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Stochastic Analysis: A Series of Lectures

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Stochastic Geometric Wave Equations

Zdzisław Brzeźniak and Martin Ondreját

Abstract. In these lecture notes we have attempted to elucidate the ideas behind the proof of the global existence of solutions to stochastic geometric wave equations whose solutions take values in a special class of Riemannian manifolds (which includes the two-dimensional sphere) published recently by the authors, see [10]. In particular, we aimed at those readers who could be frightened by the language of differential geometry.

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1. Introduction

The aim of these Lecture Notes is to present in a clear pedagogical way the results obtained by the authors in a recently published paper [10]. Let us begin with some historical background. Research on the topic of randomly perturbed (or stochastic) geometric wave equations (SGWEs) began with our 2007 paper [6] where we proved the existence and uniqueness of solutions of stochastic wave equations with a one-dimensional space variable and an arbitrary target compact Riemannian manifold. The proof from that paper was motivated by an earlier result (still unpublished) by the first named author and A. Carroll [3], see also [12], for random perturbation of the geometric heat equation (considered in Slobodetski–Besov spaces $W^{s,p}$) and [5] (considered in Slobodetski–Besov spaces $H^{1,2}$). The paper [10] is the second one and it was followed by (an earlier published) paper [8]. In [10] we extended the earlier results by proving the existence (but not uniqueness) of a solution for an arbitrary dimension of the space variable but for a restricted class of target manifolds: compact homogenous Riemannian manifolds. This class is however general enough to contain the most classical manifold: the sphere. In the current Lecture Notes we have tried to present the main ideas of the proof

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together with some explanation of differential geometry background. In the earlier mentioned paper [8] we improved the results from our first paper [6] by allowing the initial velocity to belong to the physically natural space $H^{1,2}$. In this paper the issue of uniqueness was also left open. Anyone interested in the history and results obtained for deterministic geometric wave equations can read the Introductions to the above-mentioned papers, bearing in mind the existence of a beautiful book [59] by Shatah and Struwe. Moreover, the Introductions to our earlier papers contain a lot of references to stochastic wave equations in linear spaces. For completeness, we have decided to keep a long list of references at the end of the paper.

2. Differential Geometry background

We assume that the reader is familiar with notions of a differentiable (and Riemannian) manifold, a tangent space and a vector field. From now we assume that M, or rather (M,g), is a compact Riemannian manifold. By T_pM, $p \in M$, we will denote the tangent space to M at p, and by $\pi : TM \to M$ we will denote the tangent vector bundle. The space of all smooth vector fields on M, i.e., sections of π, will be denoted by $\mathfrak{X}(M)$. The space of all smooth $\mathbb{R}$-valued functions on M will be denoted by $\mathfrak{F}(M)$. If $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth map, then by $\partial_t\gamma(t) \in T_{\gamma(t)}M$, or simply by $\gamma'(t)$, we will denote the tangent vector to γ at $t \in I$. One should recall an alternative equivalent definition of a vector field, namely a vector field on M is a smooth $\mathbb{R}$-linear map $X : \mathfrak{F}(M) \to \mathfrak{F}(M)$ such that

(D0) $X(fh) = X(f)h + fX(h)$, for all $f,h \in \mathfrak{F}(M)$.

We will exchangeably use these two different approaches to a vector field. In what follows we will use the following notation for $Y,Z \in \mathfrak{X}(M)$:

$$\langle Y,Z\rangle(p) = g_p(Y(p),Z(p)), \quad p \in M.$$

A connection on M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(D1) for each $Y \in \mathfrak{X}(M)$, the map $\mathfrak{X}(M) \ni X \mapsto \nabla_X Y \in \mathfrak{X}(M)$ is $\mathfrak{F}(M)$-linear,

(D2) for each $X \in \mathfrak{X}(M)$, the map $\mathfrak{X}(M) \ni Y \mapsto \nabla_X Y \in \mathfrak{X}(M)$ is $\mathbb{R}$-linear,

(D3) for all $X,Y \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$, $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

The vector field $\nabla_X Y$ is called the covariant derivative of Y with respect to X for the connection ∇. In view of [51, Proposition 2.2], the axiom (D1) implies that for any $Y \in \mathfrak{X}(M)$ and each $p \in M$ and each individual tangent vector $u \in T_p(M)$, a tangent vector $\nabla_u Y \in T_p(M)$ is well defined. To be precise, $\nabla_u Y = \nabla_X Y(p)$, for every $X \in \mathfrak{X}(M)$ such that $X(p) = u$. A fundamental result due to Levi-Civita is, see [51, Theorem 3.11], there exists a unique connection ∇ on M, called the Levi-Civita connection such that for all $X,Y,Z \in \mathfrak{X}(M)$,

(D4) $[X,Y] = \nabla_X Y - \nabla_Y X$

and

(D5) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Let us also recall the following result about differentiating along a curve. For a smooth map $\gamma : I \to M$ we will denote by $\mathfrak{X}(\gamma)$ the space of all smooth vector fields on γ and if $V \in \mathfrak{X}(M)$ then $(V_\gamma)(t) = V(\gamma(t))$, $t \in I$. By $\mathfrak{F}(I)$ we will denote the space $C^\infty(I,\mathbb{R})$. One can show, see [51, Proposition 3.18], that if ∇ is the Levi-Civita connection on M, $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth map, then there exists a unique linear map $' : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ such that for all $h \in \mathfrak{F}(I)$, $Z, Z_2 \in \mathfrak{X}(\gamma)$, $V \in \mathfrak{X}(M)$

(i2) $(hZ)' = (\frac{dh}{dt})Z + hZ'$,

(i3) $(V_\gamma)'(t) = \nabla_{\partial_t\gamma(t)}(V)$, $t \in I$,

and

(i4) $\frac{d}{dt}\langle Z, Z_2\rangle = \langle Z', Z_2\rangle + \langle Z, Z_2'\rangle$.

We will denote $Z'(t)$ by $\nabla_{\partial_t\gamma(t)}(Z)(t)$. In particular, if $Z(t) = \partial_t\gamma(t)$, $t \in I$, is the velocity field of γ, then $\nabla_{\partial_t\gamma(t)}(\partial_t\gamma)(t)$ is called the *acceleration* of the curve γ at $t \in I$ and will be denoted in this paper by $\mathbf{D}_t\partial_t\gamma(t)$. Let us note that the time variable will sometimes be denoted by s and also that the same construction works for a space variable x.

Example 1. The Euclidean space $\overline{M} = \mathbb{R}^d$ equipped with a trivial metric g is a Riemannian manifold. For each $p \in \mathbb{R}^d$, the tangent space $T_p\mathbb{R}^d$ is naturally isometrically isomorphic to $\mathbb{R}^d$. Hence a vector field X on $\mathbb{R}^d$ is simply a function $X : \mathbb{R}^d \to \mathbb{R}^d$. A function $\overline{\nabla} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $(\overline{\nabla}_X Y)(p) := (d_pY)(X(p))$ is the corresponding Levi-Civita connection and is called the *natural* connection on $\mathbb{R}^d$. In particular, the acceleration of a smooth curve $\gamma : I \to \mathbb{R}^d$ with respect to the *natural* connection on $\overline{M} = \mathbb{R}^d$ satisfies $\overline{\nabla}_{\partial_t\gamma(t)}(\partial_t\gamma)(t) = \partial_t^2\gamma(t) = \ddot{\gamma}(t)$, $t \in I$.

Example 2. The $d-1$-dimensional unit sphere $M = S^{d-1}$ embedded into the Euclidean space $\mathbb{R}^d$ and equipped with the following Riemannian metric g: $g_p(u,v) := \langle u, v\rangle$, where $p \in M \subset \mathbb{R}^d$ and $u, v \in T_pM \subset \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^d$. Note that for $p \in M$, $T_pM = \{u \in \mathbb{R}^d : \langle u, p\rangle = 0\}$ and let us denote by $\pi_p : \mathbb{R}^d \ni u \mapsto u - \langle u, p\rangle p \in T_pM$ the orthogonal projection from the ambient space $\mathbb{R}^d$ onto the tangent space T_pM. The acceleration of a smooth curve $\gamma : I \to M$ with respect to the Levi-Civita connection satisfies

$$\overline{\nabla}_{\partial_t\gamma(t)}(\partial_t\gamma)(t) = \pi_{\gamma(t)}(\ddot{\gamma}(t)) = \ddot{\gamma}(t) + |\dot{\gamma}(t)|^2\gamma(t), \quad t \in I. \tag{2.1}$$

In the special case of $d = 3$, we can use the notion of the vector product in $\mathbb{R}^3$ and have

$$\overline{\nabla}_{\partial_t\gamma(t)}(\partial_t\gamma)(t) = -\gamma(t) \times \Big(\gamma(t) \times \ddot{\gamma}(t)\Big), \quad t \in I.$$

In our context, the integration by parts formula takes the following geometric form. If $\varphi : I \to \mathbb{R}$ is of C_0^1-class and $u : I \to M$ is C^1 and $Z \in \mathfrak{X}(M)$, then

$$\begin{aligned} &-\int_I \frac{d\varphi}{dx}(x)\langle \partial_x u(x), Z(u(x))\rangle\, dx \qquad (2.2)\\ &= \int_I \varphi(x)\langle \mathbf{D}_x\partial_x u(x), Z(u(x))\rangle\, dx + \int_J \varphi(x)\langle \partial_x u(x), \nabla_{\partial_x u(x)} Z\rangle\, dx. \end{aligned}$$

Example 3. Let us fix $p, q \in M$ and consider a set $\mathcal{M}_{p,q}$ of all continuous functions $\gamma : [0,1] \to M$ such that $\gamma(0) = p$, $\gamma(1) = q$, γ is absolutely continuous and $E(\gamma) = \int_0^1 |\partial_t\gamma(t)|^2\,dt$ is finite, where $|\partial_t\gamma(t)|^2 = g_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t))$, $t \in [0,1]$. Then, it is known that $\mathcal{M}_{p,q}$ is a Hilbert manifold and that E is a smooth map from $\mathcal{M}_{p,q}$ to $\mathbb{R}$. Using the integration by parts formula (2.2) one can prove that if $\gamma \in \mathcal{M}_{p,q}$ is a stationary point of E, then $\mathbf{D}_t\partial_t u(t) = 0$ for all $t \in (0,1)$.

According to the celebrated Nash embedding theorem, see [45], there exists an **isometric** embedding $i : M \hookrightarrow \mathbb{R}^d$ for some $d \in \mathbb{N}$. Hence M can be identified with its image in $\mathbb{R}^d$. In this case, i.e., when M is a Riemannian submanifold of $\mathbb{R}^d$, one introduces the second fundamental form $\mathbf{S}$ of the submanifold M of $\mathbb{R}^d$ in such a way that $\mathbf{S}_p : T_pM \times T_pM \to N_pM = T_pM^\perp$, $p \in M$. If ∇ is the Levi-Civita connection on M and X is a vector field on M and $\widetilde{X}$ is a smooth $\mathbb{R}^d$-valued extension of X to an $\mathbb{R}^d$ neighbourhood V of some $p \in M$, then, see see [51, p. 100],

$$(d_p\widetilde{X})(\eta) = \nabla_\eta X \oplus \mathbf{S}_p(X(p), \eta), \qquad \eta \in T_pM. \tag{2.3}$$

If $\gamma : I \to M$ is a C^1 curve and $X \in \mathfrak{X}_M(\gamma)$, $\bar\gamma = i \circ \gamma$ and $\bar X := i_*(X) \in \mathfrak{X}_{\mathbb{R}^d}(\bar\gamma)$ is defined by $\bar X(t) := (d_{\gamma(t)}i)\big(X(\gamma(t))\big)$, $t \in I$, then, see [51, Proposition 4.8], for all $t \in I$,

$$\begin{aligned} \bar\nabla_{\partial_t\bar\gamma(t)}\bar X &= \nabla_{\partial_t\gamma(t)}X \oplus \mathbf{S}_{\gamma(t)}(X(\gamma(t)), \partial_t\gamma(t)), \\ \bar X^{\cdot}(t) &= X'(t) + \nabla_{\partial_t\gamma(t)}X \oplus \mathbf{S}_{\gamma(t)}(X(\gamma(t)), \partial_t\gamma(t)), \end{aligned} \tag{2.4}$$

where $' : \mathfrak{X}_{\mathbb{R}^d} \to \mathfrak{X}_{\mathbb{R}^d}$ and $^{\cdot} : \mathfrak{X}_M \to \mathfrak{X}_M$ are the linear maps introduced earlier and $\overline{\nabla}$ is the natural connection on $\mathbb{R}^d$ as in Example 1.

