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On the observer design problem for continuous-time switched linear systems with unknown switchings

David Gómez-Gutiérrez[a,*], Sergej Čelikovský[b], Antonio Ramírez-Treviño[c], Bernardino Castillo-Toledo[c]

[a]*Intel Labs, Intel Tecnología de México, Av. del Bosque 1001, Colonia El Bajío, 45019 Zapopan, Jalisco, Mexico*
[b]*Institute of Information Theory and Automation of the Czech Academy of Sciences, P.O. Box 18, 182 08 Prague, Czech Republic*
[c]*CINVESTAV Unidad Guadalajara, Av. del Bosque 1145, Colonia El Bajío, 45019 Zapopan, Jalisco, Mexico*

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Abstract

The observer design problem for Switched Linear Systems (*SLS*) subject to an unknown switching signal is addressed in this work. Based on known observability results for *SLS*, an appropriate *SLS* observer is proposed and its convergence is analysed showing that the corresponding estimates converge in finite-time to the *SLS* state. More precisely, the observers of the continuous state evolution and the observers of the switching signal are investigated and their convergence studied separately. The main tool to analyse the observability is the well-known geometric concept of (A, B)-invariant subspaces. The developed *SLS* observers are then applied to construct synchronized chaotic generators based on the *SLS* with chaotic behavior. Finally, an example of a non-trivial chaotic *SLS* and its detailed analysis are presented to illustrate the achieved results.

1. Introduction

The goal of this paper is to address an observer design for Switched Linear Systems (*SLS*) subject to an unknown switching signal. A *SLS* may be viewed as a subclass of the class of

*Corresponding author.
E-mail addresses: david.gomez.gutierrez@intel.com (D. Gómez-Gutiérrez), celikovs@utia.cas.cz (S. Čelikovský), art@gdl.cinvestav.mx (A. Ramírez-Treviño), toledo@gdl.cinvestav.mx (B. Castillo-Toledo).

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Hybrid Dynamical Systems (*HDS*) formed by a collection of Linear Systems (*LS*) together with a time dependent exogenous switching signal. The system state evolution is then uniquely determined by a given initial condition and by the mentioned switching signal determining at each time instant the unique active linear system driving the state.

The study of the fundamental properties of *SLS* has received a great deal of attention during the last decade. In particular, *SLS* observability has been thoroughly analysed depending on whether the switching signal is known or unknown.

On one hand, if the switching signal is known, the focus has been on determining the continuous state after a finite number of switchings [1–4]. It has been shown that the observability of each *LS* is not a necessary condition [1,4]. On the other hand, if the switching signal is unknown (for instance when switchings may occur in an unpredictable way), the observability of the continuous phase and the observability of the switching signal [5–8] were shown to be mutually independent properties [7]. Moreover, if the switching signal is unknown, the observability of each *LS* is not sufficient for the overall *SLS* observability [7]. As a matter of fact, the unknown switching signal computation requires the so-called distinguishability property [5] enabling to detect the current evolving *LS* based on the input–output information only. Last but not least, for the unknown switching signal the controlled inputs also play a central role leading to two different distinguishability notions [9]: (i) distinguishability for every nonzero state trajectory [10] and (ii) distinguishability for "almost every" input [6–8]. The latter case, as expected, requires less restrictive conditions to be fulfilled, in particular, the observability of each *LS* is not required.

To be more specific, the unknown switching signal is usually estimated using the so-called "location observer" based on the *SLS* continuous-time input–output information. In [11], a multi-observer structure based on Luenberger observers together with a residual generator (similar to those used in fault detection) is used as a location observer. The reported results, however, require a careful selection of the threshold for the residue generation. Moreover, as noticed in [11] it can occur that the residue remains true after switching to another subsystem, thus missing the detection of switching. In [12], the location observer algorithm that requires the numerical computation of derivatives of inputs and outputs has been proposed as the location observer. Nevertheless, the analysis presented there is restricted to monovariable *SLS*. In [13], the location observer uses a super twisting based step-by-step observer for switched nonlinear systems that can be transformed into the normal form.

However the focus is on autonomous systems and the super twisting based step-by-step observer requires the knowledge of the bound of the state velocity. In [14] the location observer is formed by a set of Luenberger observers with an associated super twisting based differentiator [15] used to obtain the exact error signal which updates the estimate. Using the results on distinguishability for every nonzero state trajectory, the authors of [14] showed the convergence of the observer. Unfortunately, this distinguishability notion requires the observability of each *LS*.

Furthermore, multi-observer structures have been proposed in the framework of supervisory control of a class of *SLS* composed of a *LS* with an unknown parameter [16–18] and in adaptive control [19,20] where the aim is to decide, based on the size of the output estimation error, which candidate controller (from a bank of controllers) should be used in the feedback loop with the process. Unfortunately, as noticed in [21], this scheme cannot be used to recover the switching signal because the smallness of the output estimation error is not sufficient to infer the evolving *LS* [21].

The observability and the observer design problem have also been addressed in discrete-time *SLS*, see e.g. [9,22–24]. Nevertheless, as pointed out in [7], continuous and discrete–time *SLS* have notable differences requiring these two classes of systems to be studied separately.

In this framework, the aim of this paper is twofold. First, to introduce a finite-time *SLS* observer guaranteeing that the observers and controllers can be designed separately. To the best of our knowledge, no result exists allowing such separation using Luenberger multi-observers. Such a task will be addressed for a rather general class of *SLS* for which the observability of each *LS* is not required. Since the observability of the switching signal is independent from the observability of the continuous state, the proposed observer is capable of estimating the continuous state and the switching signal of the *SLS* independently. The corresponding convergence analysis is based on the global finite-time stability [25], higher-order sliding modes techniques [26] and the *SLS* observability results [7,27].

Secondly, based on the developed observers, this paper addresses the design of a synchronization scheme for several chaotic *SLS*. As a matter of fact, while dealing successfully with the issue of the unknown switching signal detection, the paper opens new possibilities for chaotic *SLS* synchronization. This is an important task enabling the possible use of *SLS* with chaotic behavior in various chaotic encryption schemes, such as chaos shift keying, in which different chaotic attractors are associated to different digital symbols, and two-channel transmission, in which the output of the chaotic *SLS* is sent to the receiver through the first channel, while the encrypted message (encrypted using the chaotic state) is conveyed by the second channel, see [28,29] and further references within there. To the best of our knowledge, even though it has been shown that new and novel chaotic behavior can be generated by *SLS* (see e.g. [30–35]), the synchronization of general classes of *SLS* with chaotic behavior (e.g. those presented in [32,31,36]) has not been reported yet in the literature.

This work is organized as follows. Section 2 introduces the basic notation and preliminaries while Section 3 reviews the observability results of *SLS* needed later on. The main paper results are presented in Sections 4 and 5. Section 4 is devoted to the *SLS* observer design and its convergence analysis while in Section 5 the later results are used it to construct an efficient synchronization scheme of *SLS* with chaotic behavior. The final section draws some conclusions and provides some outlooks for the related future research.

