting normal operator maps for quasiconvex functions. These maps satisfy important properties as semicontinuity and quasimonotonicity. Moreover, calculus rules together with necessary and sufficient optimality conditions for constrained optimization are established.

Keywords Quasiconvex function · Sublevel set · Normal operator

Mathematics Subject Classification (2000) 49J52 · 26A51 · 49J40 · 26B25 · 90C26

1 Introduction

The theory of subdifferential, in particular limiting subdifferential, has been widely developed for lower semicontinuous functions in the last decades. This tool has proved to be very useful in optimization since it allows calculus rules and necessary optimality conditions at the same time, see [10]. For convex optimization, such necessary optimality conditions turn out to be also sufficient.

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In this paper, however, we focus on the class of quasiconvex functions being important in many applications, e.g., mathematical economics. For this class the subdifferential does not provide sufficient optimality conditions, not even in the local sense. Some work has already been done to find a proper first order tool for quasiconvex analysis benefiting from convexity of the sublevel sets of quasiconvex functions. The basic idea is to deal with the normal cone to convex-valued sublevel set map instead of the normal cone to epigraph of the function in question. Such approach was pioneered in [6] where the notions of normal operator and strict normal operator were first introduced. The first operator is upper semicontinuous and the other quasimonotone, however, none of them fulfills both these important properties together. On that account, a unifying approach based on adjusted normal operator was introduced in [4]. It is upper semicontinuous and quasimonotone at the same time. Moreover, as the strict normal operator, it is capable of full characterization of the optimality conditions for quasiconvex optimization. However, it is hard to find proper calculus rules for adjusted normal operator owing to its non-local nature and this reduces its applicability. Finally let us quote also the concept of quasiconvex subdifferential defined as a mixture of subdifferential and normal operator in [8], where also several calculus rules for particular cases were derived.

Our aim in this paper is to adapt the normal operator in such a way that all the above quoted properties are satisfied and, at the same time, some calculus rules are provided for the class of lower semicontinuous quasiconvex functions. To this end, we introduce a notion of *limiting sublevel set*, and consequently *limiting normal operator* as its point-wise polar. We show that such normal operator is outer semicontinuous and quasimonotone at the same time. We employ the concept of outer semicontinuity instead of upper semicontinuity as it fits better the case of unbounded cone-valued maps [11]. Then we establish calculus rules for the two main operations being stable for the class of quasiconvex functions. Namely for the composition of a quasiconvex and a non-decreasing function, and the computation of maximum of finite family of quasiconvex functions. Moreover necessary and sufficient optimality conditions (local and global) are provided following recent works [4, 9]. We also establish a clear relationship between limiting normal operator and limiting subdifferential.

The organization of this paper is as follows. Section 2 contains notation, introduction and several preliminary results. Section 3 is the core of the paper. The concepts of limiting sublevel set and limiting normal operator maps are developed there, together with respective semicontinuity and optimality conditions. In Section 4, calculus rules are presented, and finally we established relationship between limiting normal operator and limiting subdifferential in Section 5.

2 Preliminaries

Let us first introduce the basic elements of modern variational analysis. For simplicity, we deal only with finite-dimensional case and finite valued functions.

The following notation will be used. Given a set $C \subset \mathbb{R}^m$, the *conic hull* is denoted as $\mathrm{cone}\{C\} \equiv \bigcup_{\lambda \geq 0} \lambda C$, then set C is called a *cone* if $C = \mathrm{cone}\{C\}$. The *convex hull* of set C is denoted by $\mathrm{conv}\{C\}$. Next, for sets $A, B \subset \mathbb{R}^m$ we define $A + B \equiv \{a + b : a \in A, b \in B\}$, and for any point $x \in \mathbb{R}^m$ symbol $\mathrm{dist}(A, x) \in \mathbb{R}$ stands for $\inf\{\|a - x\| : a \in A\}$. Finally the *(negative) polar cone* C° of set $C \subset \mathbb{R}^m$ is

$$C^\circ \equiv \left\{ y \in \mathbb{R}^m : \langle y, x \rangle \leq 0, \ \forall x \in C \right\}.$$

It is a convex closed cone. For any nonempty subset $C \subset \mathbb{R}^m$, it holds

$$C^{\circ\circ} \equiv \left(C^{\circ}\right)^{\circ} = \overline{\text{conv}}\{\text{cone}\{C\}\} \tag{1}$$

according to bipolar theorem. Thus, for a closed convex cone $K \subset \mathbb{R}^m$ we have $K^{\circ\circ} = K$. Now, for a set-valued map $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ we define *outer limit* and *inner limit* of M at a point $\bar{x} \in \mathbb{R}^m$ as

$$\operatorname*{Limsup}_{x \to \bar{x}} M(x) \equiv \left\{y \in \mathbb{R}^n : \exists\, x_k \to \bar{x},\ \exists\, y_k \in M(x_k),\ y_k \to y\right\}$$

and

$$\operatorname*{Liminf}_{x \to \bar{x}} M(x) \equiv \left\{y \in \mathbb{R}^n : \forall\, x_k \to \bar{x},\ \exists\, y_k \in M(x_k),\ y_k \to y\right\},$$

respectively. Let us observe that if M is cone-valued then $\operatorname{Limsup}_{x \to \bar{x}} M(x)$ and $\operatorname{Liminf}_{x \to \bar{x}} M(x)$ are cones.

A set-valued map $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be:

- *outer semicontinuous* at $\bar{x} \in \mathbb{R}^m$ if $\operatorname{Limsup}_{x \to \bar{x}} M(x) \subset M(\bar{x})$;
- *inner semicontinuous* at $\bar{x} \in \mathbb{R}^m$ if $\operatorname{Liminf}_{x \to \bar{x}} M(x) \supset M(\bar{x})$.

Note that M is outer semicontinuous at $\bar{x}$ if and only if $\operatorname{Limsup}_{x \to \bar{x}} M(x) = M(\bar{x})$ while, when M is closed-valued, it is inner semicontinuous at $\bar{x}$ if and only if $\operatorname{Liminf}_{x \to \bar{x}} M(x) = M(\bar{x})$. Note that actually the outer semicontinuity of a set-valued map corresponds to the fact that its graph is closed, and that the inner semicontinuity is often referred to as lower semicontinuity; for more details see [11, Theorem 5.7].

Let us finally recall classical definitions of tangent and normal cone to a subset at a given point: for any closed subset $C \subset \mathbb{R}^m$ and $\bar{x} \in C$ the *tangent cone* $T_C(\bar{x})$ to C at point $\bar{x}$ is

$$T_C(\bar{x}) \equiv \operatorname*{Limsup}_{\lambda \searrow 0} \frac{C - \bar{x}}{\lambda}, \tag{2}$$

and the *limiting normal cone* $N_C(\bar{x})$ to C at $\bar{x}$ is given by

$$N_C(\bar{x}) \equiv \operatorname*{Limsup}_{x \underset{C}{\to} \bar{x}} T_C(x)^{\circ} \tag{3}$$

where $x \underset{C}{\to} \bar{x}$ means $x \to \bar{x}$ with $x \in C$. Note that whenever C is a closed subset then the map $x \mapsto N_C(x)$ is outer semicontinuous on C. In order to work with nonsmooth functions $f : \mathbb{R}^m \to \mathbb{R}$, one of the most popular approaches is to define the so-called *limiting subdifferential* $\partial f(x)$ of f at $x \in \mathbb{R}^m$ as follows $\partial f(x) \equiv \{v \in \mathbb{R}^m : (v, -1) \in N_{\text{epi}\, f}(x, f(x))\}$. If moreover the considered function is not Lipschitz at x then it is convenient to define the *singular subdifferential* $\partial^{\infty} f(x) \equiv \{v \in \mathbb{R}^m : (v, 0) \in N_{\text{epi}\, f}(x, f(x))\}$. Calculus and properties of these subdifferentials are widely developed in [10]. Finally, let us observe that for any $x \in \mathbb{R}^m$ and any lower semicontinuous $f : \mathbb{R}^m \to \mathbb{R}$, it holds

$$\text{cone}\{\partial f(x)\} \cup \partial^{\infty} f(x) = \text{Proj}_{\mathbb{R}^m}\left(N_{\text{epi}\, f}(x, f(x))\right), \tag{4}$$

where $\text{Proj}_{\mathbb{R}^m}$ is a canonical projection onto $\mathbb{R}^m$, see [11, Theorem 8.9].