In particular, but see also [51, Corollary 4.8], by applying the equality from Example 1 we infer that for any smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$,

$$\langle \partial_{tt}\gamma(t), \partial_t\gamma(t)\rangle = \langle \partial_{tt}\gamma(t) - \mathbf{S}_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t)), \partial_t\gamma(t)\rangle = 0, \quad t \in I. \tag{2.5}$$

3. Homogenous Riemannian manifold

We now present the standing assumption for the remaining part of the paper.

Assumption 3.1. *Let us assume that M is a compact Riemannian manifold and G compact Lie group, with the unit element denoted by e, such that G acts transitively by isometries on M, i.e., there exists a smooth map*

$$\pi : G \times M \ni (g,p) \mapsto gp \in M \tag{3.1}$$

such that

(i) *$\pi(e,p) = p$ and $\pi(g_0g_1, p) = \pi\big(g_0\pi(g_1,p)\big)$, for all $p \in M$ and $g_0, g_1 \in G$,*
(ii) *there exists $p_0 \in M$ such that $\{\pi(g,p_0) : g \in G\} = M$,*
(iii) *for every $g \in G$, the map $\pi_g : M \ni p \mapsto \pi(g,p) \in M$ is an isometry.*

Conditions (i–ii) *are equivalent to conditions* (i-ii′), *where*

(ii′) *for all $p_0 \in M$ such that $\{\pi(g,p_0) : g \in G\} = M$.*

In what follows we will often write gp instead of $\pi(g,p)$.

Example 4. The manifold $M = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ and the Lie group $G = SO(3) = \{A \in GL(3) : A^t A = I,\ \det A = 1\}$ satisfy Assumption 3.1.

Proposition 3.2. *Assume that M and G satisfy Assumption 3.1. Then for every $p \in M$, the stabiliser*

$$G_p = \{g \in G : \pi(g,p) = p\}$$

is a closed Lie subgroup of G and the map

$$\pi^p : G \ni g \mapsto gp \in M \tag{3.2}$$

is a locally trivial fibre bundle over M with fibre G_p, in particular, for every $p \in M$, the map π^p is a submersion.

Proof. Follows from [36, Theorem 2.20 and Corollary 2.23]. □

We deduce from the celebrated Moore–Schlafly Theorem [43] that in some sense Example 4 is general.

Theorem 3.3. *Assume that M and G satisfy Assumption 3.1. Then there exists a natural number n, an isometric embedding*

$$\Phi : M \to \mathbb{R}^n, \tag{3.3}$$

and an orthogonal representation, i.e., a smooth Lie group homomorphism,

$$\rho : G \to SO(n), \tag{3.4}$$

such that

$$\Phi(gp) = \rho(g)\Phi(p) \text{ for all } p \in M \text{ and } g \in G. \tag{3.5}$$

The above theorem implies that up to an isomorphism, we can assume the following.

Assumption 3.4. *We assume that M is a compact Riemannian submanifold of $\mathbb{R}^n$, for some $n \in \mathbb{N}$, i.e., M is a Riemannian manifold with the induced metric, and G compact Lie subgroup of $SO(n)$, with the unit element denoted by e, such that the natural action of G on M is transitive (and obviously isometric).*

The Lie algebra $\mathfrak{g} \cong T_eG$ is naturally identified with a subspace of the $so(n)$, which is the Lie algebra associated with $SO(n)$. Let us denote by ν the right-invariant Haar measure on G, the unique probability measure on G that is invariant with respect to right multiplication, i.e., satisfying

$$\int_G f(gh)\,\nu(dg) = \int_G f(g)\,\nu(dg), \quad h \in G, \quad f \in C(G).$$

Remark 3.5. Let us fix the canonical ONB $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. Then each matrix $A \in SO(n)$ can be identified with a linear operator on $\mathbb{R}^n$. This operator, also denoted by A, is an orientation preserving isometry. Analogously, every element A of $so(n)$ can be identified with a skew symmetric (i.e., skew-self-adjoint) (and hence of trace 0) operator. Let $\{A_i : i \in \mathbb{I}\}$ be a basis in $T_eG \subset so(n)$. Let us choose a smooth function $f : \mathbb{R}^n \to [0,1]$ such that $M = f^{-1}(\{0\})$ and $f^{-1}([0,1))$

is bounded and define a function

$$F : \mathbb{R}^n \ni x \mapsto \int_G f(gx)\,\nu(dg) \in \mathbb{R}. \tag{3.6}$$

We have the following results whose proofs are explained in [10], see the original papers [59], [33], [26] or [43].

Claim 1. (o) If $p \in M$, then $\mathrm{linspan}\{A_i p : i \in \mathbb{I}\} = T_pM$.

Claim 2.

(i) The function F is of C^∞-class,
(ii) $0 \le F \le 1$ and $F^{-1}([0,1))$ is bounded.
(iii) $M = F^{-1}(\{0\})$,
(iv) the function F is G-invariant, i.e., $F(gx) = F(x)$ for all $g \in G$ and $x \in \mathbb{R}^n$.

Claim 3.

(v) for every $i \in \mathbb{I}$ and $x \in \mathbb{R}^n$, $\langle \nabla F(x), A_i x\rangle = 0$,
(vi) for every $i \in \mathbb{I}$ and each $p \in M$, $A_i p \in T_pM$.

Claim 4. (vii) There exists a family $\left(h_{i,j}\right)_{i,j=1}^N$ of $C^\infty(M,\mathbb{R})$ functions such that

$$\xi = \sum_\alpha \sum_\beta h_{i,j}(p)\langle \xi, A_\alpha p\rangle A_\beta p, \qquad p \in M,\ \ \xi \in T_pM. \tag{3.7}$$

Claim 5. (viii) If $\tilde{h}_{ij}$ a smooth compactly supported extension of the function h_{ij} to the whole ambient space $\mathbb{R}^n$ and, for $k = 1, \dots, N$, Y^k is the restriction to M of $\tilde{Y}^k : \mathbb{R}^n \ni x \mapsto \sum_{j=1}^N \tilde{h}_{kj}(x) A^j x \in \mathbb{R}^n$, then

$$\xi = \sum_{k=1}^N \langle \xi, A^k p\rangle\, Y^k p,\ p \in M,\ \xi \in T_pM. \tag{3.8}$$

Identity (3.8) is a close reminiscence of formula (7) in [33, Lemma 2].

Remark 3.6. Let us note here that the condition (vi) is a consequence of the condition (v) if the normal space $(T_pM)^\perp$ is one dimensional (which is not assumed here), e.g., if $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$.
Let us also observe that Claim 1 implies part (vi) of Claim 3.

Example 5. It follows from Remark 3.5 that the properties listed in Claims 1–4 are satisfied when $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$, see for instance [60]. This can be seen as follows. For $i, j \in \{1, \dots, n\}$ such that $i < j$ let A^{ij} be a skew-symmetric linear operator in $\mathbb{R}^n$ whose matrix in the canonical basis $\{e_1, \dots, e_n\}$ is equal to $[a_{kl}^{ij}]_{k,l=1}^n$, where

$$a_{kl}^{ij} = \begin{cases} 1, & \text{if } (k,l) = (i,j), \\ -1, & \text{if } (k,l) = (j,i), \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Let a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\varphi(x) = 0$ iff $x = 1$ and $\varphi(x) = 1$ iff $x \in [0, \frac{1}{2}] \cup [2, \infty)$. Define then a function $F : \mathbb{R}^n \ni x \mapsto \varphi(|x|^2) \in \mathbb{R}_+$. It is easy to verify that

$$\langle \nabla F(x), A^{ij}x \rangle = 0, \text{ for every } x \in \mathbb{R}^n, A^{ij}p \in T_p\mathbb{S}^{n-1} \text{ if } p \in \mathbb{S}^{n-1}, \tag{3.10}$$

$$\xi = \sum_{1\le i<j\le n} \langle \xi, A^{ij}p \rangle A^{ij}p \text{ if } p \in \mathbb{S}^{n-1},\ \xi \in T_p\mathbb{S}^{n-1}. \tag{3.11}$$

Example 6. In the case $n = 3$ the three matrices A^{ij} from Example 5 can be relabeled as $(A_i)_{i=1}^3$, so we have

$$A_i x = x \times e_i, \quad x \in \mathbb{R}^3,\ i = 1, 2, 3.$$

Let us note that now formula (3.11) takes the particularly nice form

$$\xi = \sum_{i=1}^{3} \langle \xi, p \times e_i \rangle\ p \times e_i, \quad \text{if } p \in \mathbb{S}^2 \text{ and } \langle \xi, p \rangle = 0. \tag{3.12}$$

One can prove directly the following useful later result.

$$-|\xi|^2 p = \sum_{i=1}^{3} \langle \xi, p \times e_i \rangle\ \xi \times e_i, \quad \text{if } p \in \mathbb{S}^2 \text{ and } \langle \xi, p \rangle = 0. \tag{3.13}$$

In view of Claim 5 and identity (3.12) we may put $h_{kj} = \delta_{kj}1$ and so, with $\phi : \mathbb{R} \to \mathbb{R}$ being a C_0^∞ function such that $\text{supp}\,(\phi) = [-1, 1]$, $\phi^{-1}(\{0\}) = [-\frac{1}{2}, \frac{1}{2}]$, $\tilde{h}_{jk}(x) = \phi(|x|^2 - 1)\delta_{jk}$. Then, we infer that $\tilde{Y}^k(x) = A_k x$, for $x \in \mathbb{R}^3$ and $Y^k p = A_k p = p \times e_k$ for $p \in M = S^2$. In particular, formula (3.8) takes the form

$$\xi = \sum_{k=1}^{3} \langle \xi, p \times e_k \rangle\, p \times e_k,\ p \in M,\ \xi \in T_pM \tag{3.14}$$

which coincides with (3.12). Let us observe that

$$d_x\tilde{Y}^k(y) = A_k y = y \times e_k, \quad x, y \in \mathbb{R}^3.$$

The following lemma will prove most useful in the *identification* part of the proof of the existence of a solution.

Lemma 3.7 ([10, Lemma 5.4]). *For every $p \in M$ we have*

$$\mathbf{S}_p(\xi, \xi) = \sum_{k=1}^{N} \langle \xi, A^k p \rangle\, d_p Y^k(\xi), \quad \xi \in T_pM, \tag{3.15}$$

where $d_pY^k(\xi) := d_p\tilde{Y}^k(\xi)$ and $d_p\tilde{Y}^k$ is the Fréchet derivative of the map $\tilde{Y}^k$ at p.

In the framework of Example 6 we have

$$\mathbf{S}_p(\xi, \xi) = \sum_{k=1}^{3} \langle \xi, p \times e_k \rangle\, \xi \times e_k, \quad p \in S^2, \xi \in T_pS^2,$$

Hence, by taking into account formula (3.13) we infer that

$$\mathbf{S}_p(\xi, \xi) = -|\xi|^2 p, ;\ p \in S^2, \xi \in T_pS^2. \tag{3.16}$$

4. Itô formula in the L^2_{loc} space

4.1. The Wiener process

Given a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ is a filtration, an $\mathscr{S}'$-valued process $W = \big(W_t\big)_{t\geq 0}$ is called a spatially homogeneous Wiener process with a spectral measure μ which, throughout the paper we always assume to be positive, symmetric and to satisfy $\mu(\mathbb{R}^d) < \infty$, if and only if the following three conditions are satisfied (with $L^2(\mu) = L^2(\mathbb{R}^d, \mu; \mathbb{C})$):

- $W\varphi := \big(W_t\varphi\big)_{t\geq 0}$ is a real $\mathbb{F}$-Wiener process, for every $\varphi \in \mathscr{S}$;
- $W_t(a\varphi + \psi) = aW_t(\varphi) + W_t(\psi)$ almost surely for all $a \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $\varphi, \psi \in \mathscr{S}$;
- $\mathbb{E}\{W_t\varphi_1 W_t\varphi_2\} = t\langle\widehat{\varphi}_1, \widehat{\varphi}_2\rangle_{L^2(\mu)}$ for all $t \geq 0$ and $\varphi_1, \varphi_2 \in \mathscr{S}$.

