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On Todorcevic orderings

by

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Abstract. The Todorcevic ordering $\mathbb{T}(X)$ consists of all finite families of convergent sequences in a given topological space X. Such an ordering was defined for the special case of the real line by S. Todorcevic (1991) as an example of a Borel ordering satisfying ccc that is not σ-finite cc and even need not have the Knaster property. We are interested in properties of $\mathbb{T}(X)$ where the space X is taken as a parameter. Conditions on X are given which ensure the countable chain condition and its stronger versions for $\mathbb{T}(X)$. We study the properties of $\mathbb{T}(X)$ as a forcing notion and the homogeneity of the generated complete Boolean algebra.

1. Introduction. In this paper, we will describe a general method of constructing an ordering from a topological space. When we look at orderings from the point of view of the forcing method, the orderings satisfying the countable chain condition (ccc) are of special interest. Our method will yield mostly such orderings. The ordering is obtained in the following way. For any topological space we can consider the set of finite unions of converging sequences such that limit points are outside of this union. This set is ordered by reverse inclusion. We obtain an ordering which was considered first by Todorcevic [Tod91] for the special case of the real numbers. Some ideas of the construction can already be found in the Galvin–Hajnal example (see [CN82]).

For technical reasons, we slightly modify Todorcevic's original definition as follows.

Definition 1.1. For a topological space X, the *Todorcevic ordering* $\mathbb{T}(X)$ is the set of all partial functions p from X to $\{0,1\}$ such that the domain of p is the union of a finite set and a finite union of sets which are each a convergent sequence including the limit point; in the latter case p assigns to each point of the sequence value 0 and to the limit point value 1;

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formally (throughout this paper we follow the standard set-theoretic notation, see e.g. [JechST]),

$$\mathbb{T}(X) = \big\{p : X \to \{0,1\} : \mathrm{dom}(p) = X_0 \,\dot\cup\, X_1 \,\dot\cup \cdots \dot\cup\, X_n, \text{ where}$$

(i) $|X_0| < \omega$ (with $p{\restriction}X_0$ arbitrary) and
(ii) for each $1 \le i \le n$, X_i is a convergent sequence including its limit point x_i with $p(x_i) = 1$ and $p(x) = 0$ for all $x \in X_i \setminus \{x_i\}\big\}$.

The set is ordered by inverse extension, i.e. $p \le q$ if $p \supseteq q$.
The set $\mathrm{supp}(p) = p^{-1}(1)$ is called the *support* of p.

The topological space is a parameter for the ordering $\mathbb{T}(X)$.

It is clear from the definition that the domain $\mathrm{dom}(p)$ of each element p of $\mathbb{T}(X)$ is at most countable and compact, and the support of p is finite. The fact that a set $A = \{x_i\}_{i<\omega}$ is converging to the point x is denoted by $A \to x$ or $x_i \to x$. Since our ordering was defined in terms of convergent sequences, we are interested only in *sequential spaces*, i.e. spaces with the topology determined by its countable converging sequences. Any topological space has a sequential modification, that is, the given space with the topology generated by all its convergent sequences. For a topological space and its sequential modification we obtain the same Todorcevic ordering. So we make the general assumption that *all topological spaces in this paper will be sequential spaces with the unique limit property*, i.e. (1) a set $Z \subseteq X$ is closed if and only if $x_i \in Z$ and $x_i \to x$ implies $x \in Z$, and (2) $x_i \to x$ and $x_i \to y$ implies $x = y$. Such spaces are necessarily T_1.

We here make the following remark. Given a topological space X, we consider the set of all countable and compact subsets with only finitely many accumulation points and order this set by extensions which do not change isolated points into accumulation points. In the case of a Hausdorff space without isolated points this ordered set can be densely embedded into the order $\mathbb{T}(X)$. Note also that in this case this is the separative quotient of Todorcevic's original definition in [Tod91].

Concerning the ordered set $\mathbb{T}(X)$, we use the standard terminology. By $p \perp r$ we denote the fact that p and r are *orthogonal*, i.e. there is no $q \in \mathbb{T}(X)$ such that $q \le p, r$. Conversely, $p \parallel r$ means that p and r are not orthogonal. A set of pairwise orthogonal elements is called an *antichain*. An ordered set is said to satisfy the *κ-cc* if it contains no antichain of size κ. The ω_1-chain condition is also called the *countable chain condition* (ccc). By $\mathbb{T}(X){\restriction}p = \{q \in \mathbb{T}(X) : q \le p\}$ we denote the restriction of $\mathbb{T}(X)$ to p.

We observe that the ordering $\mathbb{T}(X)$ is *separative*, i.e. for any $p, q \in \mathbb{T}(X)$ with $p < q$ there is an $r < q$ such that $r \perp p$. Any separative ordering can be densely embedded in a complete Boolean algebra which is uniquely determined up to isomorphism. Let $\mathbb{B}(X)$ be this complete Boolean algebra obtained from $\mathbb{T}(X)$.

Our set-theoretic terminology is standard. By $A \subseteq^* B$ ($A =^* B$ resp.) we denote the fact that $|A \setminus B| < \omega$ ($|A \triangle B| < \omega$ resp.)

An embedding of an ordering $\mathbb{Q}$ into an ordering $\mathbb{P}$ is called *regular* if maximal antichains are mapped to maximal antichains. A criterion for a mapping ρ from a separative order $\mathbb{Q}$ to $\mathbb{P}$ to be a regular embedding is the following:

(1) $q_1 \leq q_2$ implies $\rho(q_1) \leq \rho(q_2)$,
(2) $q_1 \perp q_2$ implies $\rho(q_1) \perp \rho(q_2)$,
(3) for all $r \in \mathbb{P}$ there is $q \in \mathbb{Q}$ such that there is no $q' \in \mathbb{Q}$ with $q' \leq q$ and $\rho(q') \perp r$.

We conclude this section with a basic and useful fact.

LEMMA 1.2. *If Y is a closed subspace of X then $\mathbb{T}(Y)$ is a regular suborder of $\mathbb{T}(X)$.*

Proof. We have to check point (3) of the criterion of regularity. Let r be in $\mathbb{T}(X)$. Then $q = r{\restriction}Y$ is as required, and $q \in \mathbb{T}(Y)$ since Y was assumed to be closed. ∎

We remark that for the same reason, $\mathbb{T}(Y){\restriction}(p{\restriction}Y)$ is a regular suborder of $\mathbb{T}(X){\restriction}p$ for any $p \in \mathbb{T}(X)$.

2. The countable chain condition. The orderings $\mathbb{T}(X)$ are interesting as examples of ccc orderings. It will turn out that $\mathbb{T}(X)$ is ccc for most X, but still it can happen that $\mathbb{T}(X)$ is not ccc. We are going to make this more precise. First we note that $\mathbb{T}(X)$ is in any case ω_2-cc. We look now at the ccc for $\mathbb{T}(X)$. It would be convenient to have an easy criterion in terms of X for $\mathbb{T}(X)$ to be ccc. We will give a necessary condition and a sufficient condition on X for $\mathbb{T}(X)$ to be ccc, both easily verifiable.

DEFINITION 2.1.

(1) We say that a space X satisfies *condition* 1 if
$$(\forall x \in X)(\forall M \in [X]^{\omega_1})(\exists M' \in [M]^{\omega_1})(\forall A \in [M']^{\omega})\ (A \not\to x).$$
(2) We say that X satisfies *condition* 2 if
$$(\forall x \in X)(\forall M \in [X]^{\omega_1})(\exists A \in [M]^{\omega})\ (A \not\to x).$$

It is clear that condition 1 is stronger than condition 2. We observe that condition 2 is satisfied if and only if it is not possible to embed the one-point compactification of an uncountable discrete set into the space (in the context of sequential spaces with the unique limit property). Conditions 1 and 2 are rather similar, but they are not the same. As is known [Sim80], there exists a maximal almost disjoint system $\mathcal{A}$ in $[\omega_1]^\omega$ and its partition $\mathcal{A} = \mathcal{A}_0 \dot\cup \mathcal{A}_1$ such that for any uncountable $M \subseteq \omega_1$ the restriction of neither $\mathcal{A}_0$ nor $\mathcal{A}_1$ to M is a maximal antichain. We can then define a sequential space $X = \omega_1 \cup \{x\}$ where ω_1 is discrete and $A \in [\omega_1]^\omega$ converges to x for all $A \in \mathcal{A}_0$. This space satisfies condition 2 but not condition 1.

We will now relate conditions 1 and 2 for a space X to the fact that $\mathbb{T}(X)$ is ccc:

Theorem 2.2.

(a) *Condition* 1 *for a space* X *implies that* $\mathbb{T}(X)$ *is ccc.*
(b) *If* $\mathbb{T}(X)$ *is ccc then condition* 2 *holds for* X.
(c) (PFA) *The ordering* $\mathbb{T}(X)$ *is ccc if and only if condition* 2 *holds for* X.