2. Notation and preliminaries

2.1. Linear systems

A *LS* $\Sigma(A,B,C)$ or simply Σ is represented by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

where $x \in \mathcal{X}\mathbb{R}^n$ is the state vector, $u \in \mathcal{U} = \mathbb{R}^p$ is the control input, $y \in \mathcal{Y} = \mathbb{R}^q$ the output signal and A, B, C are constant matrices of appropriate dimensions. The input function space is denoted by $\mathcal{U}_f$ and is considered to be $L_p(\mathcal{U})$. Through the paper $\mathcal{B}$ stands for $\Im B$ and $\mathcal{K}$ for $\ker C$. A subspace $\mathcal{S} \subset \mathcal{X}$ is called A-invariant if $A\mathcal{S} \subset \mathcal{S}$ and (A,B)-invariant if there exists a state feedback $u(t) = Fx(t)$ such that $(A+BF)\mathcal{S} \subset \mathcal{S}$ or equivalently if $A\mathcal{S} \subset \mathcal{S} + \mathcal{B}$. The class of (A, B)-invariant subspaces contained in a subspace $\mathcal{L} \subset \mathcal{X}$ is denoted by $\Im(A,B;\mathcal{L})$. It is well known that this class is closed under addition, hence, there exists a supremal element denoted $\sup \Im(A,B;\mathcal{L})$. An algorithm for computing $\sup \Im(A,B;\mathcal{L})$ is presented in [37].

2.2. Switched linear systems

A *SLS* $\Sigma_{\sigma(t)} = \langle \mathcal{F}, \sigma \rangle$ is a *HDS* where $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_m\}$ is a collection of *LS* and $\sigma : [t_0, \infty) \rightarrow \{1, \ldots, m\}$ is a switching signal determining, at each time instant, the evolving *LS* $\Sigma_i \in \mathcal{F}$. The state equation of the *SLS* is

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad y(t) = C_{\sigma(t)}x(t), \quad x(t_0) = x_0, \quad \sigma(t_0) = \sigma_0. \tag{2}$$

where $\sigma(t)$ is assumed to be unknown and generated exogenously. If $\sigma(t) = i$, $i \in \{1, \ldots, m\}$ then the evolving *LS* is $\Sigma_i(A_i, B_i, C_i)$.

The notation, $x_i(t, x(\tau_1), u_{[\tau_1,\tau_2]})$ stands for the state trajectory $x(t)$ obtained when $\sigma(t) = i$, $\forall t \in [\tau_1, \tau_2]$ and the input $u_{[\tau_1,\tau_2]}$ (i.e. the restriction of the functions $u(t)$ to $[\tau_1, \tau_2]$) is applied to the system with initial condition $x(\tau_1) \in \mathcal{X}$, i.e.

$$x_i(t, x(\tau_1), u_{[\tau_1,\tau_2]}) = e^{A_i(t-\tau_1)}x(\tau_1) + \int_{\tau_1}^{t} e^{A_i(t-\varsigma)}B_i u(\varsigma)\, d\varsigma, \quad t \in [\tau_1, \tau_2]. \tag{3}$$

In a similar way, the notation $y_i(t, x(\tau_1), u_{[\tau_1,\tau_2]})$ stands for the output of Eq. (3).

Through this work we make the following assumptions on Eq. (2).

2.2.1. Assumptions

The initial condition of the *SLS* (2) is bounded, i.e. $\| x_0 \| < \delta$ with a known constant δ. A minimum dwell time in each *LS* is assumed, i.e. if t_{k-1} and t_k are two consecutive switching times, then $t_k - t_{k-1} > \tau_{d_k}$. However, only the dwell time for the first switching time τ_{d_1} is assumed to be known. A maximum dwell time is not considered. For simplicity, the state $x(t)$ in Eq. (2) is assumed to be continuous, i.e. at the switching time t_l, $x(t_l) = x(t_l^-)$. Zeno behavior, which is characterized by infinite switching in a finite-time interval, is excluded.

3. Observability of switched linear systems

Basic results on observability of *SLS* needed later on are reviewed here, see [5–7,27] for more details. These results are based on the geometric approach. An interested reader can find their extension to the perturbed case in [8] while the discrete-time case is treated in [9].

Definition 1. The continuous state trajectory $x(t)$ (resp. the switching signal $\sigma(t)$) of the *SLS* (2) is said to be observable if there exists a finite-time τsuch that the knowledge on the structure (2) and the input $u_{[t_0,\tau]}$ and output $y_{[t_0,\tau]}$ over a finite-time interval $t \in [t_0, \tau]$ suffices to uniquely determine the signal $x(t)$ (resp. the signal $\sigma(t)$).

SLS observability problem with unknown switching signal is critically related to the so-called distinguishability concept defined as follows.

Definition 2. Let $u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]}$ be a measurable input–output behavior of the *SLS* $\Sigma_{\sigma(t)} = \langle \mathcal{F}, \sigma \rangle$ in the time interval $[\tau_1, \tau_2]$ and let $\Sigma_i, \Sigma_j \in \mathcal{F}$, then the *LS* Σ_i and Σ_j are said to be $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-indistinguishable, if there exist state trajectories (not necessarily starting from the same initial condition) $x_i(t, x(\tau_1), u_{[\tau_1,\tau_2]})$ of Σ_i and $x_j(t, x'(\tau_1), u_{[\tau_1,\tau_2]})$ of Σ_j, such that

$$y_i(t, x(\tau_1), u_{[\tau_1,\tau_2]}) = y_j(t, x'(\tau_1), u_{[\tau_1,\tau_2]}) = y_{[\tau_1,\tau_2]}.$$

Otherwise, Σ_i and Σ_j are said to be $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-distinguishable. The indistinguishability subspace, $\mathcal{W}_{ij} \subseteq \mathbb{R}^{2n}$ of Σ_i, Σ_j is defined as

$$\mathcal{W}_{ij} = \left\{ \begin{bmatrix} x_0^T \\ x_0'^T \end{bmatrix}^T : \exists u(t), \tau > t_0 \text{ such that } y_i(t, x_0, u_{[t_0,\tau]}) = y_j(t, x_0', u_{[t_0,\tau]}) \right\}. \tag{4}$$

In [27], it has been shown that the indistinguishability subspace, $\mathcal{W}_{ij}$, is equal to the supremal (A_{ij}, B_{ij})-invariant subspace contained in $\mathcal{K}_{ij} = \ker C_{ij}$ (see also [9]), denoted as sup $\Im(A_{ij}, B_{ij}; \mathcal{K}_{ij})$, where

$$A_{ij} = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \quad B_{ij} = \begin{bmatrix} B_i \\ B_j \end{bmatrix}, \quad C_{ij} = \begin{bmatrix} C_i & -C_j \end{bmatrix}. \tag{5}$$

This implies that the extended *LS* $\Sigma_{ij}(A_{ij}, B_{ij}, C_{ij})$ with matrices (5) gives zero output in the interval $[\tau_1, \tau_2]$. Thus, the inputs for which two *LS* may become indistinguishable are closely related to the systems zeros of $\Sigma_{ij}(A_{ij}, B_{ij}, C_{ij})$ [8].

Clearly, the *LS* Σ_i and Σ_j are distinguishable for every nonzero state trajectory iff $\mathcal{W}_{ij} = 0$ [27].

However, since the *LS* becomes indistinguishable for particular inputs [10,27], less restrictive conditions can be obtained by studying the observability for almost every input [7,9].

Definition 3. On a functional space a property is said to hold "almost everywhere" or "for almost every" in the sense of prevalence [38] if the set of exceptions is a Haar null set, also known as a "shy set". For instance, every proper subspace of the function space is a shy set [38], this property is used in [8] to show that the property holds for almost every input, by showing that it only fails to hold on a proper subspace of the input space.

Although we do not report here the formal definition of a shy set $S \subset V$, the following important properties give an insight on its meaning [38]:

- A set $S \subset \mathbb{R}^n$ is shy if and only if it has Lebesgue measure zero. Thus, the concept of shy set is a natural extension of the concept of null set because both concepts coincides in $\mathbb{R}^n$.
- A shy set has no interior. Thus, "almost every" implies dense.
- Every countable set in V and every proper subspace of V are shy with respect to V.