Remark 4.1. The reader is referred to the works by Peszat and Zabczyk [53, 54] and Brzeźniak and Peszat [11] for further details on spatially homogeneous Wiener processes.

Let us denote by $H_\mu \subseteq \mathscr{S}'$ the reproducing kernel Hilbert space of the $\mathscr{S}'$-valued random vector $W(1)$, see, e.g., [23]. Then W is an H_μ-cylindrical Wiener process. Moreover, see [53] and [11], then the following result identifying the space H_μ is known.

Proposition 4.2.

$$H_\mu = \{\widehat{\psi\mu} : \ \psi \in L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu)\},$$

$$\langle\widehat{\psi\mu}, \widehat{\varphi\mu}\rangle_{H_\mu} = \int_{\mathbb{R}^d} \psi(x)\overline{\varphi(x)}\, d\mu(x), \quad \psi, \varphi \in L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu).$$

See [46] for a proof of the following lemma that states that under some assumptions, H_μ is a function space and that multiplication operators are Hilbert–Schmidt from H_μ to L^2.

Lemma 4.3. *Assume that $\mu(\mathbb{R}^d) < \infty$. Then the reproducing kernel Hilbert space H_μ is continuously embedded in the space $C_b(\mathbb{R}^d)$ and for any $g \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ and a Borel set $D \subseteq \mathbb{R}^d$, the multiplication operator $m_g = \{H_\mu \ni \xi \mapsto g \cdot \xi \in L^2(D)\}$ is Hilbert–Schmidt. Moreover, there exists a universal constant c_μ such that*

$$\|m_g\|_{\mathscr{T}_2(H_\mu, L^2(D))} \leq c_\mu \|g\|_{L^2(D)}. \tag{4.1}$$

4.2. Itô formula

In general, neither mild nor weak solutions of SPDEs are semimartingales on their state spaces. Hence, if we need to apply smooth transformations, the Itô formula cannot be applied directly and certain approximations need to be done to justify the formal Ansatz. The aim of this section is to formulate such an Ansatz which is in fact a special form of an Itô formula, see [9]. The regularity assumptions on the processes make this a new and hopefully interesting result. It is certainly crucial for our purposes, see the proof of Theorem 5.4. This result shows the key idea of the main existence result of this paper.

To this end, let us introduce the trilinear form

$$\langle u, v\rangle_{\varphi} := \int_{\mathbb{R}^d} \langle u(x), v(x)\rangle_{\mathbb{R}^m}\, \varphi(x)\, dx \tag{4.2}$$

defined for φ, u and v such that the integral on the RHS of (4.2) converges.

Lemma 4.4. *Assume that $q \in (1,2]$, for $d = 1,2$ and $q \in [\frac{2d}{d+2}, 2]$ for $d \geq 3$. Assume that U is a separable Hilbert space. Assume that*

(i) *h_0 is a progressively measurable $L^q_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$-valued process,*
(ii) *$h_1,\dots,h_d$ are progressively measurable $L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$-valued processes and*
(iii) *g is an $\mathscr{L}(U, L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k))$-valued process such that for every $\xi \in U$, $g\xi$ is $L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$-valued progressively measurable.*

Assume that the processes u, v and z are, respectively,
(iv) *adapted $H^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^n)$-valued weakly continuous,*
(v) *progressively measurable $L^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^n)$-valued,*
(vi) *progressively measurable $L^q_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$-valued, which moreover satisfy the following integrability condition. For every $T > 0$, $\mathbb{P}$-almost surely,*

$$\int_0^T \left\{ \|v(s)\|^2_{L^2(B_T;\mathbb{R}^n)} + \|z(s)\|^2_{L^q(B_T;\mathbb{R}^k)} + \|g(s)\|^2_{\mathscr{T}_2(U;L^2(B_T;\mathbb{R}^k))} \right\} ds < \infty, \tag{4.3}$$

$$\int_0^T \left\{ \|h_0(s)\|_{L^q(B_T;\mathbb{R}^k)} + \sum_{k=1}^d \|h_k(s)\|_{L^2(B_T;\mathbb{R}^k)} \right\} ds < \infty. \tag{4.4}$$

Assume finally that for each $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and for every $t \geq 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \langle u(t), \varphi\rangle &= \langle u(0), \varphi\rangle + \int_0^t \langle v(s), \varphi\rangle\, ds, \\ \langle z(t), \varphi\rangle &= \langle z(0), \varphi\rangle + \int_0^t \left\{ \langle h_0(s), \varphi\rangle + \sum_{k=1}^d \langle h_k(s), \partial_{x_k}\varphi\rangle \right\} ds \\ &\quad + \int_0^t \langle \varphi, g(s)\, dW\rangle. \end{aligned} \tag{4.5}$$

Let $Y : \mathbb{R}^n \to \mathbb{R}^k$ be a C^2-class function such that

$$Y' \text{ is bounded.} \tag{4.6}$$

Then for every $t \geq 0$ *and each* $\varphi \in \mathscr{D}(\mathbb{R}^d)$, $\mathbb{P}$*-a.s.,*

$$\begin{aligned}\big\langle z(t), Y(u(t))\big\rangle_\varphi &= \big\langle z(0), Y(u(0))\big\rangle_\varphi + \int_0^t \big\langle h_0(s), Y(u(s))\big\rangle_\varphi\, ds \\ &+ \sum_{k=1}^d \int_0^t \big\langle h_k(s), Y(u(s))\big\rangle_{\partial_{x_l}\varphi}\, ds + \sum_{k=1}^d \int_0^t \big\langle h_k(s), Y'(u(s))\partial_{x_k} u(s)\big\rangle_\varphi\, ds \\ &+ \int_0^t \big\langle z(s), Y'(u(s))v(s)\big\rangle_\varphi\, ds + \int_0^t \big\langle g(s)\, dW, Y(u(s))\big\rangle_\varphi.\end{aligned} \tag{4.7}$$

Remark 4.5. The assumption on the exponent q is motivated by the need to have the bilinear map

$$H^1(B_R) \times L^q(B_R) \ni (u,h) \mapsto \int_{B_R} u(x)h(x)\, dx \in \mathbb{R}$$

where $B_R \subset \mathbb{R}^d$, bounded.

Remark 4.6. Roughly speaking, the result says that, if $du = v\, dt$ and

$$dz = \big[h_0 - \sum_{j=1}^d \partial_{x_j} h_j\big]\, dt + g dW = \big[h_0 - \operatorname{div} h\big]\, dt + g dW$$

where $h = (h_1, \dots, h_n)$, then, with $\langle z, u\rangle = \langle z, u\rangle_\varphi$, we have

$$\begin{aligned} d\langle z, Y(u)\rangle &= \langle z, dY(u)\rangle + \langle dz, Y(u)\rangle \\ &= \langle g\, dW, Y(u)\rangle + \Big[\langle z, Y'(u)v\rangle + \langle h_0, Y(u)\rangle \\ &\quad + \langle h, Y'(u)\nabla u\rangle - \langle \operatorname{div}(hY(u)), 1\rangle\Big]\, dt. \end{aligned}$$

5. The main result

Roughly speaking our main result states that for each reasonable initial data the equation

$$\begin{cases} \partial_{tt} u = \Delta u + \mathbf{S}_u(\partial_t u, \partial_t u) - \sum_{k=1}^d \mathbf{S}_u(u_{x_k}, u_{x_k}) + f_u(Du) + g_u(Du)\,\dot{W}, \\ (u(0), \partial_t u(0)) = (u_0, v_0) \end{cases} \tag{5.1}$$

has a weak solution both in the PDE and in the Stochastic senses. By a weak solution to equation (5.1) in the PDE sense we mean a process that satisfies a variational form identity with a certain class of test functions. By a weak solution in the Stochastic Analysis sense to equation (5.1) we mean a stochastic basis, a spatially homogeneous Wiener process (defined on that stochastic basis) and a continuous adapted process z such that (5.1) is satisfied, see the formulation of Theorem 5.4 below. We recall that $\mathbf{S}$ is the second fundamental tensor/form of the isometric embedding $M \subseteq \mathbb{R}^n$.

Definition 5.1. A continuous map $\lambda : TM \to TM$ is a vector bundles homomorphism iff for every $p \in M$ the map $\lambda_p : T_pM \to T_pM$ is linear.

In our two previous papers [6, 7] we introduced and discussed two different notions of a solution, the intrinsic and the extrinsic, to following the stochastic geometric wave equation

$$\begin{aligned} \mathbf{D}_t \partial_t u = \sum_{k=1}^{d} \mathbf{D}_{x_k} \partial_{x_k} u + f(u, \partial_t u, \partial_{x_1} u, \dots, \partial_{x_d} u) \\ + g(u, \partial_t u, \partial_{x_1} u, \dots, \partial_{x_d} u)\, \dot{W}, \end{aligned} \tag{5.2}$$

In the framework of those papers we proved that these two notions are equivalent. Contrary to those papers, in the present article as well as in [10], we only deal with the extrinsic solutions, since we refer to the ambient space $\mathbb{R}^n$. Hence, since we do not introduce (neither use) an alternative notion of an intrinsic solution, we will not use the adjective "extrinsic". We will discuss these issues in a subsequent publication.

Assumption 5.2. *We assume that f_0, g_0 are continuous functions on M, $f_1, \dots, f_d$, $g_1, \dots, g_d$ are continuous vector bundles homomorphisms and f_{d+1}, g_{d+1} are continuous vector fields on M. We set, for $(\xi_i)_{i=0}^{d} \in [T_pM]^{d+1}$*

$$f(p, \xi_0, \dots, \xi_d) = f_0(p)\xi_0 + \sum_{k=1}^{d} f_k(p)\xi_k + f_{d+1}(p), \quad p \in M, \tag{5.3}$$

$$g(p, \xi_0, \dots, \xi_d) = g_0(p)\xi_0 + \sum_{k=1}^{d} g_k(p)\xi_k + g_{d+1}(p), \quad p \in M. \tag{5.4}$$

In the following, we will use the following notation.

$$\mathscr{H} = H^1(\mathbb{R}^d, \mathbb{R}^n) \oplus L^2(\mathbb{R}^d, \mathbb{R}^n), \quad \mathscr{H}_{\text{loc}}(\mathbb{R}^n) = H^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^n) \oplus L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^n),$$
$$\mathscr{H}_{\text{loc}}(M) := \{(u, v) \in \mathscr{H}_{\text{loc}}(\mathbb{R}^n) : \ u(x) \in M, \ v(x) \in T_{u(x)}M \text{ for a.e. } x \in \mathbb{R}^d\}.$$

The strong, resp. weak, topologies on $\mathscr{H}_{\text{loc}}(M)$, are by definition the traces of the strong, resp. weak topologies on $\mathscr{H}_{\text{loc}}$. In particular, a function $u : [0, \infty) \to \mathscr{H}_{\text{loc}}(M)$ is weakly continuous, iff u is weakly continuous viewed as a $\mathscr{H}_{\text{loc}}$-valued function.

Definition 5.3. Suppose that Θ is a Borel probability measure on $\mathscr{H}_{\text{loc}}(M)$. A system $\mathfrak{U} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, z)$ consisting of a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, a spatially homogeneous Wiener process W and an adapted, weakly-continuous $\mathscr{H}_{\text{loc}}(M)$-valued process $z = (u, v)$ is called a weak solution to equation (5.2) if and only if for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the following equalities hold $\mathbb{P}$-a.s., for all $t \geq 0$

$$\langle u(t), \varphi \rangle = \langle u(0), \varphi \rangle + \int_0^t \langle v(s), \varphi \rangle \, ds, \tag{5.5}$$

$$\begin{aligned}\langle v(t),\varphi\rangle = \langle v(0),\varphi\rangle &+ \int_0^t \left\langle \mathbf{S}_{u(s)}\left(v(s),v(s)\right),\varphi\right\rangle \\ &+ \int_0^t \langle f(z(s),\nabla u(s)),\varphi\rangle\, ds + \int_0^t \langle u(s),\Delta\varphi\rangle\, ds \\ &- \sum_{k=1}^d \int_0^t \left\langle \mathbf{S}_{u(s)}\left(\partial_{x_k}u(s),\partial_{x_k}u(s)\right),\varphi\right\rangle \\ &+ \int_0^t \langle g(z(s),\nabla u(s))\, dW,\varphi\rangle\,,\end{aligned} \tag{5.6}$$

where we assume that all integrals above are convergent and we use the notation (5.3)–(5.4).