Let us recall from [1, Theorem 1.1.8] the following interesting relationship between polarity and semicontinuity of set-valued maps.

Proposition 1 [1, Theorem 1.1.8] *For all set-valued maps $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and all $\bar{x} \in \mathbb{R}^m$ one has*

$$\operatorname*{\mathit{Liminf}}_{x \to \bar{x}} M(x) \subset \left(\operatorname*{\mathit{Limsup}}_{x \to \bar{x}} M(x)^{\circ}\right)^{\circ}. \tag{5}$$

Note that we do not assume closed-valuedness of the set-valued map M as only finite-dimensional case is considered, for details see [11, Proposition 4.4].

Corollary 1 *If* $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is inner semicontinuous at* $\bar{x}$ *then the polar map* $x \mapsto M(x)^\circ$ *is outer semicontinuous at* $\bar{x}$.

Proof From the inner semicontinuity of M we have $M(\bar{x}) \subset \operatorname{Liminf}_{x\to\bar{x}} M(x)$. By applying polarity to both sides of (5) and using bipolar relation (1) it holds

$$\begin{aligned}\operatorname{Limsup}_{x\to\bar{x}} M(x)^\circ \supset M(\bar{x})^\circ \supset \; & \left(\operatorname{Limsup}_{x\to\bar{x}} M(x)^\circ\right)^{\circ\circ}\\ = \; & \overline{\operatorname{conv}}\left\{\operatorname{cone}\left\{\operatorname{Limsup}_{x\to\bar{x}} M(x)^\circ\right\}\right\}\\ = \; & \overline{\operatorname{conv}}\left\{\operatorname{Limsup}_{x\to\bar{x}} M(x)^\circ\right\} \supset \operatorname{Limsup}_{x\to\bar{x}} M(x)^\circ.\end{aligned}$$

Thus all inclusions are equalities and so M° is outer semicontinuous at $\bar{x}$. □

3 Limiting Approach to Quasiconvex Analysis

3.1 Classical Sublevel Approaches

As observed in [4–6], the derivative and the subdifferential map (in any sense of nonsmooth analysis) does not take into account that a function is quasiconvex. Let us recall that a function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be quasiconvex if for any $x \in \mathbb{R}^m$ the sublevel set $S_f(x)$ defined by

$$S_f(x) \equiv \{y \in \mathbb{R}^n : f(y) \leq f(x)\}$$

is a convex subset of $\mathbb{R}^m$. The subdifferentials, being based on the epigraph of the function, do not take advantage of this convexity of the sublevel sets. On the contrary, the recent developement of quasiconvex analysis, see e.g. [2, 7, 9], is focused on the *sublevel set map* and its associated *normal operator*. The present work follows this line. The *normal operator* of a function f is a set-valued map $N_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined as the point-wise polar to the (shifted) sublevel set of f, that is

$$N_f(x) \equiv \left(S_f(x) - x\right)^\circ.$$

It is well known that the gradient map of a differentiable quasiconvex function is quasimonotone [12, Proposition 4.12]. This quasimonotonicity property also holds true for the normal operator N_f of a lower semicontinuous function [2, Proposition 5.10]. Let us recall that a map $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is *quasimonotone* if implication

$$\left\langle x^\star, y - x\right\rangle > 0 \Rightarrow \left\langle y^\star, y - x\right\rangle \geq 0$$

holds for all $x, y \in \mathbb{R}^m$, $x^\star \in M(x)$, $y^\star \in M(y)$.

Next, we observe that the set-valued map S_f is closed-valued providing that f is lower semicontinuous on $\mathbb{R}^m$. Another "sublevel set map" is the so-called *strict sublevel set map* $S_f^<$ defined as follows

$$S_f^<(x) \equiv \{y \in \mathbb{R}^m : f(y) < f(x)\}.$$

In this paper we will intensively use its closed-valued variant defined at all $x \in \mathbb{R}^m$ as $\bar{S}_f^<(x) \equiv \overline{S_f^<(x)}$. Note that quasiconvexity of a function f is characterized by convex-valuedness of the map $S_f(\cdot)$ (or the map $\bar{S}_f^<(\cdot)$). But in opposite to S_f, the closed strict sublevel map can have empty values. In fact, condition $\bar{S}_f^<(x) = \emptyset$ directly characterizes global minimum of f.

Lemma 1 (Inner semicontinuity of $\bar{S}_f^<$) *For a lower semicontinuous function f, the closed strict sublevel set map $\bar{S}_f^<$ is inner semicontinuous on $\mathbb{R}^m$.*

Proof Without loss of generality we can assume that $\bar{x} \in \mathbb{R}^m$ is such that $\bar{S}_f^<(\bar{x})$ is not empty and we have to show $\bar{S}_f^<(\bar{x}) \subset \operatorname{Liminf}_{x \to \bar{x}} \bar{S}_f^<(x)$. First consider any $y \in S_f^<(\bar{x})$ and take arbitrary sequence $x_n \to \bar{x}$. We search for a sequence (y_k) converging to y and satisfying $y_k \in \bar{S}_f^<(x_k)$ for all k large enough. From lower semicontinuity of f there exists some $k_0 \in \mathbb{N}$ such that $f(y) < f(x_k)$ for all $k > k_0$, which may be equivalently written as $y \in S_f^<(x_k)$. Thus we shown $S_f^<(\bar{x}) \subset \operatorname{Liminf}_{x \to \bar{x}} \bar{S}_f^<(x)$. To complete the proof we apply closure on both sides of this inclusion and recall that inner-limit is closed by definition. □

Contrary to $\bar{S}_f^<$, the level set map S_f is not inner semicontinuous in general. As an example consider function $f(x) \equiv \min\{0, x\}$. Then $S_f(0) = \mathbb{R}$ but $\operatorname{Liminf}_{x \to 0} S_f(x) = \mathbb{R}^-$. Then the outer semicontinuity of N_f at 0 may be in jeopardy, see Corollary 1. Indeed, N_f is not outer semicontinuous at 0. Note that such a lack of outer semicontinuity of N_f for a general quasiconvex function f was already observed in [6].

Similarly as for the sublevel set, one can consider the *strict normal operator map* $N_f^<$ associating to any $x \notin \operatorname{argmin} f$, the set

$$N_f^<(x) \equiv \left(\bar{S}_f^<(x) - x\right)^\circ .$$

According to [6, Proposition 2.1], gph $N_f^<$ is closed, that is $N_f^<$ is outer semicontinuous. There are, however, well-known examples of lower semicontinuous quasiconvex functions such that N_f is not outer semicontinuous and $N_f^<$ is not quasimonotone, see [6, Example 2.2] and [4, Example 2.1], respectively.