Definition 4. The continuous state trajectory $x(t)$ (resp. switching signal $\sigma(t)$) of the *SLS* (2) is said to be observable for almost every input if almost every input ensure the computation of the continuous state $x(t)$ (resp. switching signal $\sigma(t)$).

Clearly, an input $u_{[\tau_1,\tau_2]}$ can be used to make the *LS* $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-distinguishable iff $u_{[\tau_1,\tau_2]}$ drives every state trajectory of Σ_{ij} outside the indistinguishability subspace. Then, it follows that this holds iff $\mathcal{B}_{ij} \nsubseteq \mathcal{W}_{ij}$ [27]. Now, due to the superposition property of the extended *LS* Σ, the set of exceptions,

$$\left\{ u_{[t_0,\tau]} : \exists\ x_0, x_0', y_i(t, x_0, u_{[t_0,\tau]}) = y_j\left(t, x_0', u_{[t_0,\tau]}\right) \right\} \tag{6}$$

i.e. the set of inputs for which the evolving *LS* cannot be determined, is a subspace of $\mathcal{U}_f$, in which $u(t)$ takes values. Since $\mathcal{B}_{ij} \nsubseteq \mathcal{W}_{ij}$ implies the existence of a smooth input steering the state trajectory outside $\mathcal{W}_{ij}$, then Eq. (6) is a proper subspace of $\mathcal{U}_f$, and thus a shy set with respect to

$\mathcal{U}_f$ [38]. Hence, almost every input $u_{[t_0,\tau]}$ guarantees the $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-distinguishability between Σ_i and Σ_j, under this situation the *LS* are simply said to be distinguishable for almost every input. Thus, we can state the following proposition.

Proposition 5. *If $\mathcal{B}_{ij} \nsubseteq \mathcal{W}_{ij}$ then for almost every input and for every nonzero time interval $[\tau_1, \tau_2]$, $y_i(t) \neq y_j(t)$ holds almost everywhere in $[\tau_1, \tau_2]$.*

Thus, if the *LS* are distinguishable for almost every input the equality $y_i(\bar{t}) = y_j(\bar{t})$ cannot hold during a nonzero time interval, i.e. $y_i(\bar{t}) = y_j(\bar{t})$ then $y_i(\bar{t}^+) \neq y_j(\bar{t}^+)$.

If the *LS* are observable and distinguishable, then the continuous state can also be estimated. However, if every $[x^T \ x'^T]^T \in \mathcal{W}_{ij}$ is of the form $x = x'$ (i.e. $\mathcal{W}_{ij}$ is symmetric), then the continuous state trajectory for which the *LS* become indistinguishable is the same in both systems and the continuous state can be estimated even though the *LS* are indistinguishable, clearly, in this case the observability of each *LS* is required. From the previous discussion, the next theorem follows easily.

Theorem 6 (*Gómez-Gutiérrez et al. [10], Babaali and Pappas [7]*). *Let $\Sigma_{\sigma(t)} = \langle \mathcal{F}, \sigma \rangle$ be a SLS with no maximum dwell time set. Then*

1. *The switching signal $\sigma(t)$ of $\Sigma_{\sigma(t)}$ is observable for almost every input iff $\forall \Sigma_i, \Sigma_j \in \mathcal{F}$, $i \neq j$ Σ_i and Σ_j are distinguishable for almost every input, i.e. $\mathcal{B}_{ij} \nsubseteq \mathcal{W}_{ij}$.*
2. *The continuous state $x(t)$ of $\Sigma_{\sigma(t)}$ is observable for almost every input iff each LS $\Sigma_i \in \mathcal{F}$ is observable and $\forall \Sigma_i, \Sigma_j \in \mathcal{F}$ $i \neq j$, either $\mathcal{B}_{ij} \nsubseteq \mathcal{W}_{ij}$ or $\mathcal{W}_{ij}$ is symmetric.*

3.1. Illustrative example

Consider, the *SLS* $\Sigma_{\sigma(t)} = \langle \mathcal{F}, \sigma \rangle$ with $\mathcal{F} = \{\Sigma_1(A_1, B_1, C_1),\ \Sigma_2(A_2, B_2, C_2)\}$ with $\sigma(t)$ an exogenous switching signal, where Σ_1 and Σ_2 with matrices

	A	*B*	*C*
Σ_1	$\begin{bmatrix} 0.9 & 15 & 0 \\ -15 & 0.9 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T$
Σ_2	$\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -1 & 20 \\ 0 & -20 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T$

do not have property being observable for every nonzero input as $\mathcal{W}_{12} \neq 0$. For instance, since $\lambda = -\frac{1}{2}$ is a system zero of the extended *LS* Σ_{12} then, when Σ_1 is evolving and the input $u(t) = e^{-(1/2)t}$ is applied to the initial condition $x_0 = [0\ 0\ -1]^T$ produces the same output $y(t) = -e^{-(1/2)t}$ as when Σ_2 is evolving and the input $u(t) = e^{-(1/2)t}$ is applied to the initial condition $x_0=[-1\ 0\ 0]^T$. Thus, Σ_1 and Σ_2 are $(e^{-(1/2)t}, -e^{-(1/2)t})-$ indistinguishable. However, Σ_1 and Σ_2 are $\left(-\frac{3}{2}e^{-t}, e^{-t}\right)$-distinguishable because the input–output behavior $\left(-\frac{3}{2}e^{-t}, e^{-t}\right)$ can only be produced by Σ_1 when the input $u(t)=-\frac{3}{2}e^{-t}$ is applied to the initial condition $x_0=[0\ 0\ 1]^T$.

However, since $\mathcal{B}_{12}\nsubseteq\mathcal{W}_{12}$, the *SLS* is observable for almost every input. In particular, Σ_1 and Σ_2 cannot become indistinguishable if the input $u(t)=1$ is applied, because $\lambda=0$ is not a system zero of the extended system Σ_{12} [8].

4. Observer design

The proposed *SLS* observer uses a bank of extended finite-observers designed one for each *LS* in the *SLS*. Thus, let us first introduce the extended finite-time observers for *LS*.

4.1. The extended finite-time observer for multivariable LS

The finite-time observer for each *LS* is based on the result of global finite-time stability of scalar systems presented in [25], this observer is described in the following for the multivariable case. This finite-time observer is preferred because it provides a simple way to choose the observer's gains. However, different gain selection can be used following the results on high-order sliding modes presented in [26].

Let $r=n-\dim(\mathcal{N})$, where $\mathcal{N}$ is the unobservable subspace of the *LS*, and assume that C is composed of linearly independent rows, then, by a similarity transformation T, Σ can be transformed into the observable/unobservable form with the observable subsystem expressed in a multivariable observer form. This transformation leads to the similar system

$$\dot{\overline{x}}=\begin{bmatrix}\overline{A}_o & 0\\ \overline{A}_{21} & \overline{A}_{\overline{o}}\end{bmatrix}, \quad \begin{bmatrix}\overline{x}_o\\ \overline{x}_{\overline{o}}\end{bmatrix}+\begin{bmatrix}\overline{B}_o\\ \overline{B}_{\overline{o}}\end{bmatrix}u, \quad \overline{y}=\begin{bmatrix}\overline{C}_o & 0\end{bmatrix}\begin{bmatrix}\overline{x}_o\\ \overline{x}_{\overline{o}}\end{bmatrix} \tag{7}$$

where $x=T\overline{x}$ and the state vector $\overline{x}_o=[\overline{x}_1\cdots\overline{x}_r]^T$ represents the observable variables. The subsystem $\overline{\Sigma}_o(\overline{A}_o, \overline{B}_o, \overline{C}_o)$ is observable [39,40] with matrices