Proof. (a) Suppose that $\{p_\alpha\}_{\alpha<\omega_1}$ is an uncountable antichain in $\mathbb{T}(X)$. We may assume that $\{\mathrm{supp}(p_\alpha)\}_{\alpha<\omega_1}$ is a Δ-system with kernel $\Delta = \{x_k\}_{k<\bar{k}}$ and all elements have the same size $\bar{k}+\bar{n}$. That means that $\mathrm{supp}(p_\alpha)$ has the form $\{x_k\}_{k<\bar{k}} \cup \{x^\alpha_n\}_{n<\bar{n}}$. Moreover, we can assume (after inductive selection) that $\mathrm{dom}(p_\beta) \cap \mathrm{supp}(p_\alpha) = \Delta$ for $\beta < \alpha$.

It now follows from the definition of $\mathbb{T}(X)$ and the fact that the p_α's are orthogonal that for any $\beta < \alpha < \omega_1$ we can find an $x \in X$ such that $p_\beta(x) = 1$ and $p_\alpha(x) = 0$. This x is therefore equal to an $x^\beta_{n(\beta,\alpha)}$ for some $n(\beta,\alpha) < \bar{n}$.

Applying condition 1 to x_k and $\{x^\alpha_n\}_{\alpha<\omega_1}$ for some $k < \bar{k}$ and $n < \bar{n}$, we obtain an uncountable $S \subseteq \omega_1$ such that for no $A \in [S]^\omega$ does $\{x^\alpha_n\}_{\alpha\in A}$ converge to x_k. After repeating this procedure $\bar{k} \times \bar{n}$ times for all $k < \bar{k}$ and $n < \bar{n}$, we get an $S \in [\omega_1]^{\omega_1}$ such that no countable subset of $\{x^\alpha_n\}_{\alpha\in S,\, n<\bar{n}}$ converges to any x_k. For any $\alpha \in S$ and $A \in [S]^\omega$ with $\alpha > \sup(A)$ we see therefore that $\{x^\beta_{n(\beta,\alpha)}\}_{\beta\in A}$ is a finite union of sequences converging to some x^α_n, $n < \bar{n}$. In order to simplify notation, we assume $S = \omega_1$.

Let us now consider the first $\omega + \bar{n} + 1$ elements of the antichain. By a pigeon-hole principle, there are different $n_1, n_2 \le \bar{n}$ and $A \in [\omega]^\omega$ such that $n(i, \omega + n_1) = n(i, \omega + n_2) =: n(i)$ for all $i \in A$. But then we can find an $A' \in [A]^\omega$ such that $\{x^i_{n(i)}\}_{i\in A'}$ converges to some $x^{\omega+n_1}_n$ and simultaneously to some $x^{\omega+n_2}_{n'}$, contradicting the unique limit property.

(b) Let x and $M = \{x_\alpha\}_{\alpha<\omega_1}$ be such that condition 2 is violated. Define $p_\alpha \in \mathbb{T}(X)$ by $p_\alpha(x) = p_\alpha(x_\alpha) = 1$ and $p_\alpha(x_\beta) = 0$ for $\beta < \alpha$. We have $\{x_\beta\}_{\beta<\alpha} \to x$ by assumption, hence $p_\alpha \in \mathbb{T}(X)$. Also $p_\beta \perp p_\alpha$ for $\beta < \alpha$ since $p_\beta(x_\beta) = 1$ and $p_\alpha(x_\beta) = 0$. The set $\{p_\alpha\}_{\alpha<\omega_1}$ is an uncountable antichain in $\mathbb{T}(X)$.

(c) Let $\{p_\alpha\}_{\alpha<\omega_1}$ be an uncountable subset of $\mathbb{T}(X)$ and proceed as in the first paragraph of the proof of (a) to obtain pairwise different $\{x_k\}_{k<\bar{k}}$, $\{x^\alpha_n\}_{\alpha<\omega_1, n<\bar{n}}$.

For any $n < \bar{n}$ and $k < \bar{k}$ we consider the ideal I_{nk} generated by all finite subsets of ω_1 and all sets $A \in [\omega_1]^\omega$ for which there is some $\alpha < \omega_1$ with $\alpha > \sup(A)$ such that $\{x_n^\beta\}_{\beta\in A} \subseteq \mathrm{dom}(p_\alpha)$ and $\{x_n^\beta\}_{\beta\in A}$ converges to x_k. Since the domain of p_α is finite or a finite union of converging sequences, this ideal is in fact ω_1-generated. We can now apply the following dichotomy for ω_1-generated ideals which holds under PFA [Tod11, Theorem 62]: There is an $S \in [\omega_1]^{\omega_1}$ such that either $[S]^\omega \subseteq I_{nk}$ or $[S]^\omega \cap I_{nk} = \emptyset$. In the first case we would violate condition 2 with $x = x_k$ and $M = \{x_n^\alpha\}_{\alpha\in S}$, so the second case must hold. After repeating this procedure $\bar{k} \times \bar{n}$ times for all $k < \bar{k}$ and $n < \bar{n}$, we get an $S_1 \in [S]^{\omega_1}$ such that $\mathrm{dom}(p_\alpha) \cap \{x_n^\beta\}_{\beta\in S_1,\, n<\bar{n}}$ is finite or a finite union of sequences converging to some x_n^α, $n < \bar{n}$, for any $\alpha \in S_1$.

We can again apply the dichotomy for ω_1-generated ideals, now to the ideal I_n generated by all finite subsets of S_1 and all sets $A \in [S_1]^\omega$ for which there is some $\alpha < \omega_1$ with $\alpha > \sup(A)$ and $n' < \bar{n}$ such that $\{x_n^\beta\}_{\beta\in A} \subseteq \mathrm{dom}(p_\alpha)$ and $\{x_n^\beta\}_{\beta\in A}$ converges to $x_{n'}^\alpha$. We obtain an $S_2 \in [S_1]^{\omega_1}$ such that

$$\text{either} \quad [S_2]^\omega \subseteq I_n \quad \text{or} \quad [S_2]^\omega \cap I_n = \emptyset.$$

Assume the first alternative holds true. We proceed by induction on $i < \omega$ to find $A_i \in [S_2]^\omega$, $n_i < \bar{n}$, and $\alpha_i \in S_2$ with $\alpha_i \geq \sup(A_i)$ such that $\{x_n^\beta\}_{\beta\in A_i}$ converges to $x_{n_i}^{\alpha_i}$ and $\alpha_i < \min(A_{i+1})$. Let $\alpha = \sup_{i<\omega}(\alpha_i)$. Then $\{x_n^\beta\}_{\beta\in S_2\cap\alpha}$ is not a finite union of converging sequences, so $S_2 \cap \alpha \notin I_n$, a contradiction, and the second alternative must hold. Again, we can repeat the procedure $\bar{n}$ times and obtain an $S_3 \in [S_2]^{\omega_1}$ such that for all $\alpha \in S_3$ the set $\mathrm{dom}(p_\alpha) \cap \{x_n^\beta\}_{\beta\in S_3,\, n<\bar{n}}$ does not contain a converging sequence, i.e. it is finite. Using the pressing down lemma, we find an $S_4 \in [S_3]^{\omega_1}$ such that even $\mathrm{dom}(p_\alpha) \cap \{x_n^\beta\}_{\beta\in S_4,\, n<\bar{n}} \subseteq \mathrm{supp}(p_\alpha)$. That means that no pair p_α, p_β with $\alpha, \beta \in S_4$ is orthogonal, so the starting set $\{p_\alpha\}_{\alpha<\omega_1}$ was not an antichain. ∎

Assertion (c) of the preceding theorem cannot be proved absolutely, as stated by the following theorem.