Thus a new concept of sublevel set, the *adjusted sublevel set* and the associated *adjusted normal operator* has been considered in [4, 5] to overcome this difficulty: for any function $f : \mathbb{R}^m \to \mathbb{R}$ the adjusted sublevel set $S_f^a(x)$ at a point $x \in \mathbb{R}^n$ is

$$S_f^a(x) \equiv \begin{cases} S_f(x) \cap \mathbb{B}(S_f^<(x), \rho_x) & \text{if} \quad x \notin \operatorname{argmin} f, \\ S_f(x) & \text{otherwise,} \end{cases}$$

where argmin f denotes a set of global minimizers of function f, and

$$N_f^a(x) \equiv \left(S_f^a(x) - x\right)^\circ ,$$

where $\mathbb{B}(A, \rho)$ denotes closed set $\{y \in \mathbb{R}^m : \operatorname{dist}(A, y) \le \rho\}$ and $\rho_x \equiv \operatorname{dist}(S_f^<(x), x)$. Note that $\rho_x = 0$ is equivalent to $x \in \bar{S}_f^<(x)$ and therefore $S_f^a(x) = \bar{S}_f^<(x)$. For any function $f : \mathbb{R}^m \to \mathbb{R}$ and any $x \in \mathbb{R}^m$ one has

$$\bar{S}_f^<(x) \subset S_f^a(x) \subset S_f(x).$$

Note that for any lower semicontinuous quasiconvex function f all these sublevel set maps are closed- and convex-valued while all the introduced normal operators are cone-, closed- and convex-valued by definition. In addition, for lower semicontinuous quasiconvex function the adjusted normal operator is outer semicontinuous and quasimonotone as shown in [3, 4]. Moreover, this operator also affords additional properties like a sufficient global optimality condition, see [4, Proposition 4.1].

3.2 Limiting Sublevel Set

Even though the adjusted normal operator is the first normal operator satisfying both quasi-monotonicity and outer semicontinuity, it lacks calculus rules because of its non-local nature. This has been our ultimate motivation for the introduction of the new notions of limiting sublevel set and limiting normal operator.

Definition 1 (Limiting sublevel set) For a lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R}$ we define the *limiting sublevel set map* $S^l_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ as follows

$$S^l_f(\bar{x}) \equiv \operatorname*{Liminf}_{x\to\bar{x}} S_f(x), \quad \forall \bar{x} \in \mathbb{R}^m.$$

Note the limiting sublevel set map S^l_f is closed-valued by definition. We further observe that the lower limit in the above definition may be restricted as follows.

Lemma 2 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function, then*

$$S^l_f(\bar{x}) = \operatorname*{Liminf}_{x \underset{S_f(\bar{x})}{\longrightarrow} \bar{x}} S_f(x). \tag{6}$$

Proof For a given point $\bar{x}$ we rewrite the right-hand side of (6) as

$$R(\bar{x}) \equiv \left\{ y \in \mathbb{R}^n : \forall\, x_k \underset{S_f(\bar{x})}{\longrightarrow} \bar{x}\ \exists\, y_k \to y \text{ s.t.} \forall k\, y_k \in S_f(x_k) \right\}.$$

From definition we have $S^l_f(\bar{x}) \subset R(\bar{x})$ and we will complete the proof by showing the opposite inclusion.

First, note that $R(\bar{x}) \subset S_f(\bar{x})$. Indeed, taking any $y \in R(\bar{x})$ and considering sequence (x_k) such that $x_k = \bar{x}$ for all k, there exists a sequence (y_k) such that $y_k \to y$ with $y_k \in S_f(\bar{x})$ for all k. Thus also $y \in S_f(\bar{x})$ since this set is closed.

Now, consider arbitrary $y \in R(\bar{x})$ and $\tilde{x}_k \to \bar{x}$. To verify $y \in S^l_f(\bar{x})$ we have to find $y_k \to y$ such that $y_k \in S_f(\tilde{x}_k)$. Put $x_k \equiv \tilde{x}_k$ when $\tilde{x}_k \in S_f(\bar{x})$ and $x_k \equiv \bar{x}$ otherwise. Since $y \in R(\bar{x})$ we know that there exists a sequence (y_k) such that $y_k \to y$ and $y_k \in S_f(x_k)$. To complete the proof we realize $f(x_k) = \min\{f(\tilde{x}_k), f(\bar{x})\} \le f(\tilde{x}_k)$ for all k, thus also $y_k \in S_f(\tilde{x}_k)$. □

The limiting definition, inspired by the similar concept of limiting subdifferential, see e.g. [10], turns out to have the following very easy and natural equivalent explicit formulation for any lower semicontinuous function.

Theorem 1 (Explicit formula for $S^l_f(x)$) *Let f be a lower semicontinuous function and $x \in \mathbb{R}^m$. Then*

$$S^l_f(x) = \begin{cases} \bar{S}^<_f(x) & \text{if } x \in \bar{S}^<_f(x), \\ S_f(x) & \text{otherwise.} \end{cases}$$

In particular one always has $\bar{S}^<_f(x) \subset S^a_f(x) \subset S^l_f(x) \subset S_f(x)$.

Note that $x \in \bar{S}_f^<(x)$ means that x is not a local minimum of f. Thus Theorem 1 can be equivalently rephrased as "$S_f^l(x) = S_f(x)$ if x is a local minimum of f and $S_f^l(x) = \bar{S}_f^<(x)$ otherwise".

Proof First, observe that according to the definition and since f is lower semicontinuous, one has $S_f^l(\bar{x}) = \mathrm{Liminf}_{x\to\bar{x}}\ S_f(x) \subset S_f(\bar{x})$ for any $\bar{x} \in \mathbb{R}^m$. Moreover if $\bar{x} \in \mathbb{R}^m$ is such that $\bar{x} \notin \bar{S}_f^<(\bar{x})$ then, there exists open neighbourhood U of $\bar{x}$ such that for all $x \in U$ we have $f(x) \geq f(\bar{x})$. This is equivalent to $S_f(x) \supset S_f(\bar{x})$ for all $x \in U$ and thus $S_f^l(\bar{x}) = S_f(\bar{x})$.

Now take any $z \in S_f^<(\bar{x})$ and any sequence (x_k) converging to $\bar{x}$. By lower semicontinuity of f, one immediately has $f(x_k) > f(z)$ for k large enough, implying $z \in S_f(x_k)$. Thus finally $z \in S_f^l(\bar{x})$. And since $S_f^l(\bar{x})$ is closed by definition, we immediately have $\bar{S}_f^<(\bar{x}) \subset S_f^l(\bar{x})$.

Finally let us consider $\bar{x} \in \mathbb{R}^m$ and $y \in S_f^l(\bar{x})$ such that $\bar{x} \in \bar{S}_f^<(\bar{x})$. Thus there exists a sequence $(\tilde{x}_k)$ converging to $\bar{x}$ with $f(\tilde{x}_k) < f(\bar{x})$ for all k. By expressing $S_f^l(\bar{x})$ in the following way

$$S_f^l(\bar{x}) = \bigcap_{x_k\to\bar{x}} \operatorname*{Liminf}_{k\to\infty} S_f(x_k),$$

see, e.g. [11, Equation 5(1)], we can deduce the existence of a sequence (y_k) converging to y and such that $y_k \in S_f(\tilde{x}_k)$ for any k. In other words

$$f(y_k) \leq f(\tilde{x}_k) < f(\bar{x})$$

and so $y_k \in \bar{S}_f^<(x)$ implying $y \in \bar{S}_f^<(x)$ as $\bar{S}_f^<(x)$ is closed by definition. And the proof of this equivalent definition of the limiting sublevel set is complete.

To prove the stated inclusions we observe in the definition of $S_f^a(x)$ that $\rho_x = 0$ provided $\bar{x} \in \bar{S}_f^<(\bar{x})$, thus $S_f^a(x) = \bar{S}_f^<(x) = S_f^l(x)$. Otherwise, we see $\rho_x > 0$ and so $S_f^a(x) \subset S_f(x) = S_f^l(x)$. □

As for the classical sublevel set S_f and strict sublevel set $S_f^<$, the convexity of the limit sublevel set characterizes the quasiconvexity of a lower semicontinuous function.

