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Construction of P^1 Gradient from P^0 Gradient by Averaging

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Abstract. Construction of nodal and element-wise linear (known as P^1) gradient field from element-wise constant (known as P^0) gradient field obtained by the P^1 finite element methods on defined triangular mesh is based on works of J. Dalík et al. and it is briefly explained and numerically tested in this contribution. Nodal value of P^1 gradient is computed by averaging of P^0 gradients on elements sharing the node in a common patch.

Keywords: Poisson's Equation, Gradient Averaging, FEM
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INTRODUCTION

We suppose Ω is a bounded domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$ and $\mathscr{T}_h$ a triangular mesh of Ω consisting of closed triangles. We denote by $\mathscr{N}_h, \mathscr{E}_h$ the sets of nodes and edges of the mesh. For any node $N \in \mathscr{N}_h$ we define a patch of elements meeting at N as

$$\mathscr{T}_h^N = \{T \in \mathscr{T}_h : N \in T\}.$$

We consider a function $u \in H_0^1(\Omega)$ and its approximation $u_h \in P^1(\mathscr{T}_h)$ obtained for instance by the finite element method when solving an elliptic boundary value problem. The approximation u_h is searched in the space $P^1(\mathscr{T}_h)$ of scalar continuous and element-wise linear basis functions defined over $\mathscr{T}_h$. Nodal values $u_{h,i} = u_h(N_i)$ for $N_i \in \mathscr{N}_h$ (with a given ordering of nodes) define u_h uniquely. The values of u_h outside triangular nodes are interpolated by a linear combination

$$u_h(x,y) = \sum_{i=1}^{|\mathscr{N}_h|} u_{h,i}\varphi_i(x,y)$$

of global finite element basis functions φ_i defined over nodal patches. The gradient vector

$$\nabla u_h = (\nabla_x, \nabla_y)u_h \in P^0(\mathscr{T}_h) \times P^0(\mathscr{T}_h)$$

is constant on every triangle $T \in \mathscr{T}_h$. It might be of interest to find a higher order gradient approximation in applications. A focus of this paper is the construction of an averaged gradient

$$\mathscr{G}u_h = (\mathscr{G}_x, \mathscr{G}_y)u_h \in P^1(\mathscr{T}_h) \times P^1(\mathscr{T}_h)$$

following results of [1] and its efficient implementations. More details related to the construction can be found in [2, 3]. For $N \in \mathscr{N}$, nodal gradient values at the node N are searched as linear combinations

$$\mathscr{G}_x u_h(N) = \sum_{T_i \in \mathscr{T}_h^N} w_{x,i}(\nabla_x u_h|T_i), \qquad \mathscr{G}_y u_h(N) = \sum_{T_i \in \mathscr{T}_h^N} w_{y,i}(\nabla_y u_h|T_i) \tag{1}$$

of element-wise constant gradients in the corresponding patch $\mathscr{T}_h^N$ (with a given ordering of elements in the patch). There are two sets of weights $w_{x,i} = w_x|T_i, w_{y,i} = w_y|T_i$ for derivatives with respect to x and y components, whose values decide about the quality of $\mathscr{G}u_h$. A comparison of P^0 and P^1 gradient approximations is illustrated in Figure 1.

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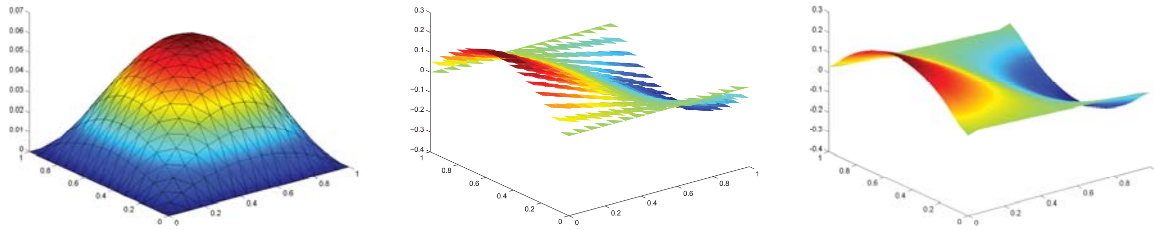

FIGURE 1. An example of a P^1 approximation v_h obtained by the finite element method (left), its P^0 gradient field $\nabla_x v_h$ (middle) and the reconstructed P^1 gradient field $\mathscr{G}_x v_h$ (right). For simplicity, only x-components of gradient fields are visualized.

RECONSTRUCTION OF P^1 GRADIENT

A simple analysis shows the gradient of a linear function $u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$ can be evaluated exactly by (1) for all coefficients $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ if and only if

$$\sum_{T_i \in \mathscr{T}_h^N} w_{x,i} = 1, \qquad \sum_{T_i \in \mathscr{T}_h^N} w_{y,i} = 1. \tag{2}$$

Solutions of two systems of equations (2) are not unique and the choice of weights satisfying (2) leads to the first-order approximation scheme. In order to evaluate the gradient of a quadratic function $u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$ exactly for all coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$, a more general system of linear equations is needed.