$$\overline{A}_o=\begin{bmatrix}\mathcal{A}_{r_1} & \cdots & \mathcal{O}_{r_1,r_q}\\ \vdots & \ddots & \vdots\\ \mathcal{O}_{r_q,r_1} & \cdots & \mathcal{A}_{r_q}\end{bmatrix}, \quad \overline{C}_o=\begin{bmatrix}\mathcal{C}_{1,r_1} & \cdots & 0_{1,r_q}\\ \vdots & \ddots & \vdots\\ \mathcal{O}_{1,r_1} & \cdots & \mathcal{C}_{1,r_q}\end{bmatrix}$$

where $0_{1,r_i}$ is a $1\times r_i$ zero matrix, $\mathcal{O}_{r_i,r_j}$ is a $r_i\times r_j$ zero matrix with possible nonzero first column and $\mathcal{A}_{r_i}$ is a $r_i\times r_i$ matrix, $\mathcal{C}_{1,r_i}$ is a $1\times r_i$ matrix of the form

$$\mathcal{A}_{r_i}=\begin{bmatrix}* & 1 & 0 & \cdots & 0\\ * & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & 0 & 0 & \cdots & 1\\ * & 0 & 0 & \cdots & 0\end{bmatrix}, \quad \mathcal{C}_{1,r_i}=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$$

where the entries marked by "*" represent possible nonzero values, The $\overline{B}_o$ matrix has no particular structure. Notice that, $\overline{\Sigma}_o$ is formed by q single output subsystems of dimensions $r_1,\ldots,r_q$ (with $\sum_{i=1}^{q} r_i=r$), coupled only by the measured variables. The similarity transformation leading to the multivariable observer form can be found in [39,40].

Thus, an observer is designed for each block $k=1,\ldots,q$, with an additional variable ξ, which is used as a measurement of the effort required by the observer to maintain a zero output estimation error, this signal will be useful in the detection of the evolving *LS*. The observer for the k-th block is given as follows:

$$\begin{aligned}
\dot{\tilde{x}}_1^k &= a_1^k(y) + \tilde{x}_2^k + \rho k_1 \lceil e_1^k \rfloor^{\alpha_1} + \overline{B}_1^k u \\
\dot{\tilde{x}}_2^k &= a_2^k(y) + \tilde{x}_3^k + \rho^2 k_2 \lceil e_1^k \rfloor^{\alpha_2} + \overline{B}_2^k u \vdots \\
\dot{\tilde{x}}_{r_k}^k &= a_{r_k}^k(y) + \rho^{r_k} k_{r_k} \lceil e_1^k \rfloor^{\alpha_{r_k}} + \overline{B}_{r_k}^k u + \xi^k \\
\dot{\xi}^k &= -\rho^{r_k+1} k_{r_k+1} \lceil e_1^k \rfloor^{\alpha_{r_k+1}}
\end{aligned} \tag{8}$$

where $e_1^k = \overline{x}_1^k - \tilde{x}_1^k$, $\lceil e_1^k \rfloor^{\alpha_j} = |e_1^k|^{\alpha_j} \operatorname{sign}(e_1^k)$ and $a_j^i(y),\ j=1,\ldots,r_i$ is a known linear combination of the measured signal y, this function is associated with the entries marked by "$*$" shown in $\overline{A}_o$. Notice that, since the R subsystems are only coupled by the measured variables, the error dynamics of the subsystems are independent from each other and is given by

$$\begin{aligned}
\dot{e}_1^k &= e_2^k - \rho k_1 \lceil e_1^k \rfloor^{\alpha_1} \\
\dot{e}_2^k &= e_3^k - \rho^2 k_2 \lceil e_1^k \rfloor^{\alpha_2} \\
&\ \ \vdots \\
\dot{e}_{r_k}^k &= \xi^k - \rho^{r_k} k_{r_k} \lceil e_1^k \rfloor^{\alpha_{r_k}} \\
\dot{\xi}^k &= -\rho^{r_k+1} k_{r_k+1} \lceil e_1^k \rfloor^{\alpha_{r_k+1}}
\end{aligned} \tag{9}$$

which, according to [25, Corollary 2], ρ, k_i and α_i can be designed such that the estimation error of each block $\epsilon^i = [e_1^i \cdots e_{r_i}^i\ \xi^i]^T$ is globally finite-time stable. Notice that, Eq. (9) also coincides with the differentiation error of the high-order sliding-mode differentiator [26] when presented in the non-recursive form. Thus, a different choice of the observers parameters ρ, k_i and α_i for a finite-time convergence can be taken from [26].

Proposition 7. *Let the initial conditions of the observer (8) be taken as zero and let the continuous initial condition of the SLS (2) be bounded by δ, i.e. $\|x_0\| < \delta$, with a known constant δ, as in Assumption 2.2.1. Then for every constant τ_k the gains of Eq. (8) can be designed in such a way that the estimation error (9) convergences to the origin with upper convergence bound less than τ_k.*

Proof. Consider the error dynamics given in Eq. (9). Take $\rho \geq 1$ and consider the time-scaling $\tilde{t} = t\rho$ together with the coordinate change $\epsilon^k = P\tilde{\epsilon}^k$ with $P = \operatorname{diag}(1, \rho, \ldots, \rho^{r_k})$ and $\tilde{\epsilon}^k = [\tilde{e}_1^k \cdots \tilde{e}_{r_k}^k\ \tilde{\xi}^k]$. These transformations and time scaling take Eq. (9) into the following form:

$$\begin{aligned}
\frac{d(\tilde{e}_1^k)}{d\tilde{t}} &= \tilde{e}_2^k - k_1 \lceil \tilde{e}_1^k \rfloor^{\alpha_1} \\
&\ \ \vdots \\
\frac{d(\tilde{e}_{r_k}^k)}{d\tilde{t}} &= \tilde{\xi}^k - k_{r_k} \lceil \tilde{e}_1^k \rfloor^{\alpha_{r_k}} \\
\frac{d(\tilde{\xi}^k)}{d\tilde{t}} &= -k_{r_k+1} \lceil \tilde{e}_1^k \rfloor^{\alpha_{r_k+1}}
\end{aligned} \tag{10}$$

which does not depend on ρ. Therefore, due to the finite-time stability of Eq. (10), $\forall \delta > 0$, exists $\tilde{\tau}_k$ such that it holds

$$\tilde{\epsilon}(\tilde{t}, \tilde{\epsilon}_0) = 0, \quad \forall \tilde{\epsilon}_0 \text{ such that } \|\tilde{\epsilon}_0\| < \delta \quad \text{and} \quad \forall \tilde{t} \geq \tilde{\tau}_k$$

where $\tilde{\epsilon}_0 = \tilde{\epsilon}(t_0)$. Going back to original coordinates $\epsilon$and real time t the above implies that

$$\epsilon(t, \epsilon_0) = 0, \quad \forall \epsilon_0 \text{ such that } \|\epsilon_0\| < \delta \quad \text{and} \quad \forall t \geq \tilde{\tau}_k/\rho.$$

Indeed, the above implication is correct as the inequality $\|\epsilon_0\| < \delta$ clearly implies that $\|\tilde{\epsilon}_0\| < \delta$ due to the straightforward inequality

$$\|\tilde{\epsilon}\| \leq \|\epsilon\|, \quad \forall \rho \geq 1.$$

Therefore $\forall \delta, \tau_k > 0$ there exists $\rho(\delta, \tau_k)$ such that $\epsilon(t, \epsilon_0) = 0$, $\forall \epsilon_0$ such that $\|\epsilon_0\| < \delta$ and $\forall t \geq \tau_k$. □

So, one really has to make the assumption that the initial conditions are bounded by some $\delta > 0$, as in Assumptions 2.2.1, but for every bound δ and every time $\tau_k > 0$ there exist gains such that the error dynamics (9) goes to zero in finite time less than τ_k. Another alternative is to design the individual observers such that the convergence error coincides with the error of the uniformly convergent robust exact differentiator proposed in [41]. The main advantage of that algorithm is that it has a fixed time-convergence bound that is independent of the initial condition, where a similar argument can be made to show that an uniformly convergent observer can also be designed with upper convergence bound less than τ_k.