We will say that the system $\mathfrak{U}$ is a weak solution to the problem (5.2) with the initial data Θ, if and only if it is a weak solution to equation (5.2) and

$$\text{the law of } z(0) \text{ is equal to } \Theta. \tag{5.7}$$

Theorem 5.4. *Assume that μ is a positive, symmetric Borel measure on $\mathbb{R}^d$ such that $\mu(\mathbb{R}^d) < \infty$. Assume that M is a compact Riemannian homogeneous space. Assume that Θ is a Borel probability measure on $\mathscr{H}_{\rm loc}(M)$ and that the coefficients f and g satisfy Assumption* 5.2. *Then there exists a weak solution to problem* (5.2) *with the initial data Θ.*

Remark 5.5. We do not claim uniqueness of a solution in Theorem 5.4, cf. Freire [26] where uniqueness of solutions is not known in the deterministic case either.

Remark 5.6. Note that the solution from Theorem 5.4 satisfies only $u(t,\omega,\cdot) \in H^1_{loc}(\mathbb{R}^d,\mathbb{R}^n)$, $t \geq 0$, $\omega \in \Omega$. Hence, for $d \geq 2$, the function $u(t,\omega,\cdot)$ need not be continuous in general.

Remark 5.7. In the above theorem we assume that f_0 and g_0 are real functions and not general vector bundles homomorphisms. We do not know whether our result is true under these more general assumptions.

Theorem 5.4 states the mere existence of a solution. The next result tells us that, among all possible solutions, there certainly exists one that satisfies the "local energy estimates".

In order to make this precise we define the following family of energy functions $\mathrm{e}_{x,T}(t,\cdot,\cdot)$, $x \in \mathbb{R}^n$, $T > 0$ and $t \in [0,T]$, by for $(u,v) \in \mathscr{H}_{\rm loc}$,

$$\mathrm{e}_{x,T}(t,u,v) = \int_{B(x,T-t)} \left\{ \frac{1}{2}|u(y)|^2 + \frac{1}{2}|\nabla u(y)|^2 + \frac{1}{2}|v(y)|^2 + \mathbf{s}^2 \right\} dy. \tag{5.8}$$

In the above the constant $\mathbf{s}^2$ is defined by

$$\mathbf{s}^2 = \max\{\|f_{d+1}\|_{L^\infty(M)}, \|f_{d+1}\|^2_{L^\infty(M)} + \|g_{d+1}\|^2_{L^\infty(M)}\}. \tag{5.9}$$

Theorem 5.8. *Assume that μ, M, Θ, f and g satisfy the assumptions of Theorem 5.4. Then there exists a weak solution $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, z, W)$ of (5.2) with initial data Θ such that*

$$\mathbb{E}\left\{\mathbf{1}_A(z(0)) \sup_{s\in[0,t]} L(\mathbf{e}_{x,T}(s,z(s)))\right\} \leq 4e^{Ct}\mathbb{E}\left\{\mathbf{1}_A(z(0))L(\mathbf{e}_{x,T}(0,z(0)))\right\} \quad (5.10)$$

holds for every $T \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $t \in [0,T]$, $A \in \mathscr{B}(\mathscr{H}_{\mathrm{loc}})$ and every nonnegative nondecreasing function $L \in C[0,\infty) \cap C^2(0,\infty)$ satisfying (for some $c \in \mathbb{R}_+$)

$$tL'(t) + \max\{0, t^2L''(t)\} \leq cL(t), \qquad t > 0. \quad (5.11)$$

The constant C in (5.10) depends only on c, c_μ and on the $L^\infty(M)$-norms of $(f_i, g_i)_{i\in\{0,\dots,d+1\}}$.

Remark 5.9. We owe some explanation about the meaning of the energy inequality (5.10). First of all please note that for $z = (u,v) \in \mathscr{H}_{\mathrm{loc}}$ we have

$$\begin{aligned}
\mathbf{e}_{x,T}(0,z) &= \mathbf{e}_{x,T}(0,u,v)\\
&= \int_{B(x,T)} \left\{\frac{1}{2}|u(y)|^2 + \frac{1}{2}|\nabla u(y)|^2 + \frac{1}{2}|v(y)|^2 + \mathrm{s}^2\right\} dy\\
&= \frac{1}{2}|u|^2_{W^{1,2}(B(x,T))} + \frac{1}{2}|v|^2_{L^2(B(x,T))} + \frac{T}{2}\mathrm{s}^2\\
&= \frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T)}} + \frac{T}{2}\mathrm{s}^2.
\end{aligned} \quad (5.12)$$

Similarly, we have for $z = (u,v) \in \mathscr{H}_{\mathrm{loc}}$,

$$\mathbf{e}_{x,T}(s,z) = \frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T-s)}} + \frac{T-s}{2}\mathrm{s}^2. \quad (5.13)$$

Hence, if a system $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, z, W)$ is a solution to the problem (5.2) and $A \in \mathscr{B}(\mathscr{H}_{\mathrm{loc}})$ then the inequality (5.10) becomes

$$\begin{aligned}
&\mathbb{E}\left\{\mathbf{1}_A(z(0)) \sup_{s\in[0,t]} L\left(\frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T-s)}} + \frac{T-s}{2}\mathrm{s}^2\right)\right\}\\
&\qquad \leq 4e^{Ct}\int_A \left[L\left(\frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T)}} + \frac{T}{2}\mathrm{s}^2\right)\right] d\Theta(z).
\end{aligned} \quad (5.14)$$

In particular, if we take a function $L : \mathbb{R}_+ \ni t \mapsto \sqrt{t} \in \mathbb{R}_+$, which satisfies the inequality

$$\begin{aligned}
&\mathbb{E}\left\{\mathbf{1}_A(z(0)) \sup_{s\in[0,t]} \left(\frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T-s)}} + \frac{T-s}{2}\mathrm{s}^2\right)^{1/2}\right\}\\
&\qquad \leq 4e^{Ct}\int_A \left[\left(\frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T)}} + \frac{T}{2}\mathrm{s}^2\right)^{1/2}\right] d\Theta(z).
\end{aligned} \quad (5.15)$$

6. Some non-rigorous digressions

The wave equation on $\mathbb{R}^3$ with values in $\mathbb{R}^d$

$$u_{tt} + (m^2 - \Delta)u + \nabla f(u) = 0 \tag{6.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}$, can be written in the "gradient" form

$$u_{tt} + \nabla_u E(u) = 0 \tag{6.2}$$

where E is the energy defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u(x)|^2 + m^2 |u(x)|^2 + 2f(u(x)) \right] dx$$

and the gradient $\nabla_u E$ of E is with respect to the Hilbert space $L^2(\mathbb{R}^3)$.

Similarly, the geometric wave equation on $\mathbb{R}^3$ taking values in M

$$\mathbf{D}_t \partial_t u - \sum_{k=1}^{d} \mathbf{D}_{x_k} \partial_{x_k} u = 0, \tag{6.3}$$

where $\mathbf{D}$ is the covariant derivative on M, can also be heuristically written in the "gradient form"

$$\mathbf{D}_t \partial_t u + \nabla_u \hat{E}(u) = 0, \tag{6.4}$$

where $\hat{E}$ is the restriction of E to M-valued functions:

$$\hat{E}(u) = E(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\hat{\nabla} u(x)|^2 \, dx, \quad u \in H^{1,2}(\mathbb{R}^d, M)$$

and $\hat{\nabla}_u \hat{E}$ is the gradient of $\hat{E}$ with respect to "Riemannian" structure on $H^{1,2}(\mathbb{R}^d, M)$ with the $L^2(\mathbb{R}^3)$ inner product.

If u is a solution to problem (6.1) (or equivalently (6.2)), then the Hamiltonian

$$E(u(t)) + \frac{1}{2} \int_{\mathbb{R}^3} |u_t(t,x)|^2 \, dx$$

is constant with respect to time.

Similarly, if u is a solution to problem (6.3) (or equivalently (6.4)), then the Hamiltonian

$$\hat{E}(u(t)) + \frac{1}{2} \int_{\mathbb{R}^3} |u_t(t,x)|^2 \, dx$$

is also constant with respect to time. An heuristic proof of this fact can be easily accomplished by using formulae (i4) on page 159, (2.2) and (2.5).

7. The fundamental equivalence lemma

The main idea of the proof of Theorem 5.4 can be seen from the following result.

Proposition 7.1. *Assume that M is a compact Riemannian homogeneous space and that the coefficients f and g satisfy Assumption* 5.2. *Suppose that a system*

$$\left(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, (\mathbf{u}, \mathbf{v}) \right) \tag{7.1}$$

is a weak solution of (5.2). *Assume that* $A : \mathbb{R}^d \to \mathbb{R}^d$ *is a skew-symmetric linear operator satisfying condition* (vi) *of Claim* 3. *Define a process* $\mathbf{M}$ *by the formula*

$$\mathbf{M}(t) := \langle \mathbf{v}(t), A\mathbf{u}(t)\rangle_{\mathbb{R}^n},\ t \geq 0. \tag{7.2}$$

Then for every function $\varphi \in H^1_{\text{comp}}$ *the following equality holds almost surely:*

$$\begin{aligned}\langle \varphi, \mathbf{M}(t)\rangle = \langle \varphi, \mathbf{M}(0)\rangle - \sum_{k=1}^{d} \left\langle \partial_{x_k}\varphi, \int_0^t \langle \partial_{x_k}\mathbf{u}(s), A\mathbf{u}(s)\rangle_{\mathbb{R}^n}\, ds \right\rangle \\ + \left\langle \varphi, \int_0^t \langle f(\mathbf{u}(s), \mathbf{v}(s), \nabla\mathbf{u}(s)), A\mathbf{u}(s)\rangle\, ds \right\rangle \\ + \left\langle \varphi, \int_0^t \langle g(\mathbf{u}(s), \mathbf{v}(s), \nabla\mathbf{u}(s)), A\mathbf{u}(s)\rangle\, dW(s) \right\rangle, \quad t \geq 0.\end{aligned} \tag{7.3}$$

Conversely, assume that a system (7.1) *satisfies all the conditions of Definition* 5.3 *of a weak solution to equation* (5.2) *but* (5.6). *Suppose that there exists a function* $F : \mathbb{R}^d \to \mathbb{R}$ *and a finite sequence* $(A^i)_{i=1}^N$ *of skew symmetric linear operators in* $\mathbb{R}^n$ *satisfying Claims* 1 *to* 4. *For each* $i \in \{1, \dots, N\}$ *define a process* $\mathbf{M}^i$ *by the formula* (7.2) *with* $A = A^i$. *Suppose that for every function* $\varphi \in H^1_{\text{comp}}$ *each* $\mathbf{M}^i$ *satisfies equality* (7.3) *with* $A = A^i$ *and that*

$$\mathbf{v}(t,\omega) := \sum_{i,j=1}^{N} h_{ij}(\mathbf{u}(t,\omega))\mathbf{M}^i(t,\omega)A^j\mathbf{u}(t,\omega), \quad \omega \in \Omega, t \geq 0. \tag{7.4}$$

Then the process (u, v) *satisfies the equality* (5.6).

Remark 7.2. Let us note that neither formula (7.4) nor (7.2) contains the gradient term $\nabla\mathbf{u}$.