Lemma 3 (Characterization of quasiconvexity in terms of S_f^l) *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function. Then f is quasiconvex if and only if $S_f^l(x)$ is convex for all $x \in \mathbb{R}^m$.*

Proof Convexity of $S_f^l(x)$ for a quasiconvex function f is due to definition since lower limit of convex sets is convex. Now let us show that the convexity of $S_f^l(x)$ for all $x \in \mathbb{R}^m$ is a sufficient condition for quasiconvexity of f. Assume, for a contradiction, that $S_f^l(x)$ is convex for all $x \in \mathbb{R}^m$ and that f is not quasiconvex. Then, there exists some $\bar{x} \in \mathbb{R}^m$ such that sublevel set $S_f(\bar{x})$ is not convex. In such a case convexity of $S_f^l(\bar{x})$ implies $S_f^l(\bar{x}) = \bar{S}_f^<(\bar{x})$ due to Theorem 1, thus there exists $\tilde{x} \in S_f(\bar{x}) \setminus \bar{S}_f^<(\bar{x})$. Then, $f(\bar{x}) = f(\tilde{x})$ implies $\bar{S}_f^<(\bar{x}) = \bar{S}_f^<(\tilde{x})$ and so $\tilde{x} \notin \bar{S}_f^<(\tilde{x})$. Finally, by using Theorem 1 again, we obtain $S_f^l(\tilde{x}) = S_f(\tilde{x}) = S_f(\bar{x})$, a contradiction with convexity of $S_f^l(\tilde{x})$. □

Corollary 2 (Inner semicontinuity of S^l_f) *For any lower semicontinuous function f, the limiting sublevel set map S^l_f is inner semicontinuous.*

Proof We have to show that $S^l_f(\bar{x}) = \operatorname{Liminf}_{x\to\bar{x}} S^l_f(x)$ for all $\bar{x} \in \mathbb{R}^m$. First, consider such $\bar{x} \in \mathbb{R}^m$ that $\bar{x} \in \bar{S}^<_f(\bar{x})$. We have

$$S^l_f(\bar{x}) = \bar{S}^<_f(\bar{x}) = \operatorname*{Liminf}_{x\to\bar{x}} \bar{S}^<_f(x) \subset \operatorname*{Liminf}_{x\to\bar{x}} S^l_f(x)$$

where Theorem 1, Lemma 1 and Theorem 1 again were used respectively.

Now, we have to deal with such point $\bar{x} \in \mathbb{R}^m$ that $\bar{x} \notin \bar{S}^<_f(\bar{x})$. Then $S^l_f(\bar{x}) = S_f(\bar{x})$ using Theorem 1, and there exists an open neighbourhood U of $\bar{x}$ such that for all $x \in U$ we have $f(x) \geq f(\bar{x})$. Moreover, this consideration is valid also for all $x \in U$ such that $f(x) = f(\bar{x})$, i.e., $S^l_f(x) = S_f(x) = S_f(\bar{x})$ for such points. Finally, for points $x \in U$ satisfying $f(x) > f(\bar{x})$, we have $S_f(\bar{x}) \subset \bar{S}^<_f(x) \subset S^l_f(x)$ using Theorem 1 again. Thus, for all points $x \in U$ it holds $S_f(\bar{x}) = S^l_f(\bar{x}) \subset S^l_f(x)$ and so $S^l_f(\bar{x}) = \operatorname{Liminf}_{x\to\bar{x}} S^l_f(x)$. □

For any $x, y \in \mathbb{R}^m$ and any function f one clearly has $x \in S_f(y)$ or $y \in S_f(x)$. As shown in the following lemma, the limiting sublevel set inherits of the same property providing that the function is lower semicontinuous whereas it is not true for the closed strict sublevel set $\bar{S}^<_f$.

Lemma 4 *For $x, y \in \mathbb{R}^m$ and any lower semicontinuous function f it holds $y \in S^l_f(x)$ or $x \in S^l_f(y)$.*

Proof Consider $y \notin S^l_f(x)$. Using Theorem 1 we have also $y \notin \bar{S}^<_f(x)$ and so $f(y) \geq f(x)$. For $f(y) > f(x)$ we immediately obtain $x \in \bar{S}^<_f(y) \subset S^l_f(y)$. Variant $f(x) = f(y)$ deserves more effort. Since $y \notin S^l_f(x)$, there exists a sequence $(\tilde{x}_n)$ converging to x and $\varepsilon > 0$ such that $\operatorname{dist}(S_f(\tilde{x}_n), y) > \varepsilon$ for all n. Now, we take arbitrary sequence (y_n) converging to y. For n large enough we have $|y_n - y| \leq \varepsilon$, and so we know that $y_n \notin S_f(\tilde{x}_n)$. This implies $\tilde{x}_n \in S_f(y_n)$ and the proof is finished as $\tilde{x}_n$ converges to $x \in S^l_f(y)$. □

3.3 Limiting Normal Operator

The previous theory of limiting sublevel set was build for general lower semicontiunous functions. In the sequel, we assume quasiconvexity of function f in addition. Note that this assumption will be relaxed in Section 5.

Definition 2 (Limiting normal operator) For quasiconvex lower semicontinuous function f the *limiting normal operator* is a set-valued map $N^l_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined as

$$N^l_f(x) \equiv \left(S^l_f(x) - x\right)^\circ, \quad \forall x \in \mathbb{R}^m.$$

As for a quasiconvex function f set $S^l_f(x)$ is convex and $x \in S^l_f(x)$ for all x, we further observe $\left(S^l_f(x) - x\right)^\circ = \left(T_{S^l_f(x)}(x)\right)^\circ$ due to [11, Theorem 6.9]. Therefore, the above introduced limiting normal operator is a local notion. An alternative definition of the limiting normal operator can be given in terms of upper limit of normal operator.

Theorem 2 *For any quasiconvex lower semicontinuous function f it holds*

$$N^l_f(x) = \mathop{Limsup}_{x\to\bar{x}} N_f(x). \tag{7}$$

Proof Applying Proposition 1 to the set-valued map $M(x) = S_f(x) - x$ and observing that $\mathrm{Liminf}_{x\to\bar{x}} \left\{ S_f(x) - x \right\} = \mathrm{Liminf}_{x\to\bar{x}} \left\{ S_f(x) \right\} - \bar{x}$ we obtain

$$S^l_f(\bar{x}) - \bar{x} \subset \left(\mathop{\mathrm{Limsup}}_{x\to\bar{x}} \left(S_f(x) - x\right)^\circ \right)^\circ .$$

Thus using the bipolar formula (1) we have

$$N^l_f(\bar{x}) \supset \overline{\mathrm{conv}} \left\{ \mathop{\mathrm{Limsup}}_{x\to\bar{x}} N_f(x) \right\} \supset \mathop{\mathrm{Limsup}}_{x\to\bar{x}} N_f(x).$$

To show the opposite inclusion, that is

$$\left(S^l_f(\bar{x}) - \bar{x}\right)^\circ \subset \mathop{\mathrm{Limsup}}_{x\to\bar{x}} \left(S_f(x) - x\right)^\circ \tag{8}$$

we use Theorem 1. First consider $\bar{x}$ such that $\bar{x} \notin \bar{S}^<_f(\bar{x})$. Then inclusion (8) reduces to $\left(S_f(\bar{x}) - \bar{x}\right)^\circ \subset \mathrm{Limsup}_{x\to\bar{x}} \left(S_f(x) - x\right)^\circ$ which always holds true. A more effort is needed to show (8) provided that $\bar{x} \in \bar{S}^<_f(\bar{x})$. Using Theorem 1 we rewrite (8) as

$$\left(\bar{S}^<_f(\bar{x}) - \bar{x}\right)^\circ \subset \mathop{\mathrm{Limsup}}_{x\to\bar{x}} \left(S_f(x) - x\right)^\circ.$$

