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Construction of $P^1$ Gradient from $P^0$ Gradient by Averaging

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Abstract. Construction of nodal and element-wise linear (known as $P^1$) gradient field from element-wise constant (known as $P^0$) gradient field obtained by the $P^1$ finite element methods on defined triangular mesh is based on works of J. Dalík et al. and it is briefly explained and numerically tested in this contribution. Nodal value of $P^1$ gradient is computed by averaging of $P^0$ gradients on elements sharing the node in a common patch.

Keywords: Poisson’s Equation, Gradient Averaging, FEM

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INTRODUCTION

We suppose $\Omega$ is a bounded domain in $\mathbb{R}^2$ with a polygonal boundary $\partial \Omega$ and $\mathcal{T}_h$ a triangular mesh of $\Omega$ consisting of closed triangles. We denote by $\mathcal{N}_h$, $\mathcal{E}_h$ the sets of nodes and edges of the mesh. For any node $N \in \mathcal{N}_h$ we define a patch of elements meeting at $N$ as

$$\mathcal{P}_h^N = \{ T \in \mathcal{T}_h : N \in T \}.$$  

We consider a function $u \in H^1_0(\Omega)$ and its approximation $u_h \in P^1(\mathcal{T}_h)$ obtained for instance by the finite element method when solving an elliptic boundary value problem. The approximation $u_h$ is searched in the space $P^1(\mathcal{T}_h)$ of scalar continuous and element-wise linear basis functions defined over $\mathcal{T}_h$. Nodal values $u_{h,i} = u_{h}(N_i)$ for $N_i \in \mathcal{N}_h$ (with a given ordering of nodes) define $u_h$ uniquely. The values of $u_h$ outside triangular nodes are interpolated by a linear combination

$$u_h(x,y) = \sum_{i=1}^{|\mathcal{N}_h|} u_{h,i} \varphi_i(x,y)$$

of global finite element basis functions $\varphi_i$ defined over nodal patches. The gradient vector

$$\nabla u_h = (\nabla_x, \nabla_y) u_h \in P^0(\mathcal{T}_h) \times P^0(\mathcal{T}_h)$$

is constant on every triangle $T \in \mathcal{T}_h$. It might be of interest to find a higher order gradient approximation in applications. A focus of this paper is the construction of an averaged gradient

$$\mathcal{G}_h u_h = (\mathcal{G}_x, \mathcal{G}_y) u_h \in P^1(\mathcal{T}_h) \times P^1(\mathcal{T}_h)$$

following results of [1] and its efficient implementations. More details related to the construction can be found in [2, 3]. For $N \in \mathcal{N}$, nodal gradient values at the node $N$ are searched as linear combinations

$$\mathcal{G}_x u_h(N) = \sum_{T_i \in \mathcal{P}_h^N} w_{x,i} (\nabla_x u_h|T_i), \quad \mathcal{G}_y u_h(N) = \sum_{T_i \in \mathcal{P}_h^N} w_{y,i} (\nabla_y u_h|T_i)$$

(1)

of element-wise constant gradients in the corresponding patch $\mathcal{P}_h^N$ (with a given ordering of elements in the patch). There are two sets of weights $w_{x,i} = w_x|T_i$, $w_{y,i} = w_y|T_i$ for derivatives with respect to $x$ and $y$ components, whose values decide about the quality of $\mathcal{G}_h u_h$. A comparison of $P^0$ and $P^1$ gradient approximations is illustrated in Figure 1.

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A simple analysis shows the gradient of a linear function \( u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \) can be evaluated exactly by (1) for all coefficients \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \) if and only if

\[
\sum_{T_i \in \mathcal{T}_h^N} w_{x,i} = 1, \quad \sum_{T_i \in \mathcal{T}_h^N} w_{y,i} = 1.
\]

(2)

Solutions of two systems of equations (2) are not unique and the choice of weights satisfying (2) leads to the first-order approximation scheme. In order to evaluate the gradient of a quadratic function \( u(x,y) = \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \) exactly for all coefficients \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R} \), a more general system of linear equations is needed.

Let’s assume triangle \( T \in \mathcal{T}_h^N \) consists of nodes \( N_0 = (x_0, y_0), N_1 = (x_1, y_1), N_2 = (x_2, y_2), \) where \( N_0 = N \). The nodes \( N_0, N_1, N_2 \) are ordered always anticlockwise (or possibly always clockwise). Then, a linear system of equations

\[
\begin{pmatrix}
 x_0 & y_0 & 1 \\
 x_1 & y_1 & 1 \\
 x_2 & y_2 & 1 \\
\end{pmatrix}
\begin{pmatrix}
 \nabla_x f_1 \\
 \nabla_x f_2 \\
 \nabla_x f_3 \\
\end{pmatrix}
= \begin{pmatrix}
 f_1(x_0, y_0) \\
 f_2(x_1, y_1) \\
 f_3(x_2, y_2) \\
\end{pmatrix},
\]

(3)

where \( f_1(x,y) = xy, \ f_2(x,y) = x^2, \ f_3(x,y) = y^2 \) is solved. The solution \( \nabla_x f_i, \ \nabla_y f_j \) for \( j = 1 \ldots 3 \) provide constant gradients on element \( T, c_j \) is some constant. The linear system (3) is solved for all elements \( T_i \in \mathcal{T}_h^N \) resulting in the set of constant gradients

\[
\nabla_x (f_j|_{T_i}), \quad \nabla_y (f_j|_{T_i})
\]

for \( j = 1 \ldots 3, i = 1 \ldots m \), where \( m \) denotes the number of triangles in the patch \( \mathcal{T}_h^N \). Then, we set up two linear systems of equations in form

\[
M_x w_x = d_x, \quad M_y w_y = d_y,
\]