Example 7. For $M = S^2 \hookrightarrow \mathbb{R}^3$, the formula (7.4) takes form

$$\mathbf{v}(t) := \sum_{i=1}^{3} \mathbf{M}^i(t)A^i\mathbf{u}(t) = \sum_{i=1}^{3} \mathbf{M}^i(t)\mathbf{u}(t) \times e_i \tag{7.5}$$

Proof. The proof of this result follows by applying our new Itô formula from Lemma 4.4 and using the material discussed in Section 3. Details are as follows. First let us note that

$$\langle \varphi, \mathbf{M}(t)\rangle = \big\langle v(t), Au(t)\big\rangle_{\varphi},\ \ t \geq 0.$$

Since $dAu = Av\, dt$ and

$$dv = \left[\mathbf{S}_u(v,v) + f(u,v,\nabla u) + \Delta u - \sum_{k=1}^{n} \mathbf{S}_u(\partial_{x_k}u, \partial_{x_k}u)\right] dt + g(u,v,\nabla u)\, dW$$

r applying the Itô Lemma 4.4 in the form from Remark 4.6 we get

$$d\big\langle v(t), Au(t)\big\rangle = \Big[\big\langle v(t), Av(t)\big\rangle + \big\langle \mathbf{S}_u\,(v,v) + f(z,\nabla u) + \Delta u - \sum_{k=1}^{n} \mathbf{S}_u\,(\partial_{x_k}u, \partial_{x_k}u)\,, Au(t)\big\rangle\Big]\,dt + \big\langle g(z,\nabla u)\,dW, Au(t)\big\rangle.$$

ecause A is skew-symmetric, $\big\langle v, Av\big\rangle = \int \langle v(x), Av(x)\rangle_{\mathbb{R}^n}\varphi(x)\,dx = 0$. Moreover, nce $\mathbf{S}_u\,(v,v)$ is normal to T_uM and by part (vi) of Claim 3, $Au \in T_uM$, we infer at $\big\langle \mathbf{S}_u\,(v,v)\,, Au\big\rangle = 0$ as well. Similarly, $\big\langle \mathbf{S}_u\,(\partial_{x_k}u, \partial_{x_k}u)\,, Au(t)\big\rangle = 0$. Finally, ith respect to the term containing Δu, we have

$$\begin{aligned}\big\langle \Delta u, Au\big\rangle = \big\langle \Delta u, Au\big\rangle_{\varphi} &= -\sum_k \int \partial_{x_k}u\partial_{x_k}(Au\varphi)\,dx \\ &= -\sum_k \int \partial_{x_k}u\partial_{x_k}(Au)\varphi\,dx - \sum_k \int \partial_{x_k}uAu\partial_{x_k}\varphi\,dx \\ &= -\sum_k \int \partial_{x_k}uAu\partial_{x_k}\varphi\,dx,\end{aligned}$$

nce $\partial_{x_k}u\partial_{x_k}(Au) = \langle \partial_{x_k}u, \partial_{x_k}(Au)\rangle_{\mathbb{R}^n} = \langle \partial_{x_k}u, A(\partial_{x_k}u)\rangle_{\mathbb{R}^n} = 0$ by the skew-mmetricity of A. Summarising, we proved that

$$\begin{aligned}d\big\langle v(t), Au(t)\big\rangle = &-\sum_k \big\langle \partial_{x_k}u(t), Au(t)\big\rangle_{\partial_{x_k}\varphi}\,dt \\ &+ \big\langle f(z,\nabla u), Au(t)\big\rangle\,dt + \big\langle g(z,\nabla u)\,dW, Au(t)\big\rangle.\end{aligned}$$

ie proof of (7.3) is thus complete.

We now present the proof of the converse part. It is based on the proof of mma 9.10 from [10]. Let us fix $\varphi \in \mathscr{D}(\mathbb{R}^d)$. We will show that, almost surely for ery $t \geq 0$,

$$\begin{aligned}\langle \mathbf{v}(t), \varphi\rangle = \langle \mathbf{v}(0), \varphi\rangle &+ \int_0^t \langle \mathbf{u}(s), \Delta\varphi\rangle\,ds + \int_0^t \big\langle \mathbf{S}_{\mathbf{u}(s)}\,(\mathbf{v}(s), \mathbf{v}(s))\,, \varphi\big\rangle\,ds \\ &- \sum_{k=1}^{d} \int_0^t \big\langle \mathbf{S}_{\mathbf{u}(s)}\,(\partial_{x_k}\mathbf{u}(s), \partial_{x_k}\mathbf{u}(s))\,, \varphi\big\rangle\,ds \\ &+ \int_0^t \langle f(\mathbf{u}(s), \mathbf{v}(s), \nabla\mathbf{u}(s)), \varphi\rangle\,ds \\ &+ \int_0^t \langle g(\mathbf{u}(s), \mathbf{v}(s), \nabla\mathbf{u}(s))\,dW, \varphi\rangle\,.\end{aligned} \tag{7.6}$$

Let us consider the functions h_{ij} from Claim 4 the vector fields Y^i, $i = \ldots, n$ defined in Claim 5. Let $\mathbf{M}^i$ be a process defined in a formula with A placed by A^i, i.e., define a process $\mathbf{M}$ by the formula

$$\mathbf{M}^i(t) := \langle v(t), A^iu(t)\rangle_{\mathbb{R}^n},\ t \geq 0. \tag{7.7}$$

Then by applying Lemma 4.4 to the processes $\mathbf{u}$ and $\mathbf{M}^i$ and the vector field Y^i we get the following equality, $\mathbb{P}$-almost surely for every $t \geq 0$,

$$\begin{aligned}
&\sum_{i=1}^{N} \left\langle \mathbf{M}^i(t) Y^i(\mathbf{u}(t)), \varphi \right\rangle \\
&\quad = \sum_{i=1}^{N} \left\langle \mathbf{M}^i(0) Y^i(\mathbf{u}(0)), \varphi \right\rangle + \sum_{i=1}^{N} \int_0^t \left\langle \mathbf{M}^i(s)(d_{\mathbf{u}(s)} Y^i)\big(\mathbf{v}(s)\big), \varphi \right\rangle ds \\
&\quad - \sum_{i=1}^{N} \sum_{l=1}^{d} \int_0^t \left\langle \langle \partial_{x_l} \mathbf{u}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \partial_{x_l} \varphi \right\rangle ds \\
&\quad - \sum_{i=1}^{N} \sum_{l=1}^{d} \int_0^t \left\langle \langle \partial_{x_l} \mathbf{u}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} (d_{\mathbf{u}(s)} Y^i)\big(\partial_{x_l} \mathbf{u}(s)\big), \varphi \right\rangle ds \\
&\quad + \sum_{i=1}^{N} \int_0^t \left\langle \langle f(\mathbf{u}(s), \mathbf{v}(s), \nabla \mathbf{u}(s)), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \varphi \right\rangle ds \\
&\quad + \sum_{i=1}^{N} \int_0^t \left\langle \langle g(\mathbf{u}(s), \mathbf{v}(s), \nabla \mathbf{u}(s)), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s))\, dW, \varphi \right\rangle.
\end{aligned} \tag{7.8}$$

Next, by identity (7.4) we have, for each $t \geq 0$ and $\omega \in \Omega$,

$$\sum_{i=1}^{N} \langle \mathbf{M}^i(t) Y^i(\mathbf{u}(t)), \varphi \rangle = \langle \mathbf{v}(t, \omega), \varphi \rangle \tag{7.9}$$

and by identity (3.8) we have, for each $s \geq 0$ and $\omega \in \Omega$,

$$\sum_{i=1}^{N} \sum_{l=1}^{d} \left\langle \langle \partial_{x_l} \mathbf{u}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \partial_{x_l} \varphi \right\rangle = \sum_{l=1}^{d} \left\langle \partial_{x_l} \mathbf{u}(s), \partial_{x_l} \varphi \right\rangle. \tag{7.10}$$

Furthermore, from Lemma 3.7 we infer that for each $s \geq 0$ and $\omega \in \Omega$,

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{l=1}^{d} \left\langle \langle \partial_{x_l} \mathbf{u}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} (d_{\mathbf{u}(s)} Y^i)\big(\partial_{x_l} \mathbf{u}(s)\big), \varphi \right\rangle \\
&\qquad = \sum_{l=1}^{d} \left\langle \mathbf{S}_{\mathbf{u}(s)} \Big(\partial_{x_l} \mathbf{u}(s), \partial_{x_l} \mathbf{u}(s)\Big), \varphi \right\rangle.
\end{aligned} \tag{7.11}$$

Similarly, by the identities (7.7) and (3.15) we infer that for a.e. $s \geq 0$, a.s.

$$\begin{aligned}
\sum_{i=1}^{N} \left\langle \mathbf{M}^i(s)(d_{\mathbf{u}(s)} Y^i)\big(\mathbf{v}(s)\big), \varphi \right\rangle &= \sum_{i=1}^{N} \left\langle \langle \mathbf{v}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} (d_{\mathbf{u}(s)} Y^i)\big(\mathbf{v}(s)\big), \varphi \right\rangle \\
= \left\langle \mathbf{S}_{\mathbf{u}(s)} \big(\mathbf{v}(s), \mathbf{v}(s)\big), \varphi \right\rangle &= \left\langle \mathbf{S}_{\mathbf{u}(s)} \big(\mathbf{v}(s), \mathbf{v}(s)\big), \varphi \right\rangle.
\end{aligned} \tag{7.12}$$

Moreover, by a similar argument based on (3.8) we can deal with the integrands of the last two terms on the RHS of (7.8). Indeed by the definition of Y^k given in Claim 5, we get

$$\sum_{i=1}^{N}\Big\langle\langle f(\mathbf{u}(s),\mathbf{v}(s),\nabla\mathbf{u}(s)),A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}Y^i(\mathbf{u}(s)),\varphi\Big\rangle=\Big\langle f(\mathbf{u}(s),\mathbf{v}(s),\nabla\mathbf{u}(s)),\varphi\Big\rangle, \tag{7.13}$$

$$\sum_{i=1}^{N}\Big\langle\langle g(\mathbf{u}(s),\mathbf{v}(s),\nabla\mathbf{u}(s)),A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}Y^i(\mathbf{u}(s)),\varphi\Big\rangle=\Big\langle g(\mathbf{u}(s),\mathbf{v}(s),\nabla\mathbf{u}(s)),\varphi\Big\rangle. \tag{7.14}$$

Summing up, we infer from the equality (7.8) and the other equalities which follow it that for every $t\ge 0$ almost surely

$$\begin{aligned}\langle\mathbf{v}(t),\varphi\rangle=\langle\mathbf{v}(0),\varphi\rangle&-\int_0^t\sum_{l=1}^{d}\Big\langle\partial_{x_l}\mathbf{u}(s),\partial_{x_l}\varphi\Big\rangle\,ds\\&-\int_0^t\sum_{l=1}^{d}\Big\langle\mathbf{S}_{\mathbf{u}(s)}\Big(\partial_{x_l}\mathbf{u}(s),\partial_{x_l}\mathbf{u}(s)\Big),\varphi\Big\rangle\,ds\\&+\int_0^t\Big\langle f(\mathbf{u}(s),\mathbf{v}(s),\nabla\mathbf{u}(s)),\varphi\Big\rangle\\&+\int_0^t\Big\langle g(\mathbf{u}(s),\mathbf{v}(s),\nabla\mathbf{u}(s))\,dW(s),\varphi\Big\rangle.\end{aligned}$$

This concludes the proof of Proposition 7.1. □

8. Brief description of the main steps of the existence proof

8.1. The first step

We begin with introducing a penalized and regularized stochastic wave equation

$$\begin{aligned}\partial_{tt}U^m&=\Delta U^m-m\nabla F(U^m)\\&\quad+f^m(U^m,\nabla_{(t,x)}U^m)+g^m(U^m,\nabla_{(t,x)}U^m)\,dW^m\end{aligned} \tag{8.1}$$

with law of $(U^m(0),\partial_tU^m(0))=\Theta$, where the C^∞-class function F has been defined in (3.6) and (f^m) and (g^m) are sequences of approximating smooth functions, such that

$$f_0^m,g_0^m:\mathbb{R}^n\to\mathbb{R},\,f_i^m,g_i^m:\mathbb{R}^n\to\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n),\,f_{d+1}^m,g_{d+1}^m\mathbb{R}^n\to\mathbb{R}^n,$$

such that for some $R_0>0$ $\bigcup_{i=0}^{d+1}\bigcup_{m\in\mathbb{N}}\left[\operatorname{supp}(f_i^m)\cup\operatorname{supp}(g_i^m)\right]\subset B(0,R_0)\subset\mathbb{R}^n$, and the L^∞ norms of f_{d+1}^m and g_{d+1}^m do not exceed the L^∞ norms of f_{d+1} and g_{d+1} respectively and $f_i^m\to f_i$ and $g_i^m\to g_i$ as $m\to\infty$ uniformly on $\mathbb{R}^n$.