Using $\bar{x} \in \bar{S}^<_f(\bar{x})$ and lower semicontinuity of f we find a sequence (x_k) with limit $\bar{x}$ such that $(f(x_k))$ is an increasing sequence converging to $f(\bar{x})$. One can easily show that $S^<_f(\bar{x}) = \bigcup_{k=1}^\infty S_f(x_k)$. Thus $\bar{S}^<_f(\bar{x}) = \mathrm{cl} \bigcup_{k=1}^\infty S_f(x_k)$ and so using [11, Excercise 4.3] the sequence of sets $(S_f(x_k))$ converges to $\bar{S}^<_f(\bar{x})$ in the Painlevé-Kuratowski sense. Now take arbitrary

$$w \in \left(\bar{S}^<_f(\bar{x}) - \bar{x}\right)^\circ, \tag{9}$$

or, equivalently, $w \in N_{\bar{S}^<_f(\bar{x})}(\bar{x})$ or $\bar{x} = \mathrm{Proj}_{\bar{S}^<_f(\bar{x})}(w + \bar{x})$ since $\bar{S}^<_f(\bar{x})$ is a convex closed set. Then again due to closedness and convexity of $S_f(x_k)$ for all k, we may define $\tilde{x}_k \equiv \mathrm{Proj}_{S_f(x_k)}(w + \bar{x})$ which is equivalent to $w_k \in \left(S_f(x_k) - \tilde{x}_k\right)^\circ$ for all k with w_k given by $w_k \equiv w + \bar{x} - \tilde{x}_k$. Now, thanks to the Painlevé-Kuratowski convergence of sequence $(S_f(x_k))$ and according to [11, Proposition 4.9] we have

$$\lim_{k\to+\infty} \tilde{x}_k = \mathrm{Proj}_{\bar{S}^<_f(\bar{x})}(w + \bar{x}) = \bar{x}.$$

Thus we found sequences $(\tilde{x}_k)$ and (w_k) converging to $\bar{x}$ and w, respectively, such that $w_k \in \left(S_f(x_k) - \tilde{x}_k\right)^\circ$ for any k. However, we have also $w_k \in \left(S_f(\tilde{x}_k) - \tilde{x}_k\right)^\circ$ observing $S_f(\tilde{x}_k) \subset S_f(x_k)$ since $\tilde{x}_k \in S_f(x_k)$. Thus

$$w \in \mathop{\mathrm{Limsup}}_{x\to\bar{x}} \left(S_f(x) - x\right)^\circ$$

and the proof is done since $w \in \left(\bar{S}^<_f(\bar{x}) - \bar{x}\right)^\circ$ was chosen arbitrarily in (9). □

Let us conclude this section by showing that the newly defined limiting normal operator satisfies at the same time quasimonotonicity, outer semicontinuity and -necessary and sufficient- optimality conditions, providing that the considered function is quasiconvex and lower semicontinuous.

Theorem 3 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous quasiconvex function. Then*

(i) $N_f(x) \subset N^l_f(x) \subset N^a_f(x) \subset N^<_f(x), \quad \forall x \in \mathbb{R}^m,$
(ii) N^l_f *is quasimonotone,*
(iii) N^l_f *is outer semicontinuous.*

The statement (i) of the previous theorem may be well illustrated on a simple example. Consider, for instance, the quasiconvex lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R}$ defined by $f(x) = 1$ if $x_1 \in]0, 1]$ and $f(x) = x_1$ otherwise.

Proof Statement (i) is a direct consequence of Theorem 1. Now from (i) and quasimonotonicity of N^a_f, see [4, Proposition 3.3], we immediately deduce (ii) since any operator included in a quasimonotone operator is quasimonotone. The proof of (iii) is due to Theorem 2. □

Now consider the minimization problem

$$\min f(x) \qquad \text{subject to} \qquad x \in K, \tag{10}$$

where $K \subset \mathbb{R}^m$ is nonempty subset of $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ is quasiconvex. Then, necessary and sufficient optimality conditions may be stated as follows.

Theorem 4 (Necessary optimality conditions) *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous quasiconvex function, K be a nonempty convex set and $\bar{x} \in K$ be a solution to (10) which is not a local minimum of f. Then it holds*

$$0 \in N^l_f(\bar{x}) \setminus \{0\} + N_K(\bar{x}).$$

Proof First we note that $N^l_f(\bar{x}) = N^<_f(\bar{x})$ according to Theorem 1. Thus the necessary condition follows directly from [9, Theorem 3.1 (iii)] since f is quasiconvex and $\bar{x}$, being an element of $\bar{S}^<_f(\bar{x})$, cannot be a local minimizer of f on $\mathbb{R}^m$. □

Since definition of limiting normal operator is of a local nature, the respective first order sufficient conditions lead naturally to local solutions of the concerned optimization problem.

Theorem 5 (Sufficient optimality conditions) *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, K be a nonempty subset of $\mathbb{R}^m$ and $\bar{x} \in K$. Then $\bar{x}$ a is local solution to (10) if one of the following hypotheses is satified:*

(i) *point $\bar{x}$ is a solution of the Stampacchia variational inequality defined by $N^l_f(\cdot) \setminus \{0\}$ and K;*
(ii) *set K is convex and* $0 \in N^l_f(\bar{x}) \setminus \{0\} + N_K(\bar{x})$.

Let us recall that a point $x \in K$ is a solution of the Stampacchia variational inequality defined by a set-valued map F and a set K if there exists $x^* \in F(x)$ such that $\langle x^*, y - x \rangle \geq 0$, for any $y \in K$.

Proof There are two alternatives. Either $\bar{x} \notin \bar{S}_f^<(\bar{x})$, thus $\bar{x}$ is a local minimizer of f on $\mathbb{R}^m$, and so it is also local solution to (10), or $\bar{x} \in \bar{S}_f^<(\bar{x})$ holds. Then we have $N_f^l(\bar{x}) = N_f^<(\bar{x}) = N_f^a(\bar{x})$ as a consequence of Theorem 1 and definition of $N_f^a(\bar{x})$. The conclusion in case (i) follows from [5, Proposition 3.2]. Now whenever K is a convex set then both conditions of (i) and (ii) clearly coincide. □

Let us observe that, except for the case of the minimization of a convex function with respect to a convex set, first order tools like derivative or subdifferential do not provide sufficient optimality conditions, even if the objective function is quasiconvex (consider, e.g., $f(x) = x^3$, $K = [-1, 1]$, and $\bar{x} = 0$ solution to Stampacchia variational inequality defined by ∇f and K). However, we may derive necessary and sufficient conditions for global optimality in the terms of N_f^l by combining Theorem 4 and Theorem 5 (ii) as follows.

Proposition 2 (Global optimality conditions) *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, K be a nonempty convex set and $\bar{x} \in K$ which is not a local minimum of f. Then*

$$\begin{aligned} \bar{x} \text{ is a global solution to (10)} &\Leftrightarrow \bar{x} \text{ is a local solution to (10)} \\ &\Leftrightarrow 0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x}). \end{aligned}$$

Proof The only thing that remains to be proved is that, under the assumptions of the above proposition, global and local concepts of solution of (10) coincide. Indeed, if there exists $y \in K$ such that $f(y) < f(\bar{x})$ then, by quasiconvexity of f, $f(z) \le f(\bar{x})$ for any $z \in [\bar{x}, y]$. Now if $\bar{x}$ is a local minumum of f on K, one can find $\bar{z} \in]\bar{x}, y[$ such that $f(\bar{x}) = f(\bar{z}) > f(z)$, for any $z \in]\bar{z}, y]$. Now since $\bar{x}$ is not a local minimum of f and f is continuous, one can find a sequence $(\bar{x}_k)$ converging to $\bar{x}$ with $f(\bar{x}_k) < f(\bar{x})$ converging to $f(\bar{x})$. If we denote by

$$y_k = \bar{z} + \frac{\|\bar{z} - y\|}{\|\bar{z} - \bar{x}\|} (\bar{z} - \bar{x}_k),$$

then again by continuity of f, $f(y_k) < f(\bar{x}) = f(\bar{z})$ for k large enough. But since $\bar{z} \in]\bar{x}_k, y_k[$ and $f(\bar{x}_k) < f(\bar{z})$ this is a contradiction with quasiconvexity of f. □

4 Calculus Rules for Limiting Normal Operator

In order to apply the limiting normal operator to applications in quasiconvex analysis an important issue is to dispose of efficient calculus rules. In the class of quasiconvex functions, the stable operations are the composition with a non-decreasing function and maximum of a family of quasiconvex functions. This is the reason why, in this section, we deal only with these two operations.