Theorem 2.3. *In the Cohen extension there is a space X satisfying condition* 2*, but $\mathbb{T}(X)$ is not ccc.*

Proof. Let $\{e_\alpha : \alpha \to \omega\}_{\alpha<\omega_1}$ be a coherent sequence of injective mappings, i.e. $e_\alpha \restriction \beta =^* e_\beta$ for $\beta < \alpha$, and let $r : \omega \longrightarrow 2$ be a Cohen real. Define $a_\alpha = \{\beta < \alpha : r \circ e_\alpha(\beta) = 1\}$. This is also a coherent sequence, i.e. $a_\alpha \subseteq \alpha$ and $a_\alpha \cap \beta =^* a_\beta$ for $\beta < \alpha$. If S is an uncountable subset of ω_1 in the Cohen extension then it has an uncountable subset S' from the ground model. Let $\alpha < \omega_1$ be such that $|S' \cap \alpha| = \omega$. An easy density argument shows that $|S' \cap a_\alpha| = |S' \cap \alpha \setminus a_\alpha| = \omega$. Therefore in the extension,

$\circledast$ There exists a coherent sequence $\{a_\alpha\}_{\alpha<\omega_1}$ with the property that for any $S \in [\omega_1]^{\omega_1}$ there is an $\alpha < \omega$ such that

$$|S \cap a_\alpha| = |S \cap \alpha \setminus a_\alpha| = \omega.$$

We now define a topological space X in the following way. On the set $X = \omega_1 \cup \{x_0, x_1\}$ consider the sequential topology generated by $a_\alpha \to x_0$ and $\alpha \setminus a_\alpha \to x_1$ for all $\alpha < \omega_1$. This topology has the unique limit property since $\{a_\alpha\}_{\alpha<\omega_1}$ was assumed to be a coherent sequence. Property ($\circledast$) implies that X satisfies condition 2. Namely, the topology is discrete on ω_1, so the only candidates for a counterexample to condition 2 are x_0 or x_1 and $S \in [\omega_1]^{\omega_1}$. But any such S contains a sequence converging to x_0 as well as a sequence converging to x_1.

On the other hand, if we define $p_\alpha \in \mathbb{T}(X)$ by $p_\alpha(x_0) = p_\alpha(x_1) = p_\alpha(\alpha) = 1$ and $p_\alpha(\beta) = 0$ for $\beta < \alpha$ then $p_\alpha(\beta) = 0 \neq 1 = p_\beta(\beta)$, i.e. we have found an uncountable antichain, witnessing that $\mathbb{T}(X)$ is not ccc. ∎

We remark that the requirement ($\circledast$) was enough to prove Lemma 2.3. It is called *Galvin's principle* and it is also true under the assumption ♣ (club principle) [Gal77]. Theorem 2.3 therefore also holds under ♣ (for definition see e.g. [Fre84]).

Condition 1 is quite weak. It is satisfied for example by all first countable spaces (and therefore for all metric spaces) and for all countable spaces and for all linear ordered spaces (even non-sequential). As a corollary we infer that all those spaces generate Todorcevic orderings which satisfy ccc.

We apply the proof of Theorem 2.2 to obtain the following:

THEOREM 2.4. *Suppose that X satisfies condition* 1 *and let $\mathbb{Q}$ be a ccc ordering. Then $\mathbb{T}(X) \times \mathbb{Q}$ is ccc.*

Proof. Let $\langle p_\alpha, q_\alpha \rangle_{\alpha<\omega_1}$ be an antichain of length ω_1. We proceed as in the proof of Theorem 2.2(a) to thin out $\{p_\alpha\}_{\alpha<\omega_1}$ up to the application of condition 1 which results in the set S. Since $\mathbb{Q}$ is ccc, there exists a $q \in \mathbb{Q}$ such that $\{q_\alpha\}_{\alpha>\gamma,\, \alpha\in S}$ is predense under q for any $\gamma < \omega_1$. Let G be a generic ultrafilter on $\mathbb{Q}$ with $q \in G$. It follows that the set $S' = \{\alpha \in S : q_\alpha \in G\}$ is uncountable. Consider the family $\{p_\alpha\}_{\alpha\in S'}$ in $V[G]$. We repeat the rest of the proof of Theorem 2.2(a) in $V[G]$ to obtain $\alpha_0, \alpha_1 \in S'$ such that p_{α_0} and p_{α_1} are not orthogonal, therefore $\langle p_{\alpha_0}, q_{\alpha_0} \rangle$ and $\langle p_{\alpha_1}, q_{\alpha_1} \rangle$ are not orthogonal either, a contradiction. ∎

The assertion of Theorem 2.4 is of course true if $\mathbb{T}(X)$ has the Knaster property. But the Knaster property does not follow from condition 1 for X (see Theorem 3.6).

3. Properties stronger than ccc. We have shown that $\mathbb{T}(X)$ is ccc in most cases. So we can ask whether it also has stronger properties. We consider the following notions:

DEFINITION 3.1. An ordering $\mathbb{P}$ is called

(i) *σ-centered* if $\mathbb{P} = \bigcup_{n\in\omega} P_n$, each P_n being centered, i.e. any finite subset has a lower bound;
(ii) *σ-linked* if $\mathbb{P} = \bigcup_{n\in\omega} P_n$, each P_n being linked, i.e. any two elements have a lower bound;
(iii) *σ-bounded cc* if $\mathbb{P} = \bigcup_{n\in\omega} P_n$, each P_n being $n+2$-cc, i.e. there is no antichain of size $n+2$;
(iv) *σ-finite cc* if $\mathbb{P} = \bigcup_{n\in\omega} P_n$, each P_n being ω-cc, i.e. there is no antichain of infinite size;
(v) *Knaster property* if any uncountable set $X \subseteq \mathbb{P}$ has a linked uncountable subset.

These notions are all stronger than ccc and form a hierarchy from stronger to weaker ones.

We will consider several spaces X and analyse the place of $\mathbb{T}(X)$ in this hierarchy of properties.

First we observe the following:

THEOREM 3.2. *If X is countable then $\mathbb{T}(X)$ is σ-centered.*

Proof. $\mathbb{T}(X) = \bigcup_{F\in[X]^{<\omega}} \{p : \mathrm{supp}(p) = F\}$. ∎

It is obvious that any linked subset of $\mathbb{T}(X)$ is centered. Therefore $\mathbb{T}(X)$ is σ-centered if and only if it is σ-linked.

If X is discrete then $\mathbb{T}(X)$ contains only functions with a finite domain. This ordering is σ-bounded cc. But if $|X|$ is larger than continuum then $\mathbb{T}(X)$ is not σ-linked.

A linear ordered topological space X such that $\mathbb{T}(X)$ is σ-finite cc but not σ-bounded cc is constructed in [Thü14]; its Borel version has been constructed lately in [Tod14]. This solves a problem of Horn and Tarski [HT48].

We now turn to spaces X such that $\mathbb{T}(X)$ is not σ-finite cc. Let ω_1 be equipped with the order topology. Any subspace X of ω_1 satisfies condition 1, which means that $\mathbb{T}(X)$ is ccc by Theorem 2.2.

THEOREM 3.3.

(i) *If $X \subseteq \omega_1$ is stationary, then $\mathbb{T}(X)$ is not σ-finite cc.*
(ii) *If $X \subset \omega_1$ is nonstationary, then $\mathbb{T}(X)$ is σ-centered, hence σ-finite cc.*

Proof. (i) This proof is inspired by the proof of Theorem 3.5. Let $X \subset \omega_1$ be stationary. Suppose that $\mathbb{T}(X)$ is σ-finite cc, i.e. $\mathbb{T}(X) = \bigcup_{i\in\omega} P_i$, where each P_i is finite-cc. We can assume that the P_i's are disjoint.

For $p \in \mathbb{T}(X)$ and $Q \subseteq \mathbb{T}(X)$ we define $\alpha(p) = \max(\mathrm{supp}(p))$ and $\alpha(Q) = \sup\{\alpha(p) : p \in Q\}$. We say that a set Q is *good* if there is an increasing sequence $\langle \alpha_i \rangle_{i<\omega}$: such that for any $i < \omega$:

(1) $Q \cap P_i$ is an antichain,
(2) $\alpha(p) > \alpha_i$ for all $p \in Q \cap P_i$,
(3) $Q \cap P_i$ is maximal with respect to (1)–(2),
(4) $\alpha_{i+1} > \alpha(Q \cap P_i)$.

An ordinal $\alpha < \omega_1$ is called *reachable* if there is a good set Q such that $\alpha = \alpha(Q)$. (Since for any $i < \omega$ and $\beta < \omega_1$ there are $j > i$ and $p \in P_j$ such that $\alpha(p) > \beta$, it follows that there are infinitely many i such that $Q \cap P_i \neq \emptyset$ and therefore $\alpha = \sup_{i<\omega}\{\alpha_i\}$ for the corresponding increasing sequence.)

We claim that the set of reachable ordinals is closed and unbounded in ω_1. To show that it is unbounded, let $\alpha < \omega_1$ be given and put $\alpha_0 = \alpha$. We choose by induction antichains $Q_i \subseteq P_i$ such that $\alpha(p) > \alpha_i$ for all $p \in Q_i$ and Q_i is maximal with respect to this property. Since P_i is finite-cc, we have $\alpha(Q_i) < \omega_1$. Set $\alpha_{i+1} = \alpha(\bigcup_{j\le i} Q_j) + 1$. Then $\langle \alpha_i \rangle_{i<\omega}$ witnesses that $Q = \bigcup_{i<\omega} Q_i$ is good and $\alpha(Q) > \alpha$ is reachable.