Let's assume triangle $T \in \mathscr{T}_h^N$ consists of nodes $N_0 = (x_0, y_0), N_1 = (x_1, y_1), N_2 = (x_2, y_2)$, where $N_0 = N$. The nodes N_0, N_1, N_2 are ordered always anticlockwise (or possibly always clockwise). Then, a linear system of equations

$$\begin{pmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \begin{pmatrix} \nabla_x f_1 & \nabla_x f_2 & \nabla_x f_3 \\ \nabla_y f_1 & \nabla_y f_2 & \nabla_y f_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} f_1(x_0,y_0) & f_2(x_0,y_0) & f_3(x_0,y_0) \\ f_1(x_1,y_1) & f_2(x_1,y_1) & f_3(x_1,y_1) \\ f_1(x_2,y_2) & f_2(x_2,y_2) & f_3(x_2,y_2) \end{pmatrix}, \tag{3}$$

where $f_1(x,y) = xy,\ f_2(x,y) = x^2,\ f_3(x,y) = y^2$ is solved. The solution $\nabla_x f_j,\ \nabla_y f_j$ for $j = 1 \dots 3$ provide constant gradients on element T, c_j is some constant. The linear system (3) is solved for all elements $T_i \in \mathscr{T}_h^N$ resulting in the set of constant gradients

$$\nabla_x(f_j|T_i), \qquad \nabla_y(f_j|T_i)$$

for $j = 1 \dots 3, i = 1 \dots m$, where m denotes the number of triangles in the patch $\mathscr{T}_h^N$. Then, we set up two linear systems of equations in form

$$M_x w_x = d_x, \qquad M_y w_y = d_y, \tag{4}$$

where $w_x = (w_x|T_1, \dots w_x|T_m), w_y = (w_y|T_1, \dots w_y|T_m)$ and

$$M_x = \begin{pmatrix} 1 & \dots & 1 \\ \nabla_x(f_1|T_1) & \dots & \nabla_x(f_1|T_m) \\ \nabla_x(f_2|T_1) & \dots & \nabla_x(f_2|T_m) \\ \nabla_x(f_3|T_1) & \dots & \nabla_x(f_3|T_m) \end{pmatrix}, d_x = \begin{pmatrix} 1 \\ y_0 \\ 2x_0 \\ 0 \end{pmatrix}, M_y = \begin{pmatrix} 1 & \dots & 1 \\ \nabla_y(f_1|T_1) & \dots & \nabla_y(f_1|T_m) \\ \nabla_y(f_2|T_1) & \dots & \nabla_y(f_2|T_m) \\ \nabla_y(f_3|T_1) & \dots & \nabla_y(f_3|T_m) \end{pmatrix}, d_y = \begin{pmatrix} 1 \\ x_0 \\ 0 \\ 2y_0 \end{pmatrix}.$$

Remark 1 *Using special rotations and shifting of triangular nodes to the origin of the coordinate system $x - y$, it is possible [1] to reduce vectors d_x, d_y to forms $d_x = (1,0,0,0)^T, d_y = (1,0,0,0)^T$. Solution of (4) is obtained by the Moore - Penrose pseudoinverse.*

Example 1 *Lets take our triangulation $\mathscr{T}_1$ and consider the patch around the inner node $[0.5,\ 0.5]$. There are six triangles contained in this patch as depicted in Figure 2 (middle). The linear systems (4) rewrite as*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.5 & 0.5 & 0.5 & 0.5 & 1 \\ 1.5 & 0.5 & 1.5 & 0.5 & 1.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_x|T_1 \\ \vdots \\ w_x|T_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.5 & 0.5 & 1 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \begin{pmatrix} w_y|T_1 \\ \vdots \\ w_y|T_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \\ 1 \end{pmatrix}$$

and their solutions provide

$$w_x = w_y = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})^T.$$

A MODEL PROBLEM

This example is taken from [4] and numerical experiments are implemented in a Matlab code inspired by [5]. We consider a Poisson problem

$$-\triangle u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{5}$$

for the unit square domain $\Omega = (0,1) \times (0,1)$ and the right-hand side

$$f(x,y) = 2x(1-x) + 2y(1-y)$$

for all $(x,y) \in \Omega$. For this particular right-hand side, the exact solution and its gradient read