Let $\{f_i : \mathbb{R}^m \to \mathbb{R}\}_{i \in I}$ be a finite family of continuous quasiconvex functions and define

$$g(x) \equiv \max_{i \in I} f_i(x).$$

Clearly g is finite, quasiconvex and continuous on $\mathbb{R}^m$. For any x, the index set of active functions reads $I(x) \equiv \{i \in I : f_i(x) = g(x)\}$.

Theorem 6 (Limiting normal operator to maximum) *Let $\{f_i : \mathbb{R}^m \to \mathbb{R}\}_{1\in I}$ be a finite family of continuous quasiconvex functions and g be defined as above. Then*

$$N^l_g(\bar{x}) \subset \sum_{i\in I(\bar{x})} N^l_{f_i}(\bar{x})$$

if the following constraint qualification is satisfied at $\bar{x} \in \mathbb{R}^m$

$$\left\{ \underset{i\in I(\bar{x})}{\forall} v_i \in N^l_{f_i}(\bar{x}) \quad and \quad \sum_{i\in I(\bar{x})} v_i = 0 \right\} \quad \Longrightarrow \quad \underset{i\in I(\bar{x})}{\forall} v_i = 0. \tag{11}$$

Let us recall that following [10], condition (11) means that the convex sets $\{S^l_{f_i}(\bar{x})\}_{i\in I(\bar{x})}$ can not be separated. Equivalently we may say that $\bar{x}$ is not an extremal point of the system $\{S^l_{f_i}(\bar{x})\}_{i\in I(\bar{x})}$, see [10, Corollary 2.4 and Theorem 2.8]. To prove Theorem 6 the following lemma is of use.

Lemma 5 *Consider the setting of Theorem 6, then for any $i \in I(\bar{x})$ it holds*

$$S^l_{f_i}(\bar{x}) \subset \underset{\substack{x \longrightarrow \bar{x} \\ S_g(\bar{x})}}{Liminf} \{y \in \mathbb{R}^m : f_i(y) \le g(x)\}. \tag{12}$$

Proof For any $x \in \mathbb{R}^m$ we have $f_i(x) \le g(x)$ and so

$$\underset{\substack{x \longrightarrow \bar{x} \\ S_g(\bar{x})}}{\operatorname{Liminf}} \{y \in \mathbb{R}^m : f_i(y) \le g(x)\} \supset \underset{\substack{x \longrightarrow \bar{x} \\ S_g(\bar{x})}}{\operatorname{Liminf}} \{y \in \mathbb{R}^m : f_i(y) \le f_i(x)\}.$$

Now, for any $i \in I(\bar{x})$ we may use $S_{f_i}(\bar{x}) \supset S_g(\bar{x})$ and Lemma 2 to obtain

$$\underset{\substack{x \longrightarrow \bar{x} \\ S_g(\bar{x})}}{\operatorname{Liminf}} \{y \in \mathbb{R}^m : f_i(y) \le f_i(x)\} \supset \underset{\substack{x \longrightarrow \bar{x} \\ S_{f_i}(\bar{x})}}{\operatorname{Liminf}} \{y \in \mathbb{R}^m : f_i(y) \le f_i(x)\} = S^l_{f_i}(\bar{x}).$$

□

Proof of Theorem 6 We have $S_g(x) = \bigcap_{i\in I} \{y \in \mathbb{R}^m : f_i(y) \le g(x)\}$ for a fixed $x \in \mathbb{R}^m$ and therefore, taking (6) into account

$$S^l_g(\bar{x}) = \underset{\substack{x \longrightarrow \bar{x} \\ S_g(\bar{x})}}{\operatorname{Liminf}} \left\{ \bigcap_{i\in I} \{y \in \mathbb{R}^m : f_i(y) \le g(x)\} \right\}.$$

Using calculus rule for inner limit of finite intersection, e.g. [11, Theorem 4.32 (c)], $S^l_g(\bar{x}) \supset A(\bar{x}) \cap B(\bar{x})$ where

$$A(\bar{x}) \equiv \underset{\substack{x \to \bar{x} \\ S_g(\bar{x})}}{\operatorname{Liminf}} \left\{ \bigcap_{i\in I(\bar{x})} \{y \in \mathbb{R}^m : f_i(y) \le g(x)\} \right\}$$

and

$$B(\bar{x}) \equiv \underset{\substack{x \to \bar{x} \\ S_g(\bar{x})}}{\operatorname{Liminf}} \left\{ \bigcap_{i\notin I(\bar{x})} \{y \in \mathbb{R}^m : f_i(y) \le g(x)\} \right\},$$

provided that the convex sets $A(\bar{x})$ and $B(\bar{x})$ cannot be separated. To show this, let us observe that for any $i \notin I(\bar{x})$ there exists, thanks to the continuity of the functions f_i and

g, a convex neighbourhood U_i of $\bar{x}$ such that $f_i(x) < g(x)$ for any $x \in U_i$. Thus we have $U_i \subset \{y \in \mathbb{R}^m : f_i(y) \le g(x)\}$ for any $i \notin I(\bar{x})$ and any $x \in U_i$. The latter implies that $U = \cap_{i \notin I(\bar{x})} U_i$ is a convex neighbourhood of $\bar{x}$ included in $B(\bar{x})$. As a consequence, $A(\bar{x})$ and $B(\bar{x})$ cannot be separated since $\bar{x} \in A(\bar{x})$. Moreover $S_g^l(\bar{x}) \supset A(\bar{x}) \cap U$ and thus

$$N_g^l(\bar{x}) \subset ((A(\bar{x}) \cap U) - \bar{x})^\circ = (A(\bar{x}) - \bar{x})^\circ . \tag{13}$$

Next, Lemma 5 together with (11) imply that the respective limit sets in the definition of $A(\bar{x})$ can not be separated. Thus using [11, Exercise 4.32 (c)] and Lemma 5 again we have

$$A(\bar{x}) \supset \bigcap_{i \in I(\bar{x})} \operatorname*{Liminf}_{\substack{x \to \bar{x} \\ S_g(\bar{x})}} \left\{y \in \mathbb{R}^m : f_i(y) \le g(x)\right\} \supset \bigcap_{i \in I(\bar{x})} S_{f_i}^l(\bar{x}). \tag{14}$$

Finally, by combining (13) with (14) we get

$$\begin{aligned} N_g^l(\bar{x}) \subset \left(\bigcap\nolimits_{i \in I(\bar{x})} \left(S_{f_i}^l(\bar{x}) - \bar{x}\right)\right)^\circ &= N_{\cap_{i \in I(\bar{x})} \left(S_{f_i}^l(\bar{x})\right)}(\bar{x}) \\ &\subset \sum\nolimits_{i \in I(\bar{x})} N_{\left(S_{f_i}^l(\bar{x})\right)}(\bar{x}) \\ &= \sum\nolimits_{i \in I(\bar{x})} \left(S_{f_i}^l(\bar{x}) - \bar{x}\right)^\circ \\ &= \sum\nolimits_{i \in I(\bar{x})} N_{f_i}^l(\bar{x}) \end{aligned}$$

according to [11, Theorem 6.42] using (11) again. □

For any function $g : \mathbb{R}^m \to \mathbb{R}$ defined as $g \equiv \theta \circ f$ where f is a lower semicontinuous quasiconvex function and $\theta : \mathbb{R} \to \mathbb{R}$ is an increasing function it holds $N_g^l(x) = N_f^l(x)$ due to $S_g^l(x) = S_f^l(x)$, which is valid for any $x \in \mathbb{R}^m$. For the more general case of the composition with a non-decreasing function the chain rule is as follows.