It remains to show that the set of all reachable ordinals is also closed. Let $\langle \alpha^k : k < \omega \rangle$ be an increasing sequence of reachable ordinals and let Q^k be a good set witnessing together with the increasing sequence $\langle \alpha_i^k \rangle_{i<\omega}$ that α^k is reachable, $\alpha^k = \alpha(Q^k)$. We find an increasing sequence $\langle i(k) : k < \omega \rangle$ such that $\alpha_{i(k)}^{k+1} > \alpha^k$. For i with $i(k) \le i < i(k+1)$ we set $Q_i = Q^{k+1} \cap P_i$ and $\alpha_i = \alpha_i^{k+1}$. Finally, let $Q = \bigcup_{i<\omega} Q_i$. Then $\langle \alpha_i \rangle_{i<\omega}$ witnesses that Q is good and $\sup_{k<\omega}\{\alpha^k\} = \sup_{i<\omega}\{\alpha_i\} = \alpha(Q)$ is reachable.

Now assume that $X \subset \omega_1$ is stationary. Since the set of reachable ordinals is closed and unbounded, there is a reachable ordinal $\alpha \in X$ and a good set Q witnessing that. Define an element $r \in \mathbb{T}(X)$ by $r(\alpha(p)) = 0$ for all $p \in Q$ and $r(\alpha) = 1$. Since the P_i's are finite-cc, the set $Q \cap P_i$ will be finite and the only accumulation point of $\{\alpha(p) : p \in Q\}$ is α, so r was defined correctly. Such an r will be orthogonal to all $p \in Q$, but it has to be in some P_i, so $Q \cap P_i$ is not maximal in the sense of (3), hence Q is not good, a contradiction.

(ii) Suppose $X \subset \omega_1$ is nonstationary and $C \subset \omega_1$ is a club disjoint from X.

By the Hewitt–Marczewski–Pondiczery theorem, the topological space ${}^{\omega_1}\omega$, the product of ω_1 many copies of the discrete countable space, is separable. Hence there is a countable $F \subset {}^{\omega_1}\omega$ such that for each $g : K \to \omega$ with $K \subset C$ finite, there is an $f \in F$ such that $g \subset f$.

For each $\alpha \in \omega_1$ let α_C^+ denote the successor of α in C. For every $\alpha \in C$, $\mathbb{T}(X \cap [\alpha, \alpha_C^+))$ is a σ-centered ordering, because the set $[\alpha, \alpha_C^+)$ is countable (Theorem 3.2). Hence let

$$\mathbb{T}(X \cap [\alpha, \alpha_C^+)) \;=\; \bigcup_i P_i(\alpha),$$

where $P_i(\alpha)$ is a centered family for all $i \in \omega$ and $\alpha \in C$.

We claim that $\mathbb{T}(X) = \bigcup_{f \in F} P_f$, where

$$P_f = \{p \in \mathbb{T}(X) : (\exists Z \in [C \cup \{0\}]^{<\omega})\ (\mathrm{dom}(p) \subseteq \bigcup_{\alpha \in Z}[\alpha, \alpha_C^+))\ \& \\ (\forall \alpha \in Z)\ (p{\restriction}[\alpha, \alpha_C^+) \in P_{f(\alpha)}(\alpha))\}$$

is a centered family.

Since for all α and $i \in \omega$ the family $P_i(\alpha)$ is centered, we immediately see that each P_f is centered. To prove that $\mathbb{T}(X) = \bigcup_{f \in F} P_f$, take an arbitrary $p \in \mathbb{T}(X)$ and suppose that its domain $\mathrm{dom}(p)$ intersects infinitely many intervals $[\alpha, \alpha_C^+)$. Then there is an increasing sequence

$$\alpha_0 < \beta_0 < \alpha_0^+ < \alpha_1 < \beta_1 < \alpha_1^+ < \cdots, \quad \beta_i \in X,\ \alpha_i \in C.$$

It follows that $\sup \alpha_n = \sup \beta_i$, but since C is closed, we have $\sup \alpha_i \in C$, which contradicts the fact that $\sup \beta_i \in \mathrm{dom}(p) \subseteq X$ and $X \cap C = \emptyset$. ∎

Theorem 3.4. *If $X \subset \omega_1$, then the Todorcevic ordering $\mathbb{T}(X)$ has the Knaster property.*

Proof. Let $\{p_\alpha : \alpha < \omega_1\} \subset \mathbb{T}(X)$. Without loss of generality one can suppose that the $\mathrm{supp}(p_\alpha)$'s are pairwise different and $\{\mathrm{supp}(p_\alpha) : \alpha < \omega_1\}$ form a Δ-system with kernel Δ such that $\max(\mathrm{supp}(p_\beta) \setminus \Delta) \le \min(\mathrm{supp}(p_\alpha) \setminus \Delta)$ for $\beta < \alpha$. Finally, we shrink the system so that $Y = \bigcup_{\alpha < \omega_1} \mathrm{supp}(p_\alpha)$ is a nonstationary subset of ω_1. Let $C \subset \omega_1$ be a club disjoint from Y. We have $p_\alpha \perp p_\beta$ if and only if $p_\alpha{\restriction}Y \perp p_\beta{\restriction}Y$. So we can assume $\mathrm{dom}(p_\alpha) \subseteq Y$ for all $\alpha < \omega_1$. Following the proof of the previous theorem we see that $\{p_\alpha : \alpha < \omega_1\}$ is σ-centered, hence it contains a centered subfamily of size $\aleph_1$, which completes the proof. ∎

We mention still another example of a space X such that $\mathbb{T}(X)$ is not σ-finite cc. Any metric space X satisfies trivially condition 1, so $\mathbb{T}(X)$ is ccc by Theorem 3.2(a).

Theorem 3.5 ([Tod91]). *If X is an uncountable separable metric space then $\mathbb{T}(X)$ is not σ-finite cc.*

Proof. Assume for contradiction that $\mathbb{T}(X) = \bigcup_{i<\omega} P_i$ where each P_i is finite-cc. Let $D = \{x_i\}_{i<\omega}$ be dense in X. For any $i, k < \omega$ we choose a maximal antichain Q_i^k in $\{p \in P_i : \mathrm{supp}(p) \cap B_{1/i}(x_k) \neq \emptyset\}$. Since P_i is finite-cc, Q_i^k will be finite and

$$Z = \bigcup_{i,k<\omega} \bigcup \{\mathrm{supp}(p) \cap B_{1/i}(x_k) : p \in Q_i^k\}$$

will be countable. Let $x \in X \setminus Z$ and choose for all i a $k(i)$ such that $x \in B_{1/i}(x_{k(i)})$. Consider

$$Z_x = \bigcup_{i<\omega} \bigcup \{\mathrm{supp}(p) \cap B_{1/i}(x_{k(i)}) : p \in Q_i^{k(i)}\}.$$

Then Z_x is finite or a sequence with limit x and $x \notin Z \supset Z_x$. Define $q \in \mathbb{T}(X)$ by $q(x) = 1$ and $q[Z_x] = 0$. There is an i such that $q \in P_i$. But for all $p \in Q_i^{k(i)}$ we have $p \perp q$, i.e. $Q_i^{k(i)}$ is not maximal. This contradiction proves the theorem. ∎

The Knaster property is consistently the same as ccc (under MA(ω_1)). But it is also consistently strictly stronger than ccc. Let $\mathcal{N}$ be the Baire space ${}^\omega\omega$ with the standard topology and let $\mathfrak{b}$ be the minimal size of an unbounded set in the ordering $({}^\omega\omega, \leq^*)$.

THEOREM 3.6 ([Tod91]). ($\mathfrak{b} = \omega_1$) *$\mathbb{T}(\mathcal{N})$ does not have the Knaster property.*

Proof. We have to construct an uncountable subset of $\mathbb{T}(\mathcal{N})$ without an uncountable linked subset. Let $\{f_\alpha\}_{\alpha<\omega_1}$ witness $\mathfrak{b}$.