(4)

where \( w_x = (w_{x,T_1}, \ldots, w_{x,T_m}), w_y = (w_{y,T_1}, \ldots, w_{y,T_m}) \) and

\[
M_x = \begin{pmatrix}
 \nabla_x (f_1|_{T_1}) & \cdots & \nabla_x (f_1|_{T_m}) \\
 \nabla_x (f_2|_{T_1}) & \cdots & \nabla_x (f_2|_{T_m}) \\
 \nabla_x (f_3|_{T_1}) & \cdots & \nabla_x (f_3|_{T_m}) \\
\end{pmatrix}, \quad d_x = \begin{pmatrix}
 1 \\
 2 \ 0 \\
 0 \\
\end{pmatrix}, \quad M_y = \begin{pmatrix}
 \nabla_y (f_1|_{T_1}) & \cdots & \nabla_y (f_1|_{T_m}) \\
 \nabla_y (f_2|_{T_1}) & \cdots & \nabla_y (f_2|_{T_m}) \\
 \nabla_y (f_3|_{T_1}) & \cdots & \nabla_y (f_3|_{T_m}) \\
\end{pmatrix}, \quad d_y = \begin{pmatrix}
 1 \\
 0 \\
 0 \\
\end{pmatrix}.
\]

**Remark 1** Using special rotations and shifting of triangular nodes to the origin of the coordinate system \( x - y \), it is possible [1] to reduce vectors \( d_x, d_y \) to forms \( d_x = (1,0,0,0)^T, d_y = (1,0,0,0)^T \). Solution of (4) is obtained by the Moore - Penrose pseudoinverse.

**Example 1** Let’s take our triangulation \( \mathcal{T}_1 \) and consider the patch around the inner node \([0.5, 0.5]\). There are six triangles contained in this patch as depicted in Figure 2 (middle). The linear systems (4) rewrite as

\[
\begin{pmatrix}
 1 & 1 & 1 & 1 & 1 & 1 \\
 0 & 0.5 & 0.5 & 0.5 & 0.5 & 1 \\
 1.5 & 0.5 & 1.5 & 0.5 & 1.5 & 0.5 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
 w_{x,T_1} \\
 \vdots \\
 w_{x,T_6} \\
\end{pmatrix}
= \begin{pmatrix}
 1 \\
 0.5 \\
 1 \\
 0 \\
\end{pmatrix}.
\]

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\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0.5 & 0.5 & 1 & 0 & 0.5 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0.5 & 0.5 & 1.5 & 1.5 & 1.5
\end{pmatrix}
\begin{pmatrix}
w_x | T_1 \\
w_x | T_2 \\
w_x | T_3 \\
w_x | T_4 \\
w_x | T_5 \\
w_x | T_6
\end{pmatrix} =
\begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
0.5 & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot \\
0.5 & \cdot & \cdot & \cdot
\end{pmatrix}
\]
and their solutions provide
\[w_x = w_y = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)^T.\]

**A MODEL PROBLEM**

This example is taken from [4] and numerical experiments are implemented in a Matlab code inspired by [5]. We consider a Poisson problem

\[-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega\]

for the unit square domain \(\Omega = (0, 1) \times (0, 1)\) and the right-hand side

\[f(x, y) = 2x(1 - x) + 2y(1 - y)\]

for all \((x, y) \in \Omega\). For this particular right-hand side, the exact solution and its gradient read

\[u = x(1 - x)y(1 - y), \quad \nabla u = ((1 - 2x)y(1 - y), x(1 - x)(1 - 2y))\]

for all \((x, y) \in \Omega\). The square geometry is discretized using a sequence of nested uniform triangular meshes \(T_0, \ldots, T_{10}\) (containing 2, 8, 32, \ldots, 2097152 elements and 4, 9, 25, \ldots, 1050625 nodes), the first three triangulations are displayed in Figure 2. A discrete approximation \(v \in P^1(T_h)\) of the exact solution \(u\) is computed by the finite element method. Then, \(P^0\) gradient \(\nabla v\) is computed and \(P^1\) gradient \(\tau = Gv\) is reconstructed. Figure 1 shows the discrete solution \(v\), its gradient \(\nabla v\) and the reconstructed gradient \(\tau\) computed on \(T_4\). For any triangulation mesh we compare squared \(L^2\)-norms

\[||\nabla u - \nabla v|| = \int_{\Omega} (\nabla u - \nabla v) \cdot (\nabla u - \nabla v) d\Omega, \quad ||\nabla u - \tau|| = \int_{\Omega} (\nabla u - \tau) \cdot (\nabla u - \tau) d\Omega\]

measuring the error of the finite elements approximation, and the error of our gradient reconstruction. Figure 3 indicates a linear convergence of \(||\nabla u - \nabla v||\) and a quadratic convergence of \(||\nabla u - \tau||\) measured with respect to the number of mesh nodes. The linear convergence of \(||\nabla u - \nabla v||\) of the finite elements method is typical for \(P^1\) elements and the quadratic convergence of \(||\nabla u - \tau||\) coincides with the approximation properties of \(P^1\) reconstructed gradients as proved in [1].

**FIGURE 2.** Uniform triangulations \(T_0\) (left), \(T_1\) (middle), and \(T_2\) (right).
FIGURE 3. Error of elementwise constant gradient $\| \nabla u - \nabla v \|$ and error of reconstructed elementwise nodal linear gradient $\| \nabla u - \tau \|$ versus the number of mesh nodes of the considered uniform triangular mesh.

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