Notice that, whenever $\epsilon(t) = [e_y^T(t)\ \xi^T(t)]^T = 0$ the observer becomes an exact copy of the observable subsystem of its associated *LS* producing the same input–output information as the evolving *LS*. We next show that under distinguishability conditions the *i*-th observer will give $\epsilon(t) = 0$ whenever Σ_i is evolving and $\epsilon(t) \neq 0$ otherwise, thus allowing to infer the evolving *LS*.

Roughly speaking, the additional variable ξ can be seen as a measure of the effort of the observer to maintain a zero output estimation error. Thus, when Σ_i is evolving the observer will converge to $e_y^T(t) = 0$ with $\xi^T(t) = 0$. Furthermore, it is possible to compute the values that the *i*-th observer will take under the evolution of Σ_j. However, for the sake of brevity such analysis is not presented here. We would like to highlight that by simple using multi-observers structures of finite-time observers, without this additional variable, more than one observer may give zero output estimation error, provided that they are robust observers.

Thus, using the distinguishability property the evolving system can be inferred from the signal $\epsilon(t)$ of each observer, as stated below.

4.2. SLS observer

Let $\Sigma_{\sigma(t)} = \langle \mathcal{F}, \sigma \rangle$ be a *SLS* with minimum dwell time τ_{d_1} for the first switching time then an observer $\hat{\Sigma}_{\hat{\sigma}(t)}$ for $\Sigma_{\sigma(t)}$ is designed as depicted in Fig. 1, where

- A global finite-time observer is designed for the observable subsystem of each *LS* $\Sigma \in \mathcal{F}$ with an extra variable ξ (as described in Section 4.1) and with upper time convergence bound $\tau_k \ll \tau_{d_1}$. The estimation of the observable variables of Σ_k is given by $\tilde{x}_{o,k}$.
- If for the *LS* observer $\tilde{\Sigma}_k$, $e_y = y(t) - \tilde{y}(t) = 0$ and $\xi(t) = 0$ for all $t \in [\tau, \tau + \varepsilon]$ with $\varepsilon \approx 0$ and some time τ then as long as $\epsilon(t) = [e_y^T\ \xi^T(t)]^T = 0$, $\hat{\sigma}(t) = k$.
- If $\hat{\sigma}(\bar{t}) = k$ then the next switching time is detected as the minimum time $t_l > \bar{t}$ satisfying $\epsilon_k(t_l) = 0$ and $\epsilon_k(t_l^+) \neq 0$.

- If $\hat{\sigma}(t)=i$ and the switching time t_l is detected, then the variables of the k-th observer, $k=1, \ldots, m$, are reinitialized, $\tilde{x}_k(t_l)$ as $T_j^{-1}T_i\tilde{x}_i(t_l)$ and ξ_k as zero,[1] where $\tilde{x}_k=[\tilde{x}_{o,k}^T \ \ \tilde{x}_{\overline{o},k}^T]^T$ with $\tilde{x}_{\overline{o},k}=0$.
- If for the *LS* observer $\tilde{\Sigma}_k$, $\epsilon(t)=0$ then $\hat{x}(t)=T_k\tilde{x}_k(t)$.

In Fig. 1 the *SLS* is depicted for the estimation of both, the continuous state and the switching signal. In the following we show how this scheme can be used for the separate estimation of the continuous state and the switching signal.

4.3. Detection of the evolving linear system

Notice that when using the observability for almost every input, the observability of each *LS* is not required for the distinguishability and Σ_i and Σ_j are $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-distinguishable iff Σ_i and $\overline{\Sigma}_{o,j}$ are $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-distinguishable with $\overline{\Sigma}_{o,j}$ the observable subsystem of Σ_j (see Eq. (7)). Thus, we can establish the following lemma.

Lemma 8. *Let $\Sigma_{\sigma(t)}$ be a SLS and $\hat{\Sigma}_{\hat{\sigma}(t)}$ be its observer designed in such a way that each finite-time observer has an upper convergence bound τ_k and let $\epsilon_j(t)$ be the signal $\epsilon(t)=[e_y^T(t)\ \xi^T(t)]^T$ of the observer associated to the observable subsystem ofΣ_j. Then, if $\sigma(t)=i$, $\forall t\in[t_0,t_1)$, $t_1>\tau_k>t_0$, and Σ_i and Σ_j are $(u_{[\tau_k,t_1)}, y_{[\tau_k,t_1)})$-distinguishable then $\epsilon_j(t)=[e_y^T(t)\ \xi^T(t)]^T\neq 0$ almost everywhere in $[\tau_k,t_1)$. On the contrary, if Σ_i and Σ_j are $(u_{[\tau_k,t_1)}, y_{[\tau_k,t_1)})$-indistinguishable then $\forall t\in[\tau_k,t_1)$, $\epsilon_j(t)=0$.*

Proof. Let $\overline{\Sigma}_{o,j}(\overline{A}_{o,j},\overline{B}_{o,j},\overline{C}_{o,j})$ be the observable subsystem of Σ_j and assume by contradiction that $\epsilon_j(t)=0$ for all $t\in[\tau_k,t_1)$ which implies that the finite-time observer associated to Σ_j becomes a copy of $\overline{\Sigma}_{o,j}$ and its observable variables $\tilde{x}_{o,j}(t)$ are such that $y_i(t)-y_j(t)\ =C_ix(t)-C_o ,j\tilde{x}_{o,j}(t)=0$ for all $t\in[\tau_k,t_1)$. Therefore Σ_i and $\Sigma_{o,j}$ as well as Σ_i and Σ_j are $(u_{[\tau_k,t_1)}, y_{[\tau_k,t_1)})$-indistinguishable, which contradicts the initial assumption. Thus, if Σ_i is evolving then the observer associated to the observable subsystem of Σ_j gives $\epsilon(t)\neq 0$ almost everywhere in $[\tau_k,t_1)$, according to Proposition 5. The second part of the theorem follows trivially from Eq. (9). □

Since according to Theorem 6 the observability of the switching signal for almost every input requires the *LS* in $\mathcal{F}$ to be distinguishable from each other, then the evolving *LS* can be detected after the finite-time τ_k, because only the observer associated with the evolving *LS* gives $\epsilon(t)=[e_y^T(t)\ \xi^T(t)]^T=0$ for all $t\in[\tau_k,t_1)$, as formally stated in the following proposition.

Proposition 9. *Let $\Sigma_{\sigma(t)}$ be a SLS and let $\hat{\Sigma}_{\hat{\sigma}(t)}$ be the proposed observer. Then if the switching signal of $\Sigma_{\sigma(t)}$ is observable for almost every input then only the observer of Σ_i will give $\epsilon(t)=0$ for all $t\in[\tau_k,t_1]$. Furthermore only $\epsilon_i(t)$ can be zero in a nonzero interval, and the index of the evolving system can be obtained by the only observer giving $\epsilon(t)=0$ for all $t\in[\tau_k,\tau_k+\varepsilon]$, with $\varepsilon\approx 0$.*

Proof. The proof follows from Lemma 8 and Proposition 5. □

[1] T_k, $k\in\{1,\ldots,m\}$ is the similarity transformation taking Σ_k into the observable/unobservable form with the observable subsystem represented in the multivariable observer form (for the multivariable observer form see [40]).