Fix a coherent sequence of injections $e_\alpha : \alpha \to \omega$, $\alpha < \omega_1$, i.e. $e_\alpha\restriction\beta =^* e_\beta$ for all $\beta < \alpha$. Define

$$F_\alpha = \{\beta < \alpha : e_\alpha(\beta) < f_\alpha(\Delta(f_\alpha, f_\beta))\}$$

where $\Delta(f_\alpha, f_\beta)$ denotes the minimal $n < \omega$ such that $f_\alpha(n) \neq f_\beta(n)$. Since e_α is an injection, for one special α and $\Delta(f_\alpha, f_\beta)$ the inequality $e_\alpha(\beta) < f_\alpha(\Delta(f_\alpha, f_\beta))$ can hold only for a finite number of $\beta < \alpha$. It follows that if F_α is infinite then $\{f_\beta\}_{\beta\in F_\alpha}$ converges to f_α. We can therefore define $p_\alpha \in \mathbb{T}(\mathcal{N})$ by $p_\alpha(f_\alpha) = 1$ and $p_\alpha(f_\beta) = 0$ if $\beta \in F_\alpha$. We claim that $\{p_\alpha\}_{\alpha<\omega_1}$ witnesses that $\mathbb{T}(\mathcal{N})$ does not have the Knaster property. Indeed, let $H \in [\omega_1]^{\omega_1}$. Find $D \in [H]^\omega$ such that $\{f_\alpha\}_{\alpha\in D}$ is dense in $\{f_\alpha\}_{\alpha\in H}$. We have $e_\alpha\restriction D =^* e_\beta\restriction D$ for all $\alpha, \beta > \sup D$. Hence we can choose $K \in [H]^{\omega_1}$ with $\inf(K) > \sup(D)$ such that $e_\alpha\restriction D = e_\beta\restriction D$ for all $\alpha, \beta \in K$. We then set $e = e_\alpha\restriction D$. If there were no $n < \omega$, $t \in {}^n\omega$ such that

$$|\{f_\alpha(n) : \alpha \in K \;\&\; f_\alpha \supset t\}| = \omega \tag{1}$$

then, as can be seen by induction, we would have $|\{f_\alpha(n) : \alpha \in K\}| < \omega$ for all $n < \omega$ and we could define $g(n) = \max\{f_\alpha(n) : \alpha \in K\}$. But this g would be an upper bound for $\{f_\alpha\}_{\alpha\in K}$. This contradicts the fact that $\{f_\alpha\}_{\alpha<\omega_1}$ is unbounded in $\leq^*$ and K is cofinal in ω_1. We can therefore find a t such that (1) holds. Since $\{f_\alpha\}_{\alpha\in D}$ is dense in $\{f_\alpha\}_{\alpha\in H}$, there must be some $\gamma \in D$ such that $f_\gamma \supset t$. From the choice of t we infer that there must be an $\alpha \in K$ such that $f_\alpha \supset t$ and $f_\alpha(n) > \max(e(\gamma), f_\gamma(n))$. But then $e_\alpha(\gamma) = e(\gamma) < f_\alpha(n) = f_\alpha(\Delta(f_\alpha, f_\gamma))$, hence $\gamma \in F_\alpha$ and therefore $p_\gamma(f_\gamma) = 1$ and $p_\alpha(f_\gamma) = 0$, i.e. $p_\alpha \perp p_\gamma$. We have proved that $\{f_\alpha\}_{\alpha<\omega_1}$ has no uncountable linked subset. ∎

We observe that the proof of Theorem 2.2(c) shows indeed more, namely that under the dichotomy for ω_1-generated ideals, condition 2 for the space

X implies that $\mathbb{T}(X)$ has the Knaster property. This is in contrast to Theorem 3.6 since $\mathcal{N}$ is a metric space and therefore satisfies condition 2 (even condition 1). As a corollary we conclude that the dichotomy for ω_1-generated ideals implies $\mathfrak{b} > \omega_1$.

If we want to have an example of a space X such that $\mathbb{T}(X)$ is not ccc it is enough to deny condition 2—the easiest example is the one-point compactification of a discrete uncountable set.

3.1. Exhaustive functions. For the sake of completeness, let us conclude this section with some notions equivalent to σ-finite cc and σ-bounded cc orderings. Those equivalents appear in [Paz07], but otherwise remain unpublished.

DEFINITION 3.7. Let $\mathbb{P}$ be an ordering and B be a Boolean algebra.

(i) A real function $f : \mathbb{P} \to \mathbb{R}$ is called *exhaustive* if $\lim_{n\to\infty} f(a_n) = 0$ for each disjoint sequence $\langle a_n : n \in \omega\rangle \in \mathbb{P}^\omega$.
(ii) A real function $f : \mathbb{P} \to \mathbb{R}$ is called *uniformly exhaustive* if for each positive $\varepsilon > 0$ there is a $k \in \omega$ such that $|\{n \in \omega : |f(a_n)| \geq \varepsilon\}| \leq k$ for every disjoint sequence $\langle a_n : n \in \omega\rangle \in \mathbb{P}^\omega$.
(iii) A nonnegative real function $f : B \to \mathbb{R}$ is called a *supermeasure* if $f(a \vee b) \geq f(a) + f(b)$ for all orthogonal $a, b \in B$.

THEOREM 3.8.

(i) *An ordering $\mathbb{P}$ carries a strictly positive exhaustive function if and only if $\mathbb{P}$ is σ-finite cc.*
(ii) *An ordering $\mathbb{P}$ carries a strictly positive uniformly exhaustive function if and only if $\mathbb{P}$ is σ-bounded cc.*
(iii) *A Boolean algebra B carries a strictly positive supermeasure if and only if B is σ-bounded cc.*

Proof. Let f be a strictly positive exhaustive function (strictly positive uniformly exhaustive function resp.) on $\mathbb{P}$. Set $P_n = \{x \in \mathbb{P} : f(x) \geq 1/(n+1)\}$ for $n \in \omega$. Part (i): Then P_n is ω-cc and $\{P_n : n \in \omega\}$ witnesses that $\mathbb{P}$ is σ-finite cc. Part (ii): Similarly, P_n is k_n-cc for some k_n and after a suitable renumbering, $\{P_n : n \in \omega\}$ witnesses that $\mathbb{P}$ is σ-bounded cc.

In the opposite direction, let $\{P_n : n \in \omega\}$ witness that $\mathbb{P}$ is σ-finite cc (σ-bounded cc resp.). Put $f(a) = \max\{1/(n+1) : a \in P_n\}$ for $a \in \mathbb{P}$. Then f is a strictly positive exhaustive function (strictly positive uniformly exhaustive function resp.) on $\mathbb{P}$. (We remark that we could assume the P_n are upward closed, so the resulting functions will be even monotone.)

Concerning (iii), one direction follows from (ii) since every supermeasure is a uniformly exhaustive function. Let now B be σ-bounded cc, so B^+ carries a strictly positive uniformly exhaustive function by (ii). For positive n set $X_n = \{x \in B^+ : 1/n \leq f(x) < 1/(n-1)\}$. Denote by k_n the maximal size of a subset of X_n with pairwise orthogonal elements. Then $k_n < \infty$ since f is uniformly exhaustive. For $x \in X_n$ set $\phi(x) = 1/k_n \cdot 2^n$ and $\phi(\mathbf{0}) = 0$. Define $\mu(a) = \sup\{\sum_{i=1}^{m} \phi(x_i) : \{x_1, \ldots, x_m\}$ is a set of pairwise orthogonal elements $x_i \le a\}$; this is everywhere finite and is a strictly positive supermeasure. ∎

4. $\mathbb{T}(X)$ as a forcing notion. We are interested in the question which forcing notions are contained in the forcing $\mathbb{T}(X)$, especially which standard reals are added by $\mathbb{T}(X)$.

First we look at Cohen forcing. For our purposes we define Cohen forcing $\mathbb{C}_\omega$ (κ Cohen reals $\mathbb{C}_\kappa$ respectively) as the set of functions f from ω (from κ respectively) to $\{0, 1\}$ with finite domain, ordered by extension.

LEMMA 4.1. *If X is the discrete space of size $\kappa \ge \omega$ then $\mathbb{T}(X)$ is equal to the forcing notion $\mathbb{C}_\kappa$.*

Proof. Since X does not contain nontrivial converging sequences, $\mathbb{T}(X)$ consists of all functions from X to $\{0, 1\}$ with finite domain. ∎

This lemma and Lemma 1.2 imply the following.

THEOREM 4.2. *If X contains a closed discrete subset of size $\kappa \ge \omega$ then $\mathbb{T}(X)$ adds κ many Cohen reals.*

But for every reasonable space X, the forcing $\mathbb{T}(X)$ adds at least one Cohen real, as becomes clear from the following lemma:

LEMMA 4.3. *Let $X = \{x_i\}_{i<\omega} \cup \{x\}$ be a convergent sequence with limit point x.*

(a) *The algebra $\mathbb{B}(X)$ is isomorphic to the Cartesian product of the completion of Cohen forcing $\mathbb{C}_\omega$ and the power set of the natural numbers, $\mathcal{P}(\omega)$.*
(b) *If $p \in \mathbb{T}(X)$ with* $\mathrm{dom}(p) = x$ *and $p(x) = 0$ then $\mathbb{T}(X)\restriction p$ is isomorphic to $\mathbb{C}_\omega$.* ∎

THEOREM 4.4. *If X has infinitely many accumulation points then $\mathbb{T}(X)$ adds a Cohen real.*

Proof. For a given $p \in \mathbb{T}(X)$ fix a converging sequence $\{x_i\}_{i<\omega}$ with limit $x \notin \mathrm{supp}(p)$. We can also assume $x_i \notin \mathrm{dom}(p)$ for all $i < \omega$. If necessary, extend p to p' by $p'(x) = 0$. Then $Y = \{x_i\}_{i<\omega} \cup \{x\}$ is closed and Lemmas 1.2 and 4.3 imply the assertion of the theorem. ∎

We turn now to another kind of generic reals. Hechler forcing is the canonical way of adding a dominating function.