Now, it follows that each approximating problem has a unique solution. Thus for every $m \in \mathbb{N}$, there exists

(i) a complete stochastic basis $(\Omega^m, \mathscr{F}^m, \mathbb{F}^m, \mathbb{P}^m)$, where $\mathbb{F}^m = (\mathscr{F}_t^m)_{t\geq 0}$;
(ii) a spatially homogeneous $\mathbb{F}^m$-Wiener process W^m with spectral measure μ;
(iii) an $\mathbb{F}^m$-adapted $\mathscr{H}_{\mathrm{loc}}$-valued weakly continuous process $Z^m = (U^m, V^m)$

such that Θ is equal to the law of $Z^m(0)$ and for every $t \geq 0$ and $\varphi \in \mathscr{D}(\mathbb{R}^d, \mathbb{R}^n)$ the following equalities hold almost surely:

$$\langle U^m(t), \varphi\rangle_{\mathbb{R}^n} = \langle U^m(0), \varphi\rangle_{\mathbb{R}^n} + \int_0^t \langle V^m(s), \varphi\rangle_{\mathbb{R}^n}\, ds \tag{8.2}$$

$$\begin{aligned}\langle V^m(t), \varphi\rangle_{\mathbb{R}^n} = \langle V^m(0), \varphi\rangle_{\mathbb{R}^n} &+ \int_0^t \langle -m\nabla F(U^m(s)) \\ &+ f^m(Z^m(s), \nabla U^m(s)), \varphi\rangle_{\mathbb{R}^n}\, ds + \int_0^t \langle U^m(s), \Delta\varphi\rangle_{\mathbb{R}^n}\, ds \\ &+ \int_0^t \langle g^m(Z^m(s), \nabla U^m(s))\, dW_s^m, \varphi\rangle_{\mathbb{R}^n}\,. \end{aligned} \tag{8.3}$$

We can assume that for each $m \in \mathbb{N}$, $Z^m(0)$ is $\mathscr{F}_0^m$-measurable $\mathscr{H}_{\mathrm{loc}}(M)$-valued random variables whose law is equal to Θ. In particular, our initial data satisfy $U_0^m(\omega) \in M$ and $V_0^m(\omega) \in T_{U_0^m(\omega)}M$ a.e. for every $\omega \in \Omega$.

In the analysis of the problem above we will use a Lyapunov type functional $\mathbf{e}_{x,T,mF}$, where $x \in \mathbb{R}^n$, $T > 0$, $m \in \mathbb{N}$, defined by, see also (5.8), by

$$\mathbf{e}_{x,T,mF}(t,u,v) = e_{x,T}(t,u,v) + m\int_{B(x,T-t)} F(u)\, dy, t \in [0,T], (u,v) \in \mathscr{H}_{\mathrm{loc}}. \tag{8.4}$$

A nondecreasing function $L \in C[0,\infty)\cap C^2(0,\infty)$ is called a *good function* iff there exists $c = c(L) > 0$ such that

$$tL'(t) + \max\{0, t^2L''(t)\} \leq cL(t), \qquad t > 0. \tag{8.5}$$

Lemma 8.1. *There exists a weak solution*

$$(\Omega^m, \mathscr{F}^m, (\mathscr{F}_t^m), \mathbb{P}^m, Z^m = (U^m, V^m), W^m)$$

to problems (8.2)–(8.3) *such that for all* $T \geq 0$, $A \in \mathscr{B}(\mathscr{H}_{\mathrm{loc}})$, $m \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}^m\left[1_A(Z^m(0)) \sup_{s\in[0,t]} L(\mathbf{e}_{x,T,mF}(s, Z^m(s)))\right] \\ &\leq 4e^{\rho t}\mathbb{E}^m\left[1_A(Z^m(0)L\left(\mathbf{e}_{x,T,mF}(0, Z^m(0))\right)\right], \ t \in [0,T], \end{aligned} \tag{8.6}$$

for every good function L. *The constant* ρ *depends on* $c(L)$, c_μ *and on* $\|f\|_{L^\infty}$, $\|g\|_{L^\infty}$.

Example. A function $L(t) = \sqrt{t}$ is a good function with $c(L) = \frac{1}{2}$. Note that $L''(t) < 0$ for $t > 0$. In this case the above energy inequality becomes

$$\mathbb{E}\left\{\mathbf{1}_A(z(0)) \sup_{s\in[0,t]} \left(\frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T-s)}} + \frac{T-s}{2}\mathrm{s}^2\right)^{1/2}\right\}$$
$$\leq 4e^{\rho t}\int_A \left[\left(\frac{1}{2}|z|^2_{\mathscr{H}_{B(x,T)}} + \frac{T}{2}\mathrm{s}^2\right)^{1/2}\right] d\Theta(z).$$

Our next ingredient is the following lemma. The spaces C_{w} and $\mathbb{L}$ will be introduced later in Subsection 9.1.

Lemma 8.2. *Assume that $r < 2$ and $r \leq \frac{d}{d-1}$. Then*

(1) *the sequence $\{U^m\}$ is tight on $C_w(\mathbb{R}_+; H^1_{\mathrm{loc}})$;*
(2) *the sequence $\{V^m\}$ is tight on $\mathbb{L} = L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$;*
(3) *and, for every $i \in \{1, \dots, N\}$, the sequence $\langle V^m, A^i U^m\rangle_{\mathbb{R}^d}$ is tight on $C_w(\mathbb{R}_+; L^r_{\mathrm{loc}})$.*

Remark 8.3. Had we been able to prove that the sequence $\{V^m\}$ is tight on $C_{\mathrm{w}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$, then part (3) of Lemma 8.2 would follow easily from part (1). However such a stronger version of part (2) is rather not true and part (3) is the most essential ingredient of the proof of the existence of a solution. The proof of this part hangs upon the special properties of the auxiliary penalisation function F listed in Claim 3.

The proof of the above lemma uses the new version of the Itô formula presented in Lemma 4.4 together with the properties of function F and operators A^i listed earlier. By applying next the Gagliardo–Nirenberg inequality and the Hölder inequality we get that for every $R > 0$, the equality

$$\begin{aligned}\langle V^m(t), A^i U^m(t)\rangle_{\mathbb{R}^n} &= \langle V^m(0), A^i U^m(0)\rangle_{\mathbb{R}^n} \\ &\quad + \sum_{k=1}^d \partial_{x_k}\left[\int_0^t \langle \partial_{x_k} U^m(s), A^i U^m(s)\rangle_{\mathbb{R}^n}\, ds\right] \\ &\quad + \int_0^t \langle f^m(Z^m(s), \nabla U^m(s)), A^i U^m(s)\rangle_{\mathbb{R}^n}\, ds \\ &\quad + \int_0^t \langle g^m(Z^m(s), \nabla U^m(s)), A^i U^m(s)\rangle_{\mathbb{R}^n}\, dW^m_s\end{aligned} \tag{8.7}$$

holds in $\mathbb{W}^{-1,r}_R$ for every $t \geq 0$, almost surely.

Let us consider, as before, the approximating sequence of processes $\left(Z^m\right)_{m\in\mathbb{N}}$, where $Z^m = (U^m, V^m)$ and the following representation of Wiener processes W^m:

$$W^m_t = \sum_i \beta^m_i(t) e_i, \qquad t \geq 0, \tag{8.8}$$

where $\beta = (\beta^1, \beta^2, \dots)$ are independent real standard Wiener processes and $\{e_i : i \in \mathbb{N}\}$ is an orthonormal basis in H_μ.

Fix r as before. Then Lemma 8.2 together with some other results which we state later on (Corollary 9.5, Proposition 9.9 and Corollary 9.2) implies that there exists

- a probability space $(\Omega, \mathscr{F}, \mathbb{P})$,
- a subsequence m_k,
- the following sequences of Borel measurable functions

$(u^k)_{k\in\mathbb{N}}$	with values in $C(\mathbb{R}_+, H^1_{\text{loc}})$	(8.9)
$(v^k)_{k\in\mathbb{N}}$	with values in $C(\mathbb{R}_+, L^2_{\text{loc}})$	
$(w^k)_{k\in\mathbb{N}}$	with values in $C(\mathbb{R}_+, \mathbb{R}^{\mathrm{N}})$	

- the following Borel random variables

v_0	with values in L^2_{loc}	(8.10)
u	with values in $C_{\text{w}}(\mathbb{R}_+; H^1_{\text{loc}})$	
$\bar{v}$	with values in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}})$	
w	with values in $C(\mathbb{R}_+, \mathbb{R}^{\mathrm{N}})$	
M^i, $i = 1, \dots, N$	with values in $C_{\text{w}}(\mathbb{R}_+; L^r_{\text{loc}})$	

such that, with the notation $z^k = (u^k, v^k)$, $k \in \mathbb{N}$ and

$$M^i_k := \langle v^k, A^i u^k \rangle_{\mathbb{R}^n}, \; i = 1, \dots, N, \; k \in \mathbb{N}, \tag{8.11}$$

the following conditions are satisfied.

(R1) $\forall k \in \mathbb{N}$,

$$\text{Law}\left((Z^{m_k}, \beta^{m_k})\right) = \text{Law}\left((z^k, w^k)\right) \text{ on } \mathscr{B}(C(\mathbb{R}_+, \mathscr{H}_{\text{loc}}) \times C(\mathbb{R}_+, \mathbb{R}^{\mathrm{N}}));$$

(R2) pointwise on Ω the following convergences hold:

$$\begin{aligned} u^k &\to u \quad \text{in} \quad C_{\text{w}}(\mathbb{R}_+; H^1_{\text{loc}}) \\ v^k &\to \bar{v} \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}}) \\ v^k(0) &\to v_0 \quad \text{in} \quad L^2_{\text{loc}} \\ M^i_k &\to M^i \quad \text{in} \quad C_{\text{w}}(\mathbb{R}_+; L^r_{\text{loc}}) \\ w^k &\to w \quad \text{in} \quad C(\mathbb{R}_+, \mathbb{R}^{\mathrm{N}}); \end{aligned} \tag{8.12}$$

(R3) the law of $(u(0), v_0)$ is equal to Θ.

In particular, the conclusions of Lemma 8.1 hold for this new system of processes.

Proposition 8.4. *If L is a good function and ρ is the constant from Lemma 8.1, then inequality (8.6) holds, i.e., for every $k \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathbb{R}^d$, $A \in \mathscr{B}(\mathscr{H}_{\text{loc}})$.*

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}_A(z^k(0)) \sup_{s\in[0,t]} L(\mathrm{e}_{x,T,m_k}(s, z^k(s)))\right] \\ &\le 4e^{\rho t}\mathbb{E}\left[\mathbf{1}_A(z^k(0)) L\left(\mathrm{e}_{x,T,m_k}(0, z^k(0))\right)\right]. \end{aligned} \tag{8.13}$$

Before we continue, let us observe that the compactness of the embedding $H^1_{\text{loc}} \hookrightarrow L^2_{\text{loc}}$ and properties (8.10) and (8.12) imply the following auxiliary result.

Proposition 8.5. *In the above framework, all trajectories of the process u belong to $C(\mathbb{R}_+, L^2_{\text{loc}})$ and for every $t \in \mathbb{R}_+$, $u^k(t) \to u(t)$ in L^2_{loc}.*

We also introduce the following filtration $\mathbb{F} = \left(\mathscr{F}_t\right)_{t\geq 0}$ of σ-algebras on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$:

$$\mathscr{F}_t = \sigma\{\sigma\{v_0, u(s), w(s) : s \in [0,t]\} \cup \{N : \mathbb{P}(N) = 0\}\}, \quad t \geq 0.$$

Our first result states, roughly speaking, that the limiting process u takes values in the set M. To be precise, we have the following.