4.4. Detection of the switching times and state reinitialization of the observers

In this subsection we show how each switching can be detected and how to completely estimate the switching signal.

Let us assume that the *LS* Σ_i is evolving and detected by the proposed observer and that at time t_1 a switching occurs from Σ_i to Σ_j. Then, the output of the evolving state trajectory is $y(t)=y_i(t,x_0,u_{[t_0,t_1)})$for $t\in[t_0,t_1)$ and $y(t)=y_j(t,x(t_1),u_{[t_1,t_2)})$ for $t\in[t_1,t_2)$. Thus, the switching time t_1 can be detected iff

$$y_j(t,x(t_1),u_{[t_1,t_2)})\neq y_i(t,x(t_1),u_{[t_1,t_2)}). \tag{11}$$

Notice that, if the switching signal is observable for almost every input, then Σ_i is detected by the *SLS* observer before the first switching time t_1 and Eq. (11) holds because Σ_i and Σ_j are distinguishable. Thus, after the switching time, Σ_i cannot produce the same input–output information than Σ_j, and according to Lemma 8 and Proposition 5 Σ_i can no longer give $\epsilon(t)=0$ during a nonzero interval. Thus, the switching time t_1 can be detected as established in the following lemma.

Lemma 10. *Let $\Sigma_{\sigma(t)}$ be a SLS and let $\hat{\Sigma}_{\hat{\sigma}(t)}$ be the proposed observer and let $\epsilon_i(t)=$ be the signal $\epsilon(t)=[e_y^T(t)\ \xi^T(t)]^T$ of the observer associated to the observable subsystem of Σ_i. Then if the switching signal of $\Sigma_{\sigma(t)}$ is observable for almost every input then the switching time is detected as the time $t>\tau_d$ for which $\epsilon_i(t)$ is no longer zero. In fact, $\epsilon_i(t)\neq 0$ almost everywhere in $[t_1,t_2)$. In a similar way any subsequent switching time can be detected.*

4.5. Estimation of the continuous state and estimation of the switching signal independently

Recall that, under the observability for almost every input, the switching signal is observable even if each individual *LS* is unobservable. However, since the continuous state cannot be completely estimated, in general, the state of the observers cannot be reinitialized as proposed and the switching signal can only be estimated after the convergence of the observers. Thus, in general, a known minimum dwell time between each switching is required to detect the evolving *LS* before another switching occurs. Fortunately, if a switching occurs from Σ_i to Σ_j and $\mathcal{N}_i=\mathcal{N}_j$ where $\mathcal{N}_k$, is the unobservable subspace of Σ_k, $k=i,j$, then at the switching time the observable variables of the *LS* Σ_j can be completely determined from the observable variables of the *LS* Σ_i. Thus, the state of the individual observers can be reinitialized as proposed, hence the state of the switching signal can be computed right after a switching occurs and the switching signal can be completely estimated in finite-time.

The estimation of the switching signal independently from the estimation of the continuous state is formally stated below.

Theorem 11. *Let $\Sigma_{\sigma(t)}$ be a SLS, $\hat{\Sigma}_{\hat{\sigma}(t)}$ be the proposed observer and let $\mathcal{N}_i=\mathcal{N}_j\ \forall\Sigma_i,\Sigma_j\in\mathcal{F}$. Then if the switching signal of $\Sigma_{\sigma(t)}$ is observable for almost every input, then for almost every input $\hat{\sigma}(t)=\sigma(t)$ for all $t>\tau_k$.*

Proof. According to Proposition 9 $\hat{\sigma}(t)=i\ \ \forall t\in[\tau_k,t_1)$ and from Lemma 10 each switching time is detected. Now, let $T_{ij}=T_j^{-1}T_i$ be a non-singular transformation such that T_k is the transformation leading to the observable/unobservable form of the *LS* Σ_k, $k=1,2$. Since

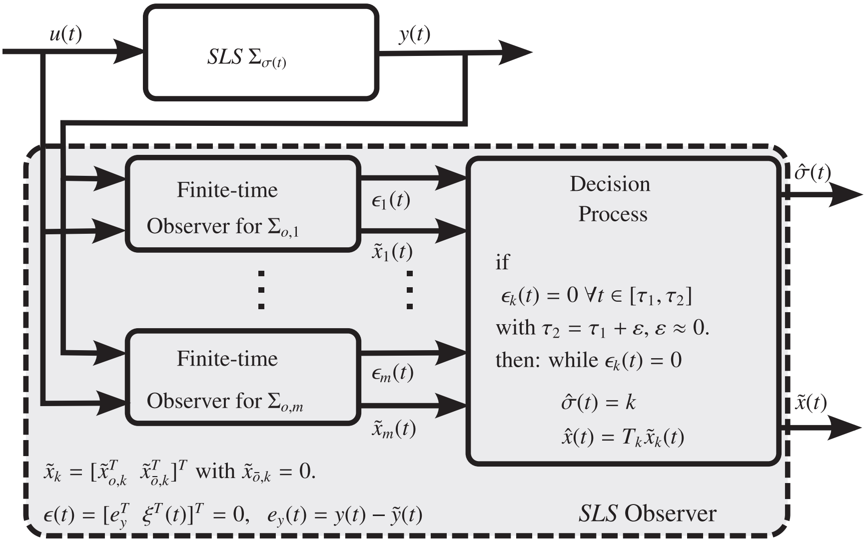

Fig. 1. *SLS* observer.

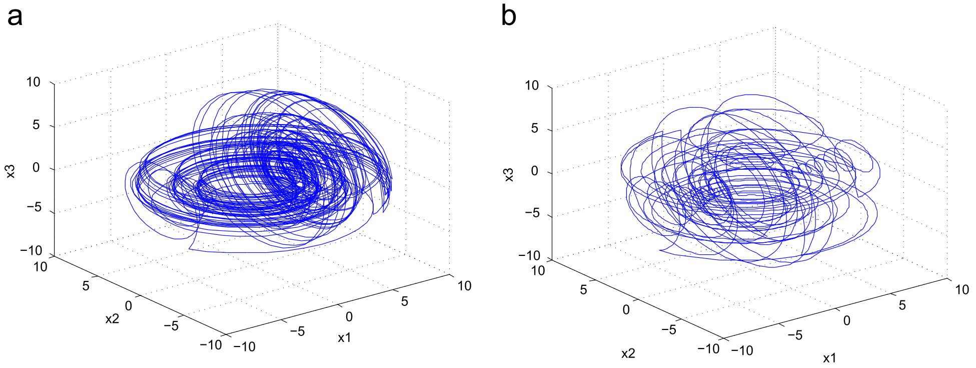

Fig. 2. Chaotic behavior of the *SLS* for different continuous and discrete initial conditions. (a) $\sigma_0 = 2$, $x_0 = [10\ 0\ 0]^T$ and $t \in [0, 20]$. (b) $\sigma_0 = 1$, $x_0 = [-1\ 1\ 2]^T$ and $t \in [0, 20]$.