DEFINITION 4.5. *Hechler forcing* is the set $D = \{\langle f, n\rangle : f \in {}^\omega\omega \ \&\ n < \omega\}$ with the ordering $\langle f_1, n_1\rangle \le \langle f_2, n_2\rangle$ if $f_1 \ge f_2$, $n_1 \ge n_2$, and $f_1\restriction n_2 = f_2\restriction n_2$.

The following theorem is formulated for $X=\mathbb{R}$, but the reader can easily verify that the same proof works for many topological spaces.

THEOREM 4.6. *$\mathbb{T}(\mathbb{R})$ adds a Hechler real.*

Proof. Let $p\in\mathbb{T}(\mathbb{R})$. Of course, $\mathrm{dom}(p)$ is nowhere dense, so there is a closed interval disjoint from it. We can assume without loss of generality $\mathrm{dom}(p)\cap[0,2]=\emptyset$. Define $p'\le p$ by $p'(0)=1$ and $p'\big(\frac{1}{k+1}\big)=0$ for $k<\omega$.

We now define an embedding ρ of Hechler forcing into $\mathbb{T}(\mathbb{R})\restriction p'$. Let

$$x(k,m)=\frac{1}{k+1}+\frac{1}{k^2+k+1}\,\frac{1}{m+1}$$

and define $s=\rho(\langle f,n\rangle)$ by $s(x(k,m))=0$ for $k<\omega$ and $m<f(k)$; $s(x(k,m))=1$ for $k<n$ and $m=f(k)$; $s(\frac{1}{k+1})=0$ for $k<\omega$; and $s(0)=1$. Clearly $s\in\mathbb{T}(\mathbb{R})$. We show the regularity of this embedding by the criterion of Section 1. Points (1) and (2) are satisfied. To prove (3), we fix $r\in\mathbb{T}(\mathbb{R})\restriction p'$. Set $n=\max\{k:(\exists m)\,(r(x(k,m))=1)\}+1$ and $f(k)=\min\{m:r(x(k,m))=1\}$ if such an m exists, and $f(k)=\max\{m: x(k,m)\in\mathrm{dom}(r)\}+1$ otherwise. The latter must exist since $r\big(\frac{1}{k+1}\big)=0$ and $\{x(k,m)\}_{m<\omega}\to\frac{1}{k+1}$. For all $\langle f',n'\rangle\le\langle f,n\rangle$ we have $\rho(\langle f',n'\rangle)\parallel r$. ∎

We turn now to another important property of the algebra $\mathbb{B}(X)$. A Boolean algebra is said to be *σ-completely generated* by a subset if the smallest σ-complete subalgebra containing this subset is the Boolean algebra itself.

THEOREM 4.7. *If X is separable then the algebra $\mathbb{B}(X)$ contains a dense σ-complete subalgebra σ-completely generated by a countable set.*

Proof. Let D be a countable dense subset of X. For $x\in X$ define $p_x^0,p_x^1\in\mathbb{T}(X)$ by $\mathrm{dom}(p_x^0)=\mathrm{dom}(p_x^1)=\{x\}$ and $p_x^0(x)=0$, $p_x^1(x)=1$. Let A be the σ-complete subalgebra of $\mathbb{B}(X)$ generated by $\{p_x^0: x\in D\}$. Note that for every $x\in X$ such that there are $x_i\in D$ with $x_i\to x$ we have

$$p_x^1=\bigvee_{j<\omega}\bigwedge_{i>j}p_{x_i}^0,$$

hence $p_x^1\in A$. Since X is sequential and D is dense, we can iterate this argument and get $p_x^1\in A$ for all $x\in X$. Since p_x^0 and p_x^1 are complements, we have $p_x^0,p_x^1\in A$ for all $x\in X$. From

$$p=\bigwedge\{p_x^0:p(x)=0\}\wedge\bigwedge\{p_x^1:p(x)=1\}$$

for any $p\in\mathbb{T}(X)$ it follows that $\mathbb{T}(X)\subseteq A$, and therefore A is dense in $\mathbb{B}(X)$. ∎

We remark that all such Boolean algebras (σ-complete Boolean algebras σ-completely generated by a countable set) are of the form $\mathrm{Borel}(\mathbb{R})/I$ for a σ-complete ideal I, and the forcing extension is determined by one real number. If $\mathbb{B}(X)$ is ccc (which holds in most cases—Theorem 2.2) then $\mathbb{B}(X)$ itself is σ-completely generated by a countable set.

In Section 2 we showed that $\mathbb{T}(X)$ is usually ccc. We ask now whether the forcing can be interesting also in the opposite case.

THEOREM 4.8. *If $\mathbb{T}(X)$ does not satisfy ccc then there is a $p \in \mathbb{T}(X)$ such that $p \Vdash \omega_1$ is collapsed.*

Proof. Suppose that $\{p_\alpha\}_{\alpha<\omega_1}$ is an uncountable antichain in $\mathbb{T}(X)$. We proceed as in the proof of Theorem 2.2(a). That means that we can assume that $\{\mathrm{supp}(p_\alpha)\}_{\alpha<\omega_1}$ is a Δ-system with kernel $\Delta = \{x_k\}_{k<\bar{k}}$ and all elements have the same size $\bar{k} + \bar{n}$. The set $\mathrm{supp}(p_\alpha)$ has the form $\{x_k\}_{k<\bar{k}} \cup \{x_n^\alpha\}_{n<\bar{n}}$ and $\mathrm{dom}(p_\beta) \cap \mathrm{supp}(p_\alpha) = \Delta$ for $\beta < \alpha$. Define $p \in \mathbb{T}(X)$ with $\mathrm{dom}(p) = \{x_k\}_{k<\bar{k}}$ by $p(x_k) = 1$. Let $p \in G$ be generic, and in $V[G]$ for $f_G = \bigcup G$ set

$$K = \{\alpha < \omega_1^V : (\forall n < \bar{n})\ (f_G(x_n^\alpha) = 1)\}.$$

We show that p forces that ω_1 is collapsed by proving that (a) K is unbounded in ω_1^V, and (b) K is of type ω.

(a) For any $\alpha < \omega_1$ and $q \le p$ there are $q' \le q$ and $\alpha' \ge \alpha$ such that $q' \Vdash \alpha' \in K$, namely $\alpha' > \sup\{\beta : (\exists n < \bar{n})\ (x_n^\beta \in \mathrm{dom}(q))\}$ and $q'(x_n^{\alpha'}) = 1$ for $n < \bar{n}$.

(b) We show that $|K \cap \alpha| < \omega$ for all $\alpha < \omega_1^V$. For any $q \le p$ there is a $q' \le q$ such that $q' \Vdash |K \cap \alpha| < \omega$, namely let $\alpha' \ge \alpha$ with $\alpha' > \sup\{\beta : (\exists n < \bar{n})\ (x_n^\beta \in \mathrm{dom}(q))\}$ and $q'(x) = p_{\alpha'}(x)$ for $x \in \mathrm{dom}(p_{\alpha'}) \setminus \mathrm{dom}(q)$. Then $q' \in \mathbb{T}(X)$. As in the proof of Theorem 2.2(a), we conclude that for any $\beta < \alpha$ we can find $n < \bar{n}$ such that $p_{\alpha'}(x_\beta^n) = 0$. It follows that $q'(x_\beta^n) = 0$ if $x_n^\beta \notin \mathrm{supp}(q)$. The assertion holds now since $\beta \notin K$ for all $\beta < \alpha$ such that $x_n^\beta \notin \mathrm{supp}(q)$.

We have found in $V[G]$ a countable cofinal subset K of ω_1^V, so ω_1 is collapsed. ∎

5. Boolean algebras. We will now examine the special case $\mathbb{T}(B)$, where B is a σ-complete Boolean algebra understood as a topological space with the canonical sequential topology τ_s defined in the following way: For a given sequence $\{\alpha_i : i < \omega\}$, let $\liminf(a_i) = \bigvee_{j<\omega} \bigwedge_{i>j} a_i$ and $\limsup(a_i) = \bigwedge_{j<\omega} \bigvee_{i>j} a_i$. Define the *algebraic limit* $\lim(a_i) = a$ if $\liminf(a_i) = \limsup(a_i) = a$ and let $A \subseteq B$ be closed if and only if $a_i \in A$ and $\lim(a_i) = a$ imply $a \in A$. This is a sequential T_1-topology with the unique limit property. It follows from the definition that the limit of any countable antichain is $\mathbf{0}$. More on this topology can be found in [Vla69], [Vla02] and [BGJ98].