Proposition 8.6. *There exists a set $Q_u \in \mathscr{F}$ such that $\mathbb{P}(Q_u) = 1$ and, for every $\omega \in Q_u$ and $t \geq 0$, $u(t,\omega) \in M$ almost everywhere on $\mathbb{R}^d$.*

Beginning of the proof. Let us fix $T > 0$ and $\delta > 0$. In view of the definition (8.4) of the function $\mathbf{e}_{0,T,m}$, the inequality (8.13) yields that for some finite constant $C_{T,\delta}$,

$$m_k \ \mathbb{E}\left[1_{B_\delta^{\mathscr{H}_T}}(z_0^k) \int_{B_{T-t}} F(u^k(t))\,dx\right] \leq C_{T,\delta}, \qquad t \in [0,T]. \tag{8.14}$$

Since $m_k \nearrow \infty$ the result follows. □

The last result suggests the following definition.

Definition 8.7. Set

$$\mathbf{u}(t,\omega) = \begin{cases} u(t,\omega), & \text{for } t \geq 0 \text{ and } \omega \in Q_u, \\ \mathbf{p}, & \text{for } t \geq 0 \text{ and } \omega \in \Omega \setminus Q_u, \end{cases} \tag{8.15}$$

where $\mathbf{p}(x) = p$, $x \in \mathbb{R}^d$ for some fixed (but otherwise arbitrary) point $p \in M$.

Let $\bar{v}$ be the $\mathbb{L}$-valued random variable as in (8.10) and (8.12). Then we proved that there exits a measurable L^2_{loc}-valued process v such that for every $\omega \in \Omega$, the function $v(\cdot,\omega)$ is a representative of $\bar{v}(\omega)$.

In the next result we show that the process $(\mathbf{u}, \mathbf{V})$ takes values in the tangent bundle TM.

Lemma 8.8. *There exists an $\mathbb{F}$-progressively measurable L^2_{loc}-valued process $\mathbf{V}$ such that* Leb $\otimes\,\mathbb{P}$*-a.e., $\mathbf{V} = v$ and, $\mathbb{P}$-almost surely,*

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \mathbf{V}(s)\,ds, \quad \text{in } L^2_{\text{loc}}, \quad \text{for all } t \geq 0.$$

Moreover $\mathbf{V}(t,\omega) \in T_{\mathbf{u}(t,\omega)}M$, Leb*-a.e. for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$. Finally, there exists an $\mathscr{F}_0$-measurable L^2_{loc}-valued random variable $\mathbf{v}_0$ such that*

$$\mathbf{v}_0 = v_0, \quad \mathbb{P} \text{ almost surely}$$

and, for every $\omega \in \Omega$,

$$\mathbf{v}_0(\omega) \in T_{\mathbf{u}(0,\omega)}M, \text{ Leb-a.e.}.$$

In order to prove the existence of a solution we will exploit the assumption that M is a compact homogenous space. The processes we have introduced so far could be identified in the following way. Before we formulate the next result let us define a process $\mathbf{v}$ by the following result.

Lemma 8.9. *There exists a $\mathbb{P}$-conegligible set $Q \in \mathscr{F}$ such that if the process $\mathbf{M}^i$ is defined by $\mathbf{M}^i = 1_Q M^i$, $i \in \{1, \ldots, N\}$, then the following properties are satisfied.*

(i) *For every $i \in \{1, \ldots, N\}$ there is an L^2_{loc}-valued $\mathbb{F}$-adapted and weakly continuous.*

(ii) *The following three identities hold for every $\omega \in Q$,*

$$\begin{aligned} \mathbf{M}^i(t,\omega) &= \left\langle \mathbf{V}(t,\omega), A^i \mathbf{u}(t,\omega) \right\rangle_{\mathbb{R}^n}, \qquad \text{for a.e. } t \geq 0, \\ \mathbf{v}_0(\omega) &= \sum_{i,j=1}^{N} h_{ij}(\mathbf{u}(0,\omega)) \mathbf{M}^i(0,\omega) A^j \mathbf{u}(0,\omega), \\ \mathbf{V}(t,\omega) &= \mathbf{v}(t,\omega), \quad \text{for a.e. } t \geq 0, \\ \mathbf{v}(t,\omega) &\in T_{\mathbf{u}(t,\omega)M}, \qquad t \geq 0, \end{aligned}$$

where

$$\mathbf{v}(t,\omega) := \sum_{i,j=1}^{N} h_{ij}(\mathbf{u}(t,\omega)) \mathbf{M}^i(t,\omega) A^j \mathbf{u}(t,\omega), \quad \omega \in \Omega, t \geq 0. \tag{8.16}$$

Moreover, with $\mathbf{z} = (\mathbf{u}, \mathbf{v})$, for every $\omega \in Q$, for almost every $t \geq 0$,

$$\begin{aligned} \lim_{k\to\infty} \left\langle f^{m_k}(z^k(t,\omega)), A^i u^k(t,\omega) \right\rangle_{\mathbb{R}^n} &= \left\langle f(\mathbf{z}(t,\omega), \nabla \mathbf{u}(t,\omega)), A^i \mathbf{u}(t,\omega) \right\rangle_{\mathbb{R}^n}, \\ \lim_{k\to\infty} \left\langle g^{m_k}(z^k(t,\omega)), A^i u^k(t,\omega) \right\rangle_{\mathbb{R}^n} &= \left\langle g(\mathbf{z}(t,\omega), \nabla \mathbf{u}(t,\omega)), A^i \mathbf{u}(t,\omega) \right\rangle_{\mathbb{R}^n}, \end{aligned}$$

where the limits are with respect to the weak topology on L^2_{loc}.

8.2. Construction of the Wiener process

The second crucial step is the following result. Its proof bears upon the identity (8.7) derived earlier in the proof of the tightness of the auxiliary processes $M^i_k := \langle V^m, A^i U^m \rangle$, see part (3) of Lemma 8.2.

Proposition 8.10. *The processes $(w_l)_{l=1}^{\infty}$ are i.i.d. real $\mathbb{F}$-Wiener processes. Moreover, if $(e_l)_{l=1}^{\infty}$ is an ONB of the RKHS H_μ then the process*

$$W\psi = \sum_{l=1}^{\infty} w_l e_l(\psi), \quad \psi \in \mathscr{S}(\mathbb{R}^d) \tag{8.17}$$

is a spatially homogeneous $\mathbb{F}$*-Wiener process with spectral measure* μ*, and for every function* $\varphi \in H^1_{\mathrm{comp}}$ *the following equality holds almost surely:*

$$\begin{aligned}\langle \varphi, \mathbf{M}^i(t)\rangle &= \langle \varphi, \mathbf{M}^i(0)\rangle - \sum_{k=1}^d \left\langle \partial_{x_k}\varphi, \int_0^t \langle \partial_{x_k}\mathbf{u}(s), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}\, ds \right\rangle \\ &\quad + \left\langle \varphi, \int_0^t \langle f(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), A^i\mathbf{u}(s)\rangle\, ds \right\rangle \qquad (8.18) \\ &\quad + \left\langle \varphi, \int_0^t \langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), A^i\mathbf{u}(s)\rangle\, dW(s) \right\rangle, \qquad t \geq 0.\end{aligned}$$

The third crucial step is to apply Proposition 7.1.

8.3. Conclusion of the proof of Theorem 5.4

Lemma 8.11. *The* L^2_{loc}*-valued process* $\mathbf{v}$ *introduced in* (8.16) *is* $\mathbb{F}$*-adapted and weakly continuous. Moreover,* $\mathbf{v}(t) \in T_{\mathbf{u}(t)}M$ *for every* $t \geq 0$ *almost surely and for every* $\varphi \in \mathscr{D}(\mathbb{R}^d)$*, equality* (7.6) *holds almost surely for every* $t \geq 0$.

Proof. Obviously the process $\mathbf{v}$ is L^2_{loc}-valued. The $\mathbb{F}$-adaptiveness and the weak continuity of $\mathbf{v}$ follows from its definition (i.e., (8.16)) and Lemma 8.9. The remaining parts follow from Proposition 7.1.

This concludes the proof of Lemma 8.11. □

To conclude the proof of the existence of a solution, i.e., the proof of Theorem 5.4 let us observe that the above equality is nothing else but (5.6). Moreover, (5.5) follows from (8.8) and (8.16). This proves that if the process $\mathbf{z} := (\mathbf{u}, \mathbf{v})$ then $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, z)$ a weak solution to equation (5.1).

Thus the description of the proof of the main result of this review is finished.

9. Some topological considerations

9.1. The Jakubowski's version of the Skorokhod representation theorem

Theorem 9.1. *Let* X *be a topological space such that there exists a sequence* $\{f_m\}$ *of continuous functions* $f_m : X \to \mathbb{R}$ *that separate points of* X*. Let us denote by* $\mathscr{S}$ *the* σ*-algebra generated by the maps* $\{f_m\}$*. Then*

(j1) *every compact subset of* X *is metrizable,*

(j2) *every Borel subset of a* σ*-compact set in* X *belongs to* $\mathscr{S}$*,*

(j3) *every probability measure supported by a* σ*-compact set in* X *has a unique Radon extension to the Borel* σ*-algebra on* X*,*

(j4) *if* (μ_m) *is a tight sequence of probability measures on* $(X, \mathscr{S})$*, then there exists a subsequence* (m_k)*, a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *with* X*-valued Borel measurable random variables* X_k*,* X *such that* μ_{m_k} *is the law of* X_k *and* X_k *converge almost surely to* X*. Moreover, the law of* X *is a Radon measure.*

Proof. See Jakubowski's paper [34]. □

Corollary 9.2. *Under the assumptions of Theorem* 9.1, *if* Z *is a Polish space and* $b : Z \to X$ *is a continuous injection, then* $b[B]$ *is a Borel set whenever* B *is Borel in* Z.

See Corollary A.2 in [50]. Since the map $F = (f_1, f_2, \dots) : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous injection, $F \circ b : Z \to \mathbb{R}^{\mathbb{N}}$ is also a continuous injection. Let us take a Borel set $B \subset Z$. Since both Z and $\mathbb{R}^{\mathbb{N}}$ are Polish spaces, we infer that $(F \circ b)[B]$ is a Borel set. Therefore $b[B] = F^{-1}[(F \circ b)[B]] \subset X$ is Borel set too. □

9.2. The space $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}})$

Let $\mathbb{L} = L^\infty_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}})$ be the space of equivalence classes $[f]$ of all measurable functions $f : \mathbb{R}_+ \to L^2_{\text{loc}} = L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ such that $\|f\|_{L^2(B_n)} \in L^\infty(0, n)$ for every $n \in \mathbb{N}$. The space $\mathbb{L}$ is equipped with the locally convex topology generated by functionals

$$f \mapsto \int_0^n \int_{B_n} \langle g(t,x), f(t,x) \rangle_{\mathbb{R}^n} \, dx \, dt, \tag{9.1}$$

where $n \in \mathbb{N}$ and $g \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^d))$.

Let us also define a space

$$Y_m = L^1((0,m), L^2(B_m)), \tag{9.2}$$

Let us recall that $L^\infty((0,m), L^2(B_m)) = Y_m^*$. Consider the following natural restriction maps:

$$\pi_m \; : \; L^2(\mathbb{R}^d) \ni g \mapsto g|_{B_m} \in L^2(B_m), \tag{9.3}$$

$$l_m \; : \; \mathbb{L} \ni f \mapsto (\pi_m \circ f)|_{[0,m]} \in (Y_m^*, w^*). \tag{9.4}$$

The following results describe some properties of the space $\mathbb{L}$.

Lemma 9.3. *A map* $l = (l_m(f))_{m \in \mathbb{N}} : \mathbb{L} \to \prod_{m \in \mathbb{N}} (Y_m^*, w^*)$ *is a homeomorphism onto a closed subset of* $\prod_{m \in \mathbb{N}} (Y_m^*, w^*)$.

Proof. The proof is straightforward. □

Corollary 9.4. *Given any sequence* $(a_m)_{m=1}^\infty$ *of positive numbers, the set*

$$\{f \in \mathbb{L} : \|f\|_{L^\infty((0,m), L^2(B_m))} \le a_m, \; m \in \mathbb{N}\} \tag{9.5}$$

is compact in $\mathbb{L}$.