$\mathcal{N}_i = \mathcal{N}_j\ \forall \Sigma_i, \Sigma_j \in \mathcal{F}$, it is easy to see that T_{ij} is of the form

$$T_{ij} = \begin{bmatrix} \mathcal{T}_{ij} & 0 \\ * & * \end{bmatrix} \quad \text{such that } \tilde{x}_{o,j} = \mathcal{T}_{ij}\tilde{x}_{o,i}.$$

Thus, if $\sigma(t) = i\ \forall t \in [t_{l-1}, t_l)$ and $\sigma(t) = j\ \forall t \in [t_l, t_{l+1})$ then, at the switching time t_l, the observable state variables of Σ_j are a linear combination only of the variables that were estimated by the previous observer. Thus, $\tilde{x}_{o,j}(t_l) = \overline{x}_{o,j}(t_l)$, with $\overline{x}_{o,j}$ the observable states of the evolving system Σ_j and, only the observer associated with the evolving *LS* Σ_j will give $\forall t \in [t_l, t_{l+1})$, $\epsilon(t) = 0$. Hence, $\hat{\sigma}(t) = \sigma(t)$ for all $t > \tau_k$. □

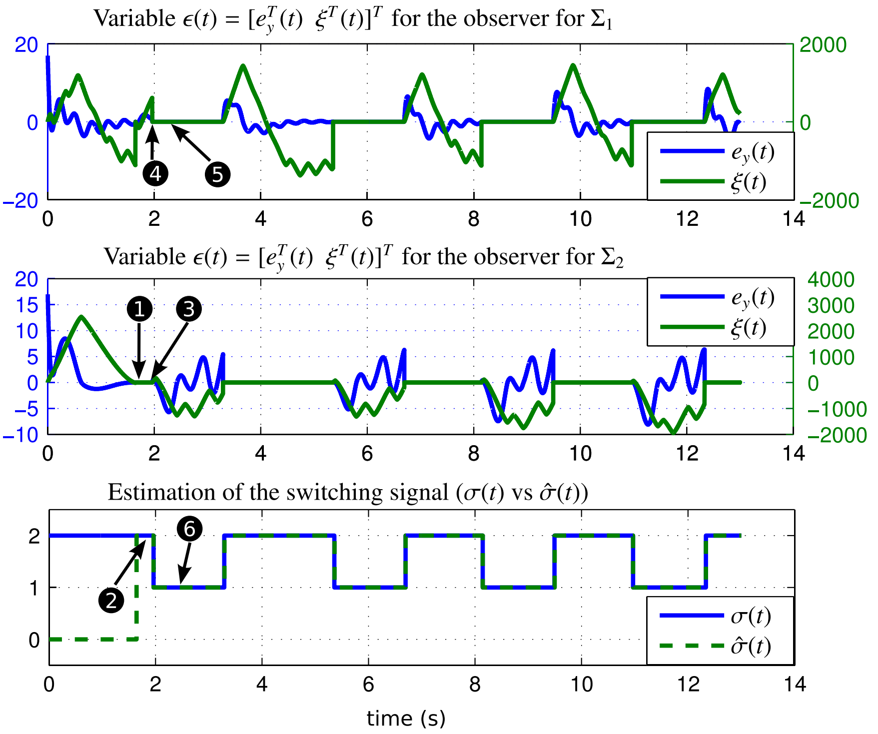

Fig. 3. Estimation of the switching signal $\sigma(t)$.

Theorem 12. *Let $\Sigma_{\sigma(t)}$ be a SLS and let $\hat{\Sigma}_{\hat{\sigma}(t)}$ be its observer. Then if the continuous state of $\Sigma_{\sigma(t)}$ is observable for almost every input, then for almost every input $\hat{x}(t) = x(t)$ for all $t > \tau_k$ with τ_k a finite time.*

Proof. Since only the continuous state of the *SLS* is observable then in the proposed *SLS* observer more than one finite-time observer may give $\epsilon(t) = [e_y^T(t)\ \xi^T(t)]^T = 0$, however, by Theorem 6 in such a case the indistinguishability subspace is symmetric which implies that the estimation of the continuous state in such observers is the same and any one gives $\hat{x}(t) = x(t)$. Furthermore, even if not every switching time is detected, an exact estimation of the continuous state is maintained because the symmetry of the indistinguishability subspace holds whenever Eq. (11) is not satisfied. Thus, the proposed *SLS* observer with its corresponding reinitialization procedure (whenever a switching is detected) gives $\hat{x}(t) = x(t)$ for all $t > \tau_k$, even if the switching signal is unknown. □

Remark 13. If noise is present, instead of detecting the evolving system by the observer with $\epsilon(t) = 0$ a threshold ς needs to be considered. Thus, detecting the evolving system if $\|\epsilon(t)\| < \varsigma$ in a short interval. However, further investigation on the conditions on the noise and the model mismatching, and its relation with ς is required and is considered as future work. However, we would like to highlight that even without this analysis the contribution has practical applications in chaotic synchronization and secure communications, for instance visible light communication systems in indoor have very high signal-to-noise-ratio (*SNR*) in the range 40–70 dB [42–44]. Under such *SNR* the effect of noise is negligible.

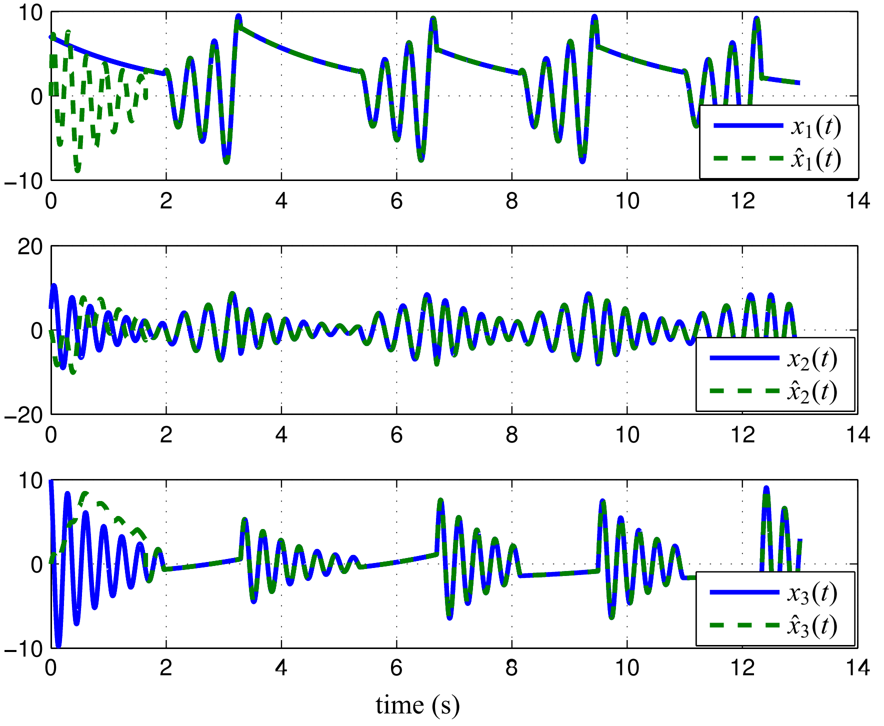

Fig. 4. Estimation of the continuous state of the *SLS*.

Remark 14. Since a finite-time estimation is achieved and no finite escapes can occur in *SLS*, the proposed observer can be designed separately from the controller.

5. Synchronization of *SLS* with chaotic behavior

SLS can exhibit highly nonlinear behavior such as chaos when a suitable switching signal is applied. This property is exploited to create new chaotic attractors as well as to synthesize by *SLS* well known chaotic systems such as the Lorenz and Rössler attractors [45]. A chaotic system can be used in encryption and secure communications, where a message encrypted using the chaotic system is transmitted through an open channel which is then decrypted by the receiver by using a synchronized copy of the same chaotic system, see [28,29] and further references within there. The synchronization problem can be naturally reformulated as an observer design problem [28,29]. As a matter of fact, for a selected chaotic *SLS*, the currently evolving *LS* and the current switching signal value may be considered to be unknown at the receiver.