$$u = x(1-x)y(1-y), \qquad \nabla u = ((1-2x)y(1-y), x(1-x)(1-2y))$$

for all $(x,y) \in \Omega$. The square geometry is discretized using a sequence of nested uniform triangular meshes $\mathscr{T}_0, \ldots, \mathscr{T}_{10}$ (containing $2, 8, 32, \ldots, 2097152$ elements and $4, 9, 25, \ldots, 1050625$ nodes), the first three triangulations are displayed in Figure 2. A discrete approximation $v \in P^1(\mathscr{T}_h)$ of the exact solution u is computed by the finite element method. Then, P^0 gradient ∇v is computed and P^1 gradient $\tau = \mathscr{G}v$ is reconstructed. Figure 1 shows the discrete solution v, its gradient ∇v and the reconstructed gradient τ computed on $\mathscr{T}_4$. For any triangulation mesh we compare squared L^2-norms

$$||\nabla u - \nabla v|| = \int_\Omega (\nabla u - \nabla v) \cdot (\nabla u - \nabla v) d\Omega, \quad ||\nabla u - \tau|| = \int_\Omega (\nabla u - \tau) \cdot (\nabla u - \tau) d\Omega$$

measuring the error of the finite elements approximation, and the error of our gradient reconstruction. Figure 3 indicates a linear convergence of $||\nabla u - \nabla v||$ and a quadratic convergence of $||\nabla u - \tau||$ measured with respect to the number of mesh nodes. The linear convergence of $||\nabla u - \nabla v||$ of the finite elements method is typical for P^1 elements and the quadratic convergence of $||\nabla u - \tau||$ coincides with the approximation properties of P^1 reconstructed gradients as proved in [1].

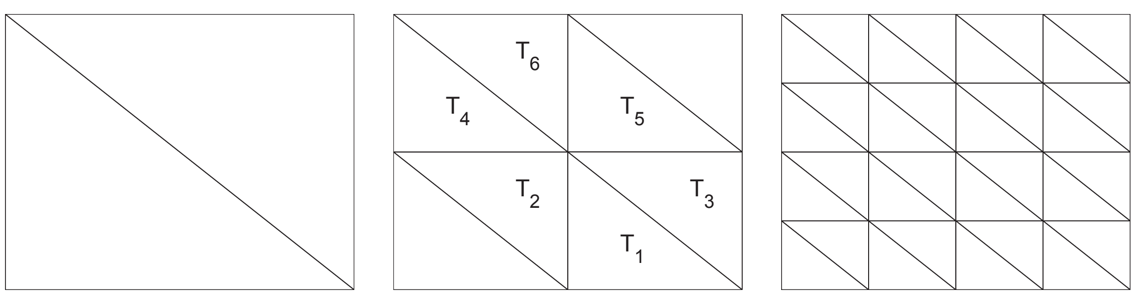

FIGURE 2. Uniform triangulations $\mathscr{T}_0$ (left), $\mathscr{T}_1$ (middle), and $\mathscr{T}_2$ (right).

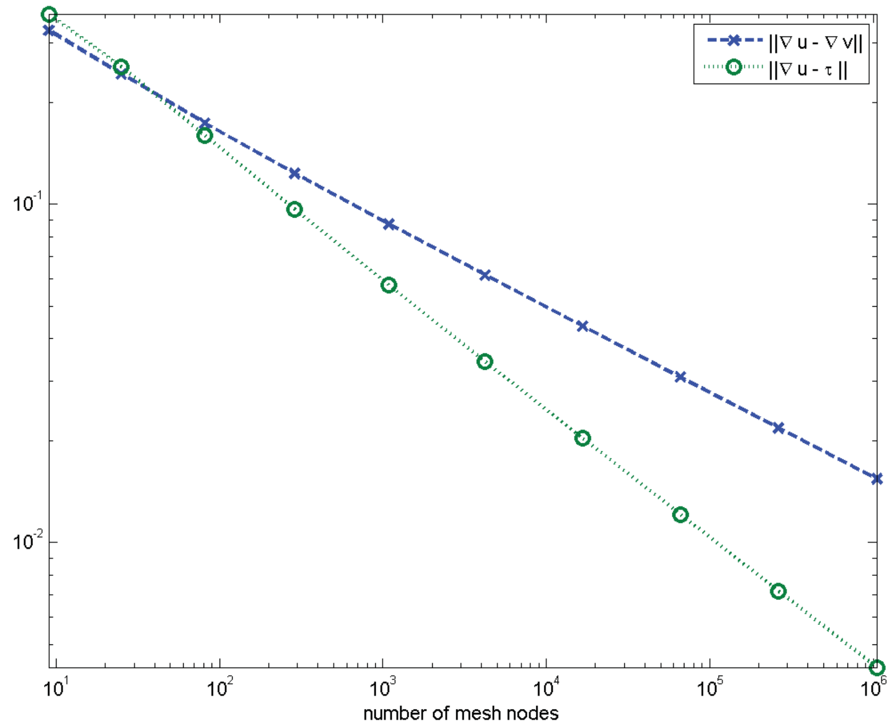

FIGURE 3. Error of elementwise constant gradient $||\nabla u - \nabla v||$ and error of reconstructed elementwise nodal linear gradient $||\nabla u - \tau||$ versus the number of mesh nodes of the considered uniform triangular mesh.

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