Proof. The proof follows immediately from Lemma 9.3 and the Banach–Alaoglu theorem since a product of compacts is a compact by the Tychonov theorem. □

Corollary 9.5. *The Skorokhod representation Theorem* 9.1 *holds for every tight sequence of probability measures defined on* $(\mathbb{L}, \sigma(\mathbb{L}^*))$, *where the* σ*-algebra* $\sigma(\mathbb{L}^*)$ *is the* σ*-algebra on* $\mathbb{L}$ *generated by* $\mathbb{L}^*$.

Proof. Since each Y_m is a separable Banach space, there exists a sequence $(j_{m,k})_{k=1}^{\infty}$, such that each $j_{m,k} : (Y_m^*, w^*) \to \mathbb{R}$ is a continuous function and $(j_{m,k})_{k=1}^{\infty}$ separate points of Y_m^*. Consequently, such a separating sequence of continuous functions exists for product space $\prod(Y_m^*, w^*)$, and, by Lemma 9.3, for the $\mathbb{L}$ as well. Existence of a separating sequence of continuous functions is sufficient for the Skorokhod representation theorem to hold by the Jakubowski theorem [34]. □

Proposition 9.6. *Let $\bar{\xi}$ be an $\mathbb{L}$-valued random variable. Then there exists a measurable L^2_{loc}-valued process ξ such that for every $\omega \in \Omega$,*

$$[\xi(\cdot,\omega)] = \bar{\xi}(\omega). \tag{9.6}$$

Proof. Let $\left(\varphi_n\right)_{n=1}^{\infty}$ be an approximation of identity on $\mathbb{R}$. Let us fix $t \geq 0$ and $n \in \mathbb{N}^*$. Then the linear operator

$$I_n(t) : \mathbb{L} \ni f \mapsto \int_0^{\infty} \varphi_n(t-s) f(s)\, ds \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \tag{9.7}$$

is well defined and for all $\psi \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^* = L^2_{\mathrm{comp}}(\mathbb{R}^d)$ and $t \geq 0$, the function $\psi \circ I_n(t) : \mathbb{L} \to \mathbb{R}$ is continuous. Hence in view of Corollary E.1 from [10] the map $I_n(t)$ is Borel measurable. We put

$$I : \mathbb{L} \ni f \mapsto \begin{cases} \lim_{n\to\infty} I_n(t)(t), & \text{provided the limit in } L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{9.8}$$

Then (by employing the Lusin Theorem [57] in case (ii)) we infer that given $f \in \mathbb{L}$

(i) the map $\mathbb{R}_+ \ni t \mapsto I_n(t) f \in L^2_{\mathrm{loc}}$ is continuous, and

(ii) $\lim_{n\to\infty} I_n(t) f$ exists in L^2_{loc} for almost every $t \in \mathbb{R}_+$ and $[I(\cdot) f] = f$.

If we next define L^2_{loc}-valued stochastic processes ξ_n, for $n \in \mathbb{N}^*$, and ξ by $\xi_n(t,\omega) = I_n(t)\big(\bar{\xi}(\omega)\big)$ and $\xi(t,\omega) = I(t)\big(\bar{\xi}(\omega)\big)$ for $(t,\omega) \in \mathbb{R}_+ \times \Omega$, then by (i) above we infer that ξ_n is continuous and so measurable. Hence the process ξ is also measurable and by (ii) above, given $\omega \in \Omega$, the function $\{\mathbb{R}_+ \ni t \mapsto \xi(t,\omega)\}$ is a representative of $\bar{\xi}(\omega)$. The proof is complete. □

9.3. The space $C_{\mathrm{w}}(\mathbb{R}_+; X)$ and a generalised Arzelà–Ascoli Theorem

If X is a locally convex space then by $C_{\mathrm{w}}(\mathbb{R}_+; X)$ we denote the space of all weakly continuous functions $f : \mathbb{R}_+ \to X$ endowed with the locally convex topology generated by the a family $\|\cdot\|_{m,\varphi}$, $m \in \mathbb{N}$, $\varphi \in X^*$, of pseudonorms defined by

$$\|f\|_{m,\varphi} = \sup_{t\in[0,m]} |\langle \varphi, f(t)\rangle|. \tag{9.9}$$

For $l \geq 0$, $R > 0$ and $p, p^* \in (1,\infty)$ satisfying $\frac{1}{p^*} + \frac{1}{p} = 1$, let $W^{l,p}(B_R) = W^{l,p}(B_R; \mathbb{R}^n)$ be the standard Sobolev space over the ball B_R. Let us recall that by $(W^{k,p}(B_R), \mathrm{w})$ we mean the space $W^{k,p}(B_R)$ endowed with the weak topology and that $W^{k,p}_{\mathrm{loc}} = W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$ is the space of all elements $u \in L^p_{\mathrm{loc}}$ whose weak derivatives up to order k belong to L^p_{loc}. The latter space is a metrizable topological vector space equipped with a natural countable family of seminorms $(p_j)_{j\in\mathbb{N}}$ defined by $p_j(u) := \|u\|_{W^{k,p}(B_j)}$, $u \in W^{k,p}_{\rm loc}$. The dual of $W^{k,p}_{\rm loc}$ can be identified with $W^{-k,p^*}_{\rm comp}$, i.e., the space of compactly supported distributions from W^{-k,p^*}. We now formulate the first of the two main results in this subsection. Their proofs are based on the second author's paper [50, Corollary B.2 and Proposition B.3].

Lemma 9.7. *The maps J and L defined by*

$$J : (W^{k,p}_{\rm loc}, w) \ni f \mapsto \big(f|_{B_m}\big)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} (W^{k,p}(B_m), w),$$

$$L : C_w(\mathbb{R}_+; W^{k,p}_{\rm loc}) \ni h \mapsto \big((h|_{B_m})|_{[0,m]}\big)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} C_w([0,m], W^{k,p}(B_m))$$

are both homeomorphisms onto closed sets.

Proof. Straightforward and hence omitted. □

Corollary 9.8. *Assume that $\gamma \in (0,1]$, $1 < r, p < \infty$, $-\infty < l \le k$ satisfy*

$$l - \frac{r}{d} \le k - \frac{d}{p}. \tag{9.10}$$

Then for any sequence $a = (a_m)_{m=1}^{\infty}$ of positive numbers the set

$$K(a) := \big\{f \in C_w(\mathbb{R}_+; W^{k,p}_{\rm loc}) : \|f\|_{L^\infty([0,m],W^{k,p}(B_m))} + \|f\|_{C^\gamma([0,m],W^{l,r}(B_m))} \le a_m,\ m \in \mathbb{N}\big\}$$

is a metrizable compact subset of $C_w(\mathbb{R}_+; W^{k,p}_{\rm loc})$.

Proposition 9.9. *The Skorokhod representation Theorem* 9.1 *holds for every tight sequence of probability measures defined on the σ-algebra generated by the family of maps*

$$\{C_w(\mathbb{R}_+; W^{k,p}_{\rm loc}) \ni f \mapsto \langle \varphi, f(t)\rangle \in \mathbb{R}\} : \ \varphi \in \mathscr{D}(\mathbb{R}^d, \mathbb{R}^n),\ t \in [0,\infty).$$

Proof of Corollary 9.8. Let us define a set A_m, $m \in \mathbb{N}$, by

$$A_m = \big\{h \in C_{\rm w}([0,m], W^{k,p}(B_m)) : \|h\|_{L^\infty([0,m],W^{k,p}(B_m))} + \|h\|_{C^\gamma([0,m],W^{l,r}(B_m))} \le a_m\big\}.$$

Then $K(a) = L^{-1}\big(\prod_m A_m\big)$. It is enough to show that each A_m is a metrizable compact in $C_{\rm w}([0,m], W^{k,p}(B_m))$. Indeed, if this is the case then $A := \prod_m A_m$ is a metrizable compact and hence, since by Lemma 9.7 the range $\mathrm{R}(L)$ of L is closed, $A \cap \mathrm{R}(L)$ is a metrizable compact. Therefore, since by Lemma 9.7 the map $L^{-1} : \mathrm{R}(L) \to C_{\rm w}(\mathbb{R}_+; W^{k,p}_{\rm loc})$ is continuous, $K(a) = L^{-1}[A \cap \mathrm{R}(L)]$ is a metrizable compact. To this end let us fix $m \in \mathbb{N}$ and let $\{\varphi_j\}$ be a dense subset of $(W^{k,p}(B_m))^*$. Denote by τ the locally convex topology on $C_{\rm w}([0,m], W^{k,p}(B_m))$ generated by the semi-norms $f \mapsto \sup_{t\in[0,m]} |\langle \varphi_j, f(t)\rangle|$. It is easy to see that

τ coincides with the original topology of $C_{\mathrm{w}}([0,m],W^{k,p}(B_m))$ on the set $\tilde{A}_m$ defined by

$$\tilde{A}_m = \{h \in C_{\mathrm{w}}([0,m],W^{k,p}(B_m)) : \|h\|_{L^\infty([0,m],W^{k,p}(B_m))} \leq a_m\}.$$

Hence the set A_m is metrizable. The compactness of A_m follows from the classical Arzelà–Ascoli Theorem in the form given in [35, Theorem 7.17, p. 233]. Let us denote by F_1, resp. F_2 the closed ball of radius a_m, resp. $(K_m \vee 1)a_m$, where K_m will be defined below, in $W^{k,p}(B_m)$, resp. $W^{l,r}(B_m)$, endowed with the weak topology. Since the spaces $W^{k,p}(B_m)$ and $W^{l,r}(B_m)$ are reflexive and separable, by the Banach–Alaoglu Theorem, see [56, Theorems 3.15 and 3.16], both F_1 and F_2 are compact. Moreover, since in view of the assumption (9.10) by the celebrated Gagliardo–Nirenberg inequalities, see, e.g., [27], $W^{k,p}(B_m) \subseteq W^{l,r}(B_m)$ continuously, the natural embedding $i : F_1 \hookrightarrow F_2$ is continuous. Let us denote by K_m the norm of the embedding $W^{k,p}(B_m) \subseteq W^{l,r}(B_m)$. Since F_1 is compact, $i(F_1)$ is compact as well and the function $i : F_1 \to i(F_1)$ is a homeomorphism. Hence, in order to prove equicontinuity of the set A_m in $C([0,m];F_1)$, it is enough to prove equicontinuity of the set A_m in $C([0,m];F_2)$. However this easily follows from the second part of the definition of the set A_m. Since for each $t \in [0,m]$, the set $\{h(t) : h \in A_m\}$ is a subset of F_1 and hence relatively compact, the claimed result follows. □

Proof of Corollary 9.9. By the Jakubowski theorem [34], it is sufficient to verify that there exists a sequence $j_k : C_{\mathrm{w}}(\mathbb{R}_+;W^{k,p}_{\mathrm{loc}}) \to \mathbb{R}$ of continuous functions that separate points of $C_{\mathrm{w}}(\mathbb{R}_+;W^{k,p}_{\mathrm{loc}})$. For, let φ_k be a countable sequence in $(W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d))^*$ separating points of $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$. Then $j_{k,q}(f) = \varphi_k(f(q))$, $k \in \mathbb{N}$, $q \in \mathbb{Q}_+$ do the job. □

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Reflections on Reflections

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Abstract. Reflection of a path is a perturbation that is sufficiently powerful to substantially change many properties of a stochastic process and yet sufficiently structured to be amenable to rigorous analysis. There seems to be no well-defined theory of reflected processes in the same sense as there is no well-defined theory of Brownian motion. Instead, the basic idea has a number of incarnations and generates many interesting questions. These notes contain a review of some directions of research concerned with reflected paths.

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1. Introduction

Reflection of a path is a perturbation that is sufficiently powerful to substantially change many properties of a stochastic process and yet sufficiently structured to be amenable to rigorous analysis. There seems to be no well-defined theory of reflected processes in the same sense as there is no well-defined theory of Brownian motion. Instead, the basic idea has a number of incarnations and generates many interesting questions. These notes contain a review of some directions of research concerned with reflected paths.

The notes are not meant to be an elementary introduction to the theory of reflected processes. They do not contain basic standard results on, for example, existence and uniqueness of solutions to the Skorokhod equation defining reflected Brownian motion in sufficiently smooth domains. Instead, the notes review four diverse topics concerned with reflected paths. The author hopes that the reader will be inspired by at least some of these research topics.

The first topic, presented in Section 2, is foundational in nature. This section is concerned with some questions related to and inspired by the definition and

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