5.1. Switching law for generating chaotic behavior

In [32] the following switching law was proposed for generating chaotic and chaotic-like behavior in *SLS*. Let $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ be an unstable and a stable scalar affine *LS*, respectively. And let $x_0^* = \frac{1}{2}\left(x_1^* + x_2^*\right)$ and $l = \frac{1}{2}\|x_1^* - x_2^*\|$, where x_1^* and x_2^* are such that $A_1 x_1^* + B_1 = 0$ and $A_2 x_2^* + B_2 = 0$, respectively.

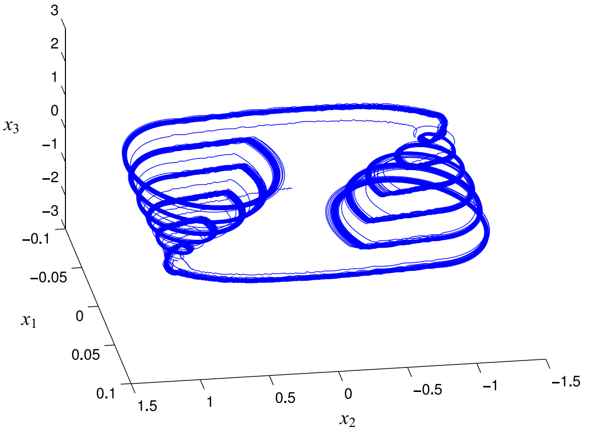

Fig. 5. Chaotic attractor generated by the SAS in Eq. (12).

Define the following three regions: $R_1 = \{x : \|x - x_0^*\| \le k\}$, $R_2 = \{x : k < \|x - x_0^*\| < m\}$, $R_3 = \{x : \|x - x_0^*\| \ge m\}$ with $l < k < m < +\infty$. Then the switching rule is constructed as follows. When Σ_1 is active, it will switch to Σ_2 at time t_j if $x(t_j) \in R_3$. Similarly, when Σ_2 is active, it will switch to Σ_1 at time t_k if $x(t_k) \in R_1$. In addition, in order to fulfill the minimum dwell time assumption, the first switching time cannot occur before the dwell time τ_d. Furthermore, the initial evolving *LS* can be either Σ_1 or Σ_2. With this switching rule $\sigma(t)$, Eq. (2) will has a chaotic behavior if the system parameters are properly chosen [32]. For a sufficient condition for a *SLS* to be chaotic see [31].

5.2. Illustrative example

Consider the *SLS* described in Section 3.1. Since, Σ_1 and Σ_2 cannot become indistinguishable if a constant input $u(t) = 1$ is applied. Thus, the continuous state and the switching signal of the affine *SLS* proposed in [32] are observable and the proposed observer can be applied where $\mathcal{F} = \{\Sigma_1(A_1, B_1, C_1), \Sigma_2(A_2, B_2, C_2)\}$ are described in Section 3.1, and $\sigma(t)$ is as described in Section 5.1 with $k=3$ and $m=10$. This *SLS* have a chaotic behavior with the strange attractor shown in Fig. 2.

The estimation of the switching signal and the continuous state is shown in Figs. 3 and 4, respectively. For this simulation, we are assuming no knowledge on the parameters k and m from Section 5.1. Clearly, that knowledge simplifies the estimation process. The parameters for both observers were chosen as $k_1 = 3.8$, $k_2 = 4.55$, $k_3 = 2.05$, $k_4 = 0.3$, $\rho = 10$ and $\alpha = 0.8$. The estimation process for the switching signal is shown in Fig. 3. According to Proposition 9, only the observer associated to the evolving system can give $\epsilon(t) = [e_y(t)\ \xi(t)]^T = 0$ for a nonzero interval which is shown in Fig. 4-❶. Hence, according to Proposition (9) the evolving *LS* Σ_2 is detected, Fig. 3-❷. After a switching occurrence the observer associated with Σ_2 can no longer maintain $\epsilon(t) = 0$, thus detecting the switching time as stated in Lemma (10), Fig. 3-❸. Whenever the switching time is detected the states of every observer are reinitialized with the final state of observer Σ_2, Fig. 3-❹. Notice that only the observer associated with the evolving *LS* will maintain $\epsilon = 0$ for a nonzero interval, Fig. 3-❺, thus detecting the next evolving *LS*, Fig. 3-❻. Finally, Fig. 4 shows that the state of the *SLS* is estimated and the synchronization occurs after a finite-time interval.

5.3. Comments on the applicability of the observer for SLS under a class of Zeno behavior

Although the focus of the paper is for *SLS* with excluded Zeno behavior, we would like to comment that the proposed observer could be used under Zeno behavior as long as the evolution during Zeno can be represented by an equivalent *AS*, as in sliding motion on a switching hyperplane (recall that sliding motion is a class of Zeno [46]). Consider for instance the chaotic *SAS* proposed in [47] which is described by

$$\dot{x} = Ax - B\,\mathrm{sign}(h), \quad h = Dx, \tag{12}$$

where

$$A = \begin{bmatrix} -(2\zeta\omega+\lambda) & 1 & 0 \\ -(2\zeta\omega\lambda+\omega^2) & 0 & 1 \\ -\lambda\omega^2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} k \\ 2k\beta\rho \\ k\rho^2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$$

and $k=1$, $\lambda=0.05$, $\omega=10$, $\rho=1$, $\beta=-1$ and $\zeta=-0.078$.

This *SAS* exhibits chaotic behavior and sliding motion [48] over the switching hyperplane, the state evolution of this system is shown in Fig. 5. For the estimation purpose an equivalent *SAS* not exhibiting Zeno behavior can be used in the observer design. Notice that sliding motion occurs since $h\dot{h}<0$ in a neighborhood of submanifold $\{x \in R^n | h(x) = 0\}$ and that in the sliding motion $h = \dot{h} = 0$ and thus the evolution is constrained in an $(A, \mathrm{Im}\, B)$-controlled invariant subspace contained in *ker D*. Using Utkin's equivalent control method [49] it can be found that during the sliding motion the evolution is equivalent to that of the system $\dot{x} = Ax + Bu_{eq}$ where $u_{eq} = Kx$ and $K = -(DB)^{-1}DA$ [47]. Thus, the evolution during the sliding motion is given by the *LS* $\dot{x} = \hat{A}x$ with $\hat{A} = (A + BK)$ and hence, by designing the proposed observer for the switched affine system

	A	b
Σ_1	A	B
Σ_2	$\hat{A}$	0
Σ_3	A	$-B$

$$\text{with} \quad \sigma(x) = \begin{cases} 1 & \text{if} \quad h > 0 \\ 2 & \text{if} \quad h = 0 \\ 3 & \text{if} \quad h < 0 \end{cases} \tag{13}$$

both $x(t)$ and the switching signal $\sigma(x)$ of Eq. (13) can be estimated. Thus, detecting also the Zeno occurrence. Notice that, if $y = x_1$ then a single observer can be designed in such a way that the error dynamics is not a switched system, otherwise the evolving *LS* and the continuous state need to be estimated from the continuous output.

6. Conclusions

An observer design for *SLS* with unknown switching signal has been presented here. The corresponding observer estimates in finite-time both the state evolution during the continuous phase and the originally unknown *SLS* switching signal. Moreover, efficiency of these observers has been demonstrated by their application to the chaotic *SLS* synchronization scheme construction being an important topic of the current research with possible applications to secure encryption. Future and ongoing research of the related topics is focused to the possible extensions of the presented results to the case when the output of the *SLS* is affected by noise while its inner dynamics is affected by (unknown) disturbances.

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