Theorem 7 (Chain rule for limiting normal operator) *Consider a lower semicontinuous quasiconvex function $f : \mathbb{R}^m \to \mathbb{R}$, a non-decreasing lower semicontinuous function $\theta : \mathbb{R} \to \mathbb{R}$, and their lower semicontinuous quasiconvex composition $g \equiv \theta \circ f$. Then the limiting normal operator $N_g^l(\bar{x})$ at any point $\bar{x} \in \mathbb{R}^m$ satisfies*

$$N_g^l(\bar{x}) \subset N_f^l(\bar{x}).$$

This inclusion becomes equality provided θ is increasing or $\bar{x} \in \bar{S}_g^<(\bar{x})$.

Proof For any $x \in \mathbb{R}^m$ and $y \in S_f(x)$, one immediately has $g(y) \le g(x)$ since θ is non-decreasing. Thus $S_f(x) \subset S_g(x)$ for any x and consequently $S_f^l(\bar{x}) \subset S_g^l(\bar{x})$ and $N_g^l(\bar{x}) \subset N_f^l(\bar{x})$. Analogously, for any $y \notin S_f^<(x)$ we obtain $y \notin S_g^<(x)$, thus $S_f^<(x) \supset S_g^<(x)$ and so $N_g^<(\bar{x}) \supset N_f^<(\bar{x})$.

The equality for the case of increasing θ was already discussed before the statement of the theorem. Finally, let us assume that $\bar{x} \in \bar{S}_g^<(\bar{x})$. In this case $N_g^<(\bar{x}) = N_g^l(\bar{x})$. Thus, together with Theorem 3 (i) and the two inclusion stated above, we complete the proof having $N_f^<(\bar{x}) \subset N_g^<(\bar{x}) = N_g^l(\bar{x}) \subset N_f^l(\bar{x}) \subset N_f^<(\bar{x})$. □

5 Relationship with Limiting Subdifferential

An important question is the relationship between the limiting normal operator and the limiting subdifferential, since many interesting properties of the latter one can be found in literature. One can wonder if the limiting normal operator of a quasiconvex function can be deduced from the respective limiting subdifferential. It is indeed natural to expect such a relationship, see, e.g., [11, Proposition 10.3], stating (in our notation) that

$$N_f(\bar{x}) \subset \operatorname{cone}\{\partial f(\bar{x})\} \cup \partial^{\infty} f(\bar{x}) \tag{15}$$

holds for a proper lower semicontinuous quasiconvex function f at any point $x \in \mathbb{R}^m$ such that $0 \notin \partial f(x)$. Moreover, inclusion (15) becomes equality if f is additionally regular.

In the forthcoming Theorem 8 it will be shown that the cone generated by the limiting subdifferential is included in the limiting normal operator. This inclusion becomes equality only at non stationary points, as emphasized by a very simple example given in Remark 1. Note that such points are usually not of interest for optimization purposes.

To prove such a relationship a generalized definition of the limiting normal operator is considered for a lower semicontinuous (possibly non quasiconvex) function. To this end we have to employ the concept of f-attentive convergence, which is essential in the analysis of lower semicontinuous functions. Recall that for a function $f : \mathbb{R}^m \to \mathbb{R}$ we say that a sequence (x_n) converges to x *f-attentively*, denoted by $x_n \underset{f}{\to} x$, if x_n converges to x and $\lim_{n\to\infty} f(x_n) = f(x)$.

Thus the norm topology is finer than the topology of f-attentive convergence, however, they coincide provided that f is continuous. Note that the limiting subdifferential $\partial f(x)$ is outer semicontinuous only with respect to f-attentive convergence, see [11, Proposition 8.7 and example below].

Definition 3 (Limiting normal operator) For a lower semicontinuous function f we define the limiting normal operator $N_f^l : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ at point $\bar{x} \in \mathbb{R}^m$ as

$$N_f^l(\bar{x}) \equiv \operatorname*{Limsup}_{x \underset{f}{\to} \bar{x}} \left(T_{S_f(x)}(x)\right)^{\circ}. \tag{16}$$

Note that N_f^l is outer semicontinuous with respect to f-attentive topology by definition for all lower semicontinuous functions. The next lemma proves that for a quasiconvex lower semicontinuous function f this definition is equivalent to the previous one, see Definition 2.

Lemma 6 *For a lower-semicontinuous quasiconvex function f the following equality holds*

$$\operatorname*{\mathit{Limsup}}_{x \to \bar{x}} \left(S_f(x) - x\right)^{\circ} = \operatorname*{\mathit{Limsup}}_{x \underset{f}{\to} \bar{x}} \left(T_{S_f(x)}(x)\right)^{\circ}. \tag{17}$$

Proof Since f is quasiconvex, it holds $\left(T_{S_f(x)}(x)\right)^{\circ} = \left(S_f(x) - x\right)^{\circ}$ at any x, see e.g. [1]. Thus, one inclusion in (17) is direct due to definition of outer limit, and we will complete the proof showing

$$L(\bar{x}) := \operatorname*{Limsup}_{x \to \bar{x}} \left(S_f(x) - x\right)^{\circ} \subset \operatorname*{Limsup}_{x \underset{f}{\to} \bar{x}} \left(S_f(x) - x\right)^{\circ} =: R(\bar{x}).$$

Take $y \in L(\bar{x})$, then there exist sequences (x_n) and (y_n) converging to $\bar{x}$ and y, respectively, such that $y_n \in \left(S_f(x) - x\right)^{\circ}$ for all n. Now denoting $\tau := \operatorname{Liminf}_{x_n \to \bar{x}} f(x_n)$ we have $\tau \geq f(\bar{x})$ due to lower semicontinuity of f. For the case of $\tau = f(\bar{x})$ there exists a subsequence of (x_n) converging f-attentively to $\bar{x}$, and so $y \in R(\bar{x})$.

Thus, we may further assume that $\tau > f(\bar{x})$ without loss of generality. Therefore we have $f(x_n) > f(\bar{x})$ for n large enough. This implies $S_f(\bar{x}) \subset S_f(x_n)$ and so $y_n \in \left(S_f(x_n) - x_n\right)^\circ \subset \left(S_f(\bar{x}) - x_n\right)^\circ$. Further, we take any $z \in S_f(\bar{x})$ and rewrite the previous formula as $\langle y_n, z - x_n\rangle \leq 0$. Now, letting $n \to \infty$ we obtain $\langle y, z - \bar{x}\rangle \leq 0$ for any $z \in S_f(\bar{x})$ and so the proof is finished since $y \in \left(S_f(\bar{x}) - \bar{x}\right)^\circ \subset R(\bar{x})$. □

Now, the concept of limiting normal operator will be compared to the classical and intensively studied subdifferential notions. Note that the condition for equality in the following theorem is substantially weaker than in the case of normal operator (15), where regularity of f is moreover assumed.

Theorem 8 (Relationship between subdifferential and N^l_f) *For a lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R}$ we have*

$$cone\{\partial f(\bar{x})\} \cup \partial^\infty f(\bar{x}) \subset N^l_f(\bar{x}), \tag{18}$$

where equality holds provided $0 \notin \partial f(\bar{x})$.