If B is an infinite σ-complete Boolean algebra then we can find a maximal antichain of size ω in it. Such a maximal antichain σ-completely generates a regular subalgebra A which is isomorphic to $\mathcal{P}(\omega)$ and is of course closed in (B,τ_s). It is known that $(P(\omega),\tau_s)$ is isomorphic to the Cantor cube 2^ω (see [BGJ98]). Therefore 2^ω embeds into (B,τ_s) for any infinite σ-complete Boolean algebra B. Lemma 1.2 and Theorem 3.5 now imply the following theorem:

THEOREM 5.1. *$\mathbb{T}((B,\tau_s))$ is not σ-finite cc for any infinite σ-complete Boolean algebra B.* ∎

We are especially interested in the connection between ccc for B and for $\mathbb{T}((B,\tau_s))$. First we note the following.

THEOREM 5.2. *Condition* 1 *for (B,τ_s) implies that B has the Knaster property.*

Proof. Assume for contradiction that B does not have the Knaster property, i.e. there is an $M\in[B]^{\omega_1}$ such that no $M'\in[M]^{\omega_1}$ is linked; that means any $M'\in[M]^{\omega_1}$ contains an orthogonal pair. The theorem of Dushnik and Miller implies now that any set $M'\in[M]^{\omega_1}$ contains a subset $A\in[M']^\omega$ of pairwise orthogonal elements. Of course, such an A converges algebraically and therefore also topologically to $\mathbf{0}$. But this means that condition 1 is violated for $\mathbf{0}$ and M. ∎

We remark that many ccc σ-complete algebras will satisfy condition 1, for example the Boolean algebras of Cohen forcing (because it has a countable dense subset) and Random forcing (the sequential topology of this algebra is in fact generated by the canonical metric). But for Hechler forcing it is undecidable in ZFC whether its sequential topology satisfies condition 1 or not.

Similarly to Theorem 5.2, condition 2 for (B,τ_s) implies that B is ccc: If D were an uncountable antichain in B then condition 2 is violated by $x=\mathbf{0}$ and $M=D$. From Theorems 2.2(b) and 4.8 we now infer the following:

THEOREM 5.3. *If the σ-complete Boolean algebra B is not ccc then the forcing notion $\mathbb{T}((B,\tau_s))$ collapses ω_1.* ∎

If B is ccc then in most cases $\mathbb{T}((B,\tau_s))$ is also ccc, but there is an example where this is not true:

EXAMPLE 5.4 (A complete ccc Boolean algebra such that $\mathbb{T}((\mathbb{B},\tau_s))$ is not ccc). Define

$$\mathbb{P}=\{p:\omega_1\to\{0,1\}:\text{o.t.}(\text{dom}(p))<\omega\times\omega\ \&\ |\text{supp}(p)|<\omega\}.$$

Here $\text{supp}(p)=p^{-1}(1)$ is used in the same sense as above, and o.t. denotes order type.

Let $\mathbb{B}$ be the Boolean completion of $\mathbb{P}$.

1. $\mathbb{B}$ *is ccc.* Suppose an uncountable antichain in $\mathbb{B}$ is given. We can assume that it is of the form $\{p_\alpha\}_{\alpha<\omega_1} \subseteq \mathbb{P}$. We argue as in 2.2(a) and improve the antichain by selection. Since in the definition of the elements of $\mathbb{P}$ the domain does not depend on the support, we can assume that the kernel of the Δ-system is empty. We obtain $\{x_n^\alpha\}_{\alpha<\omega_1,\, n<\bar{n}}$ such that without loss of generality $\operatorname{supp}(p_\alpha) = \{x_n^\alpha\}_{n<\bar{n}}$ and $\operatorname{supp}(p_\alpha) \cap \operatorname{dom}(p_\beta) = \emptyset$ for all $\beta < \alpha < \omega_1$. Moreover, we can assume that $x_n^\beta < x_{n'}^\alpha$ for $\beta < \alpha$ and $n, n' < \bar{n}$. Then for all $\alpha < \omega \times \omega$ we have $p_\alpha \perp p_{\omega\times\omega}$, so there is an $n(\alpha) < \bar{n}$ such that $p_{\omega\times\omega}(x_{n(\alpha)}^\alpha) = 0$. But this implies that the order type of $\operatorname{dom}(p_{\omega\times\omega})$ is at least $\omega \times \omega$, a contradiction.

2. $\mathbb{T}(\mathbb{B})$ *is not ccc.* This follows from Theorem 2.2(b) and the fact that $(\mathbb{B}, \tau_s)$ does not satisfy condition 2: Let $q_\alpha \in \mathbb{P}$ with $\operatorname{dom}(q_\alpha) = \{\alpha\}$ and $q_\alpha(\alpha) = 1$ and let $A \in [\omega_1]^\omega$ with o.t.$(A) < \omega \times \omega$. For any $p \in \mathbb{P}$ we define $p' \leq p$ by $p'(\alpha) = 0$ for all $\alpha \in A \setminus \operatorname{dom}(p)$. Then o.t.$(\operatorname{dom}(p')) < \omega \times \omega$, i.e. $p' \in \mathbb{P}$. But for all $\alpha \in A \setminus \operatorname{supp}(p)$ we have $p' \perp q_\alpha$ since $p'(\alpha) = \mathbf{0} \neq \mathbf{1} = q_\alpha(\alpha)$, therefore $p' \perp \limsup_{\alpha\in A}(q_\alpha)$. It follows that $\limsup_{\alpha\in A}(q_\alpha) = \mathbf{0}$ and therefore $\lim_{\alpha\in A}(q_\alpha) = \mathbf{0}$. But from this we conclude that for any $B \in [\omega_1]^\omega$ the sequence $\{q_\alpha\}_{\alpha\in B}$ converges topologically to $\mathbf{0}$ since for all $C \in [B]^\omega$ there is a $D \in [C]^\omega$ such that o.t.$(D) = \omega$, which means that $\{q_\alpha\}_{\alpha\in D}$ converges even algebraically to $\mathbf{0}$. We have proved that $x = \mathbf{0}$ and $M = \{q_\alpha\}_{\alpha<\omega_1}$ witnesses that $(\mathbb{B}, \tau_s)$ does not satisfy condition 2. ∎

The troubles from Example 5.4 lead to the following modification of the definition of condition 2 where topological convergence is replaced by algebraic convergence. Note that we can, instead of a and M, also consider $\mathbf{0}$ and $M \triangle a$. Note also that $\lim A = \mathbf{0}$ if and only if $\limsup A = \mathbf{0}$.

Definition 5.5. We say that a Boolean algebra B satisfies *condition* 2(alg) if

$$(\forall M \in [B^+]^{\omega_1})(\exists A \in [M]^\omega)\ (\limsup A \neq \mathbf{0}).$$

We are now able to prove the following.

Theorem 5.6. *The σ-complete Boolean algebra B satisfies ccc if and only if it satisfies condition* 2(alg).

Proof. $(\rightarrow)$ Let $M = \{a_\alpha\}_{\alpha<\omega_1}$ be such that condition 2(alg) is violated. This means that $\limsup\{a_\beta\}_{\beta<\alpha} = \mathbf{0}$ for any $\alpha < \omega_1$. We find for any α a $\beta(\alpha) < \alpha$ such that $\bar{a}_\alpha := a_\alpha \setminus \bigvee\{a_\beta : \beta(\alpha) < \beta < \alpha\} \neq \mathbf{0}$. By the pressing down lemma there must be a $\bar{\beta} < \omega_1$ and a stationary set S such that $\beta(\alpha) = \bar{\beta}$ for all $\alpha \in S$. But the antichain $\{\bar{a}_\alpha\}_{\alpha\in S}$ witnesses now that B is not ccc.

$(\leftarrow)$ As noted above, if M is an uncountable antichain in B then $\lim A = \mathbf{0}$ for any countable subset A, i.e. condition 2(alg) is violated. ∎

This suggests that in the case of Boolean algebras the definition of the operator $\mathbb{T}$ should also be modified by using algebraic convergence instead of the topological one. In general, it could be useful to modify the operator $\mathbb{T}$ for any space with some kind of convergence.

6. Homogeneity. A complete Boolean algebra B is called *homogeneous* if for all $a, b \in B \setminus \{0, 1\}$ there exists an automorphism $\phi : B \to B$ such that $\phi(a) = b$; it is called *weakly homogeneous* if for all $a, b \in B \setminus \{0, 1\}$ there exists an automorphism $\phi : B \to B$ such that $\phi(a)$ and b are not orthogonal. The main aim of this section is to show that for many topological spaces X the Boolean algebra $\mathbb{B}(X)$ is homogeneous. The product of many copies of a homogeneous complete Boolean algebra need not be homogeneous, but it is always weakly homogeneous. The theorem of Koppelberg and Solovay (see [Mon89, Vol. 2, Chap. 18]) says more. A complete Boolean algebra is weakly homogeneous if and only if it is isomorphic to a product C^κ of a homogeneous complete Boolean algebra C. From this we get the following corollary:

Theorem 6.1. *An infinite weakly homogeneous ccc complete Boolean algebra is homogeneous.*

Proof. Let B be weakly homogeneous. By the Koppelberg–Solovay theorem there is a homogeneous complete Boolean algebra C such that $B \cong C^\kappa$. The ccc implies $\kappa \le \omega$. If C is finite, then B is the algebra 2^ω and therefore homogeneous. Otherwise, since C is homogeneous and complete, we have $C \cong C^\kappa$, therefore $B \cong C$, and B is homogeneous. ∎

We will apply this to the algebra $\mathbb{B}(X)$. The following lemma and theorem are formulated for the open unit interval $X = (0, 1)$, but it is clear that the proof goes through for many topological spaces X.

Lemma 6.2. $\mathbb{B}((0, 1))$ *is weakly homogeneous.*

Proof. Since $\mathbb{T}(X)$ is a dense subset of $\mathbb{B}(X)$, it is enough to prove the criterion of weak homogeneity for $p, q \in \mathbb{T}((0, 1))$. We find a homeomorphism $\psi : (0, 1) \to (0, 1)$ such that $\psi[\mathrm{dom}(p)] \cap \mathrm{dom}(q) = \emptyset$. The induced isomorphism $\phi : \mathbb{T}((0, 1)) \to \mathbb{T}((0, 1))$ with $\phi(r)(x) = r(\psi^{-1}(x))$ can be extended to an automorphism on $B((0, 1))$ which is a witness of weak homogeneity, since $\mathrm{dom}(\phi(p)) \cap \mathrm{dom}(q) = \emptyset$ and therefore $\phi(p), q \ge \phi(p) \cup q \in \mathbb{T}((0, 1))$. ∎

By Theorem 2.2(a), $\mathbb{B}((0, 1))$ is ccc, hence by Theorems 6.1 and 6.2 we get:

Theorem 6.3. $\mathbb{B}((0, 1))$ *is homogeneous.*

The topological space X determines the order $\mathbb{T}(X)$ and therefore also the complete Boolean algebra $\mathbb{B}(X)$. We investigate the question to what extent this can be reversed.

THEOREM 6.4. *The orderings* $\mathbb{T}(X)$ *and* $\mathbb{T}(Y)$ *are order isomorphic if and only if the sequential spaces* X *and* Y *are homeomorphic.*

Proof. We reconstruct X from $\mathbb{T}(X)$. Two elements $p^0, p^1 \in \mathbb{T}(X)$ are said to be *complements* if $p^0 \perp p^1$ and both p^0 and p^1 are maximal under the unique biggest element of the ordering (the empty condition). Note that the only complements are pairs of the form $\mathrm{dom}(p^0) = \mathrm{dom}(p^1) = \{x\}$ for some $x \in X$ and $p^0(x) = 0$ and $p^1(x) = 1$. We denote this unique x by $\rho(\{p^0, p^1\})$. Let $\bar{X} = \{\{p^0, p^1\} : p^0$ and p^1 are complements$\}$ and define a topology on $\bar{X}$ as the strongest topology such that $\{p^0, p^1\} \in \mathrm{cl}\{\{p_i^0, p_i^1\} : i < \omega\}$, where all $\{p_i^0, p_i^1\}$'s are different from $\{p^0, p^1\}$, whenever there exists $\chi : \omega + 1 \to 2$ such that

(1) $\{p_i^{\chi(i)}\}_{i<\omega} \cup \{p^{\chi(\omega+1)}\}$ has an infimum,
(2) $\{p_i^{\chi(i)}\}_{i<\omega} \cup \{p^{1-\chi(\omega+1)}\}$ has no infimum.

We claim that the bijection $\rho : \bar{X} \to X$ is a homeomorphism. Indeed, if $Z \subset \bar{X}$ is not closed, then there are $\{p^0, p^1\} \notin Z$, $\{p_i^0, p_i^1\} \in Z$, and χ such that (1) and (2) hold. But then $\rho(\{p^0, p^1\})$ will be an accumulation point of $\{\rho(\{p_i^0, p_i^1\})\}_{i<\omega}$, i.e. $\rho[Z]$ is not closed in X. Conversely, if $Y \subset X$ is not closed, then there are $x \notin Y$ and $x_i \in Y$ such that $x_i \to x$. But then conditions (1) and (2) are met for the corresponding $\{p^0, p^1\} = \rho^{-1}(x)$, $\{p_i^0, p_i^1\} = \rho^{-1}(x_i)$ and χ with $\chi(i) = 0$ and $\chi(\omega + 1) = 1$, hence $\rho^{-1}[Y]$ is not closed in $\bar{X}$. ∎

We will see now that the same does not hold for the generated complete Boolean algebra $\mathbb{B}(X)$, i.e. we will construct nonhomeomorphic X, Y such that $\mathbb{B}(X) \cong \mathbb{B}(Y)$. The arguments of the proof of Lemma 6.2 do not apply to the closed unit interval $X = [0, 1]$. But we will see that even the following holds:

THEOREM 6.5. *The algebras* $\mathbb{B}((0, 1))$ *and* $\mathbb{B}([0, 1])$ *generated from the open and the closed unit interval are isomorphic.*

Proof. For any $p \in \mathbb{B}([0, 1])$ we will construct an $r \le p$ and a $q \in \mathbb{B}((0, 1))$ such that $\mathbb{B}([0, 1])\restriction r \cong \mathbb{B}((0, 1))\restriction q$. The latter is isomorphic to $\mathbb{B}((0, 1))$ by homogeneity. We obtain a maximal antichain R (which is countable by ccc—this follows again from 2.2(a)) such that $\mathbb{B}([0, 1])\restriction r \cong \mathbb{B}((0, 1))$ for any $r \in R$. Hence $\mathbb{B}([0, 1]) \cong \prod_{r \in R} \mathbb{B}([0, 1])\restriction r \cong \mathbb{B}((0, 1))^{|R|} \cong \mathbb{B}((0, 1))$, where the last equality follows again from homogeneity.

So, let us construct the r, q as required. We can suppose that $p \in \mathbb{T}([0, 1])$.

CASE 1: $0, 1 \notin \mathrm{supp}(p)$. Let $r \le p$ with $r(0) = r(1) = 0$ and $q = p\restriction(0, 1)$.

CASE 2: $0 \notin \mathrm{supp}(p)$ and $1 \in \mathrm{supp}(p)$ (or vice versa). We find a sequence $x_i \nearrow 1$ such that $x_i \notin \mathrm{supp}(p)$ and $x_0 = 0$. Define $r \le p$ by $r(x_i) = 0$. Fix homeomorphisms

$$\psi_{2i} : (x_{2i}, x_{2i+1}) \to \left(\frac{1}{2} - \frac{1}{i+2}, \frac{1}{2} - \frac{1}{i+3}\right),$$
$$\psi_{2i+1} : (x_{2i+1}, x_{2i+2}) \to \left(\frac{1}{2} + \frac{1}{i+3}, \frac{1}{2} + \frac{1}{i+2}\right)$$

and set $\psi = \bigcup_{j<\omega} \psi_j$. Let $q(1/2) = 1$ and $q(1/2 - 1/i) = q(1/2 + 1/i) = 0$ for $i > 2$ and $q(\psi(x)) = p(x)$ for $x \in \mathrm{dom}(p) \setminus \{0,1\}$. Then r and q are as required, as witnessed by ψ.

CASE 3: $0, 1 \in \mathrm{supp}(p)$. Choose $x \in (0,1) \setminus \mathrm{dom}(p)$. Let $r \le p$ with $r(x) = 0$. Fix homeomorphisms $\psi_1 : (0,x) \to (1/2, 1)$ and $\psi_2 : (x,1) \to (0,1/2)$ and set $\psi = \psi_1 \cup \psi_2$. Let $q(1/2) = 1$ and $q(\psi(x)) = p(x)$ for $x \in \mathrm{dom}(p) \setminus \{0,1\}$. Once more, ψ witnesses that r and q are as required. ∎

Theorem 6.5 shows that topologically different spaces, like compact and noncompact spaces, can yield isomorphic Boolean algebras. This leads to the following consideration. None of the properties discussed so far distinguishes $\mathbb{B}(X)$ for other pairs of simple spaces, such as the open unit interval and its square. It is not clear to us whether the corresponding Boolean algebras are isomorphic or not (compare [Paz07]). This raises a more general question: Which properties besides those from Definition 3.1 are able to distinguish between the Boolean algebras of the form $\mathbb{B}(X)$?

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