The previous theorem is, of course, valid also for f being additionally quasiconvex when Definition 2 of N^l_f may be used according to Lemma 6. This theorem indicates to which extent N^l_f may bring some novelty when compared with ∂f and $\partial^\infty f$. In this sense both approaches are equivalent except for (M-)stationary points, i.e. points x such that $0 \in \partial f(x)$. Nevertheless, it is important to notice that such stationary points can be very common for quasiconvex functions since they can have many "flat parts" out of the global minimum.

Remark 1 A strict inclusion in the Theorem 8 can be easily illustrated even for a quasiconvex function. Consider function $f(x) = x^3$ at $x = 0$. Then we have $N^l_f(0) = [0, \infty)$ whereas $\partial f(0) = \partial^\infty f(0) = \{0\}$. In such a case we may say that N^l_f is more informative than ∂f.

To prove Theorem 8, the following lemma will be helpful.

Lemma 7 *Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear function and $M : \mathbb{R}^k \rightrightarrows \mathbb{R}^m$ be a cone-valued outer semicontinuous set-valued map. Then, for any $\bar{x} \in R^m$*

$$L(M(\bar{x})) \subset \mathop{Limsup}_{x \to \bar{x}} L(M(x)). \tag{19}$$

This inclusion is an equality if $L^{-1}(0) \cap M(\bar{x}) = \{0\}$.

Proof Let (x_k) be any sequence converging to $\bar{x}$. According to [11, Theorems 4.26 and 4.27] one has

$$L\left(\mathop{\text{Limsup}}_{x_k \to \bar{x}} M(x_k)\right) \subset \mathop{\text{Limsup}}_{x_k \to \bar{x}} L(M(x_k)),$$

with condition of equality adopted to the case of linear map L and cone-valued outer semicontinuous map M as $L^{-1}(0) \cap M(\bar{x}) = \{0\}$. Now the conclusion follows from [11, formula 5(1)] since

$$L(M(\bar{x})) = L\left(\bigcup_{x_k\to\bar{x}} \mathrm{Limsup}_{k\to\infty} M(x_k)\right) = \bigcup_{x_k\to\bar{x}} L\left(\mathrm{Limsup}_{k\to\infty} M(x_k)\right)$$
$$\subset \bigcup_{x_k\to\bar{x}} \mathrm{Limsup}_{k\to\infty} L(M(x_k))$$
$$= \mathrm{Limsup}_{x\to\bar{x}} L(M(x)),$$

thus the proof is completed. □

To clarify the relationship of N^l_f and ∂f, one more technical lemma is needed.

Lemma 8 *For a lower semicontinuous function f it holds*

$$N_{epi\,f}(\bar{x}, f(\bar{x})) = \underset{\substack{x \to \bar{x} \\ f}}{Limsup} \left(T_{epi\,f}(x, f(x))\right)^{\circ}. \tag{20}$$

Proof We denote the right-hand side of (20) as

$$R(\bar{x}) \equiv \underset{\substack{x \to \bar{x} \\ f}}{\mathrm{Limsup}} \left(T_{\mathrm{epi}\,f}(x, f(x))\right)^{\circ}. \tag{21}$$

From the definition of $N_{\mathrm{epi}\,f}(\bar{x}, f(\bar{x}))$ it follows that $R(\bar{x}) \subset N_{\mathrm{epi}\,f}(\bar{x}, f(\bar{x}))$. Thus, we need to show that $y \in N_{\mathrm{epi}\,f}(\bar{x}, f(\bar{x}))$ implies $y \in R(\bar{x})$ to complete the proof. For such y there exist sequences $(x_n, z_n) \underset{\mathrm{epi}\,f}{\to} (\bar{x}, f(\bar{x}))$ and $(y_n) \to y$ such that $y_n \in \left(T_{\mathrm{epi}\,f}(x_n, z_n)\right)^{\circ}$ for all n. Since $\mathrm{Limsup}_{n\to\infty} f(x_n) \le f(\bar{x})$ due to $f(x_n) \le z_n$, we have also $x_n \underset{f}{\to} \bar{x}$ using lower semicontinuity of f. Observing $epi\,f - (x_n, f(x_n)) \subset epi\,f - (x_n, z_n)$ and thus $T_{\mathrm{epi}\,f}(x_n, f(x_n)) \subset T_{\mathrm{epi}\,f}(x_n, z_n)$, we conclude $y_n \in \left(T_{\mathrm{epi}\,f}(x_n, z_n)\right)^{\circ} \subset \left(T_{\mathrm{epi}\,f}(x_n, f(x_n))\right)^{\circ}$, thus showing $y \in R(\bar{x})$ and finishing the proof. □

Now, we may prove the concluding theorem of this article.

Proof of Theorem 8 For any x we observe $(S_f(x) - x) \times \mathbb{R}^+ \subset \mathrm{epi}(f) - (x, f(x))$ and thus also $T_{S_f(x)}(x) \times \mathbb{R}^+ \subset T_{\mathrm{epi}\,f}(x, f(x))$ (see e.g. [1, Table 4.3]). Then $\left(T_{S_f(x)}(x)\right)^{\circ} \times \mathbb{R}^- \supset \left(T_{\mathrm{epi}\,f}(x, f(x))\right)^{\circ}$, see, e.g. [1, Table 4.5]. Next, on both sides we apply outer limit when x tends to $\bar{x}$ f-attentively obtaining

$$N^l_f(\bar{x}) \times \mathbb{R}^- \supset \underset{\substack{x \to \bar{x} \\ f}}{\mathrm{Limsup}} \left(T_{\mathrm{epi}\,f}(x, f(x))\right)^{\circ} = N_{\mathrm{epi}\,f}(\bar{x}, f(\bar{x})).$$

where also Lemma 8 is used. By projecting this inclusion canonically from $\mathbb{R}^m \times \mathbb{R}$ to $\mathbb{R}^m$ and by using (4) we complete the proof of (18).

The opposite inclusion is more complex. Assuming $0 \notin \partial f(\bar{x})$ and recalling outer semicontinuity of the limiting subdifferential with respect to f-attentive convergence, there exists neighbourhood U of $\bar{x}$, open in f-attentive topology, such that $0 \notin \partial f(x)$ for all $x \in U$.

Then, for any $x \in U$ we may adopt [11, Proposition 10.3] to the used notation obtaining $\left(T_{S_f(x)}(x)\right)^{\circ} \subset \mathrm{cone}\{\partial f(x)\} \cup \partial^{\infty} f(x)$. We rewrite the right hand side according to (4) and apply outer limit with regards to x converging to $\bar{x}$ f-attentively thus obtaining

$$N^l_f(\bar{x}) = \underset{\substack{x \to \bar{x} \\ f}}{\mathrm{Limsup}} \left(T_{S_f(x)}(x)\right)^{\circ} \subset \underset{\substack{x \to \bar{x} \\ f}}{\mathrm{Limsup}}\, \mathrm{Proj}_{\mathbb{R}^m} \left(N_{\mathrm{epi}\,f}(x, f(x))\right). \tag{22}$$

Now since f is lower semicontinuous the map $x \mapsto N_{\mathrm{epi}\, f}(x, f(x))$ is outer semicontinuous with respect to norm convergence and thus also with respect to f-attentive convergence, which is weaker than norm convergence. Since additionally $\mathrm{Proj}_{\mathbb{R}^m}$ is linear and $N_{\mathrm{epi}\, f}(x, f(x))$ is cone-valued, we may apply Lemma 7 to right-hand side of (22). Moreover, the condition for equality is satisfied as $\mathrm{Proj}_{\mathbb{R}^m}^{-1}(0) \cap N_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x})) = \{0\}$ using the assumption $0 \notin \partial f(\bar{x})$ once more. Thus (22) turns into $N_f^l(\bar{x}) \subset \mathrm{Proj}_{\mathbb{R}^m}(N_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x})))$ proving our statement using (4). $\square$

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