Quaternion Wiener Deconvolution for Noise Robust Color Image Registration

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***Abstract*—In this letter, we propose a global method for registering color images with respect to translation. Our approach is based on the idea of representing translations as convolutions with unknown shifted delta functions, and performing Wiener deconvolution in order to recover the shift between two images. We then derive a quaternionic version of the Wiener deconvolution filter in order to register color images. The use of Wiener filter also allows us to explicitly take into account the effect of noise. We prove that the well-known algorithm of phase correlation is a special case of our method, and we experimentally demonstrate the advantages of our approach by comparing it to other known generalizations of the phase correlation algorithm.**

***Index Terms*—Clifford algebra, multivector derivative, phase correlation, quaternion, Wiener filter.**

I. Introduction

THE problem of image registration is widely recognized to have a great importance, as many different tasks in computer vision often require a set of two or more aligned images as inputs [1]. We focus our attention on global methods and in particular to the popular algorithm of phase correlation [2]. Several attempts to generalize the algorithm of phase correlation to color images are found in the literature [3], [4], however, each of these approaches has important limitations.

In [4] the authors observe that ordinary phase correlation computes the cross-correlation between two "whitened" images, thus they introduce an hypercomplex (i.e. quaternion) Fourier transform, and generalize the concept of cross-correlation between quaternionic 2D-signals. Due to the nature of the quaternionic cross-correlation, their method produces a quaternion-valued correlation spectrum that in general is not a delta function, even in the ideal case where the two images are perfectly related only by a translation. The authors show that the correlation spectrum contains information related to the color transformation that relates the two images, and then they empirically demonstrate that its magnitude usually manifests a distinct peak located in correspondence of the shift. Nonetheless, it might be undesirable to have a spectrum in which part of the radiometric information is mixed with the geometric information related to the shift.

In [3] the authors argue that the formula for calculating the phase correlation between two images is analogous to the one for calculating the cosine of the angle between two functions, interpreted as elements of a normed vector space. Based on this argument, they propose to generalize the phase correlation algorithm to color images by computing the cosine of the angle between the Fourier spectra of two multivector-valued images. A drawback of their approach is that it does not yield a single peak corresponding to the shift between the images, but rather two peaks located in positions that are centrally symmetric to each other. Thus, the authors suggest to use their method in those applications (e.g. image recognition) where only the magnitude of one of the two peaks is needed, while its spatial location is not required.

In this letter, we propose a general solution to all the aforementioned issues. The key idea of the letter is a re-formulation of the problem of translational color image registration as a particular instance of the non-blind deconvolution problem. We utilize the well-known fact that an image shift can be described as a convolution with a delta-function shifted by the same vector. Hence, the shift can be recovered by a deconvolution of the sensed image with the reference image being the convolution kernel. This approach, never reported in the literature, allows us to employ techniques previously designed for image restoration purposes. We adopted a Wiener deconvolution filter [5]. To adapt it for registration of color images, we utilize Clifford algebra and a very general concept of *multivector derivative* in order to easily derive a quaternionic version of Wiener deconvolution. This is the second contribution of the letter. Taking advantage of the Wiener filtering technique, we are able to explicitly take into account the effect of additive noise. Finally we prove that our "color phase correlation" mathematically yields a true delta function in correspondence of the translational shift.

II. Image Registration in Terms of Deconvolution

Let's consider the space of images $\Omega = \mathcal{L}^2_{\mathcal{E}}(\mathbb{R}^2)$ defined as the vector space of two-dimensional real square integrable stochastic processes. Suppose an observed image $g \in \Omega$ is given by:

$$g = f*h + n \tag{1}$$

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where f is an image, h is a convolution kernel function, and n is additive zero-mean noise uncorrelated with f. Let's recall that, given $f \in \Omega$, a delta function has the property: $f(\mathbf{x}) = (\delta * f)(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$. Hence, by considering a delta function $\delta_{\mathbf{s}} = \delta(\mathbf{x}-\mathbf{s})$ shifted by $\mathbf{s} \in \mathbb{R}^2$, and $g(\mathbf{x}) = f(\mathbf{x}-\mathbf{s}) + n(\mathbf{x})$ a shifted and noisy version of the grayscale image f, we have that:

$$g = \delta_{\mathbf{s}} * f + n \tag{2}$$

Clearly the above equation is just a special case of (1) obtained by setting $h = \delta_{\mathbf{s}}$. If we also assume that no noise is present, we can estimate $\delta_{\mathbf{s}}$ by *deconvolving* g with the inverse filter of f. This operation is easily performed in Fourier domain, in fact, taking the FT of (2) yields $G(\mathbf{u}) = e^{-i(\mathbf{u}\cdot\mathbf{s})}F(\mathbf{u})$ in virtue of the convolution theorem, and considering the inverse filter $W(\mathbf{u}) = 1/F(\mathbf{u}) = F(\mathbf{u})^*/|F(\mathbf{u})|^2$ we have:

$$G(\mathbf{u})W(\mathbf{u}) = \frac{G(\mathbf{u})F(\mathbf{u})^*}{|F(\mathbf{u})|^2} = \frac{G(\mathbf{u})F(\mathbf{u})^*}{|G(\mathbf{u})F(\mathbf{u})^*|} = e^{-i(\mathbf{u}\cdot\mathbf{s})} \tag{3}$$

where the symbol * denotes *complex conjugation*, and the second identity follows from the fact that $|G(\mathbf{u})| = |F(\mathbf{u})|$, as it can be easily verified by taking the FT of (2), and assuming no noise is present. The inverse FT of (3) obviously yields the delta function δ_s centered at $\mathbf{s}$: the location corresponding to the shift between f and g. Interestingly, the rightmost fraction of (3) is the well-known formula for calculating the phase correlation [2].

III. Noise Robust Registration

An interesting aspect of interpreting image registration in terms of deconvolution is that we can investigate the use of more advanced approaches than the simple inverse filter. In this section we consider the application of the Wiener deconvolution technique, also known as Wiener-Helstrom filtering [5]. This allows us to take into account the statistics of the noise n in Equation (2) when performing image registration. In fact, if the power spectral density $\Psi_n(\mathbf{u}) = \mathcal{E}\{|N(\mathbf{u})|^2\}$ is known, we can estimate $\delta_{\mathbf{s}}$ in (2) by convolving g with a Wiener filter w that minimizes the mean squared error term $\mathcal{E}\{\|GW - \Delta_{\mathbf{s}}\|^2\}$, where $\Delta_s(\mathbf{u}) = \mathcal{F}\{\delta_{\mathbf{s}}\}$, and the expectation $\mathcal{E}$ is over all instances of the noise. In Fourier domain such filter is given by:

$$W(\mathbf{u}) = \frac{F(\mathbf{u})^*}{|F(\mathbf{u})|^2 + \frac{\Psi_n(\mathbf{u})}{\mathcal{E}\{|\Delta_{\mathbf{s}}(\mathbf{u})|^2\}}} = \frac{F(\mathbf{u})^*}{|F(\mathbf{u})|^2 + \Psi_n(\mathbf{u})} \tag{4}$$

The last identity follows from the fact that the Fourier transform of a shifted delta function $\Delta_s(\mathbf{u}) = e^{-i(\mathbf{u}\cdot\mathbf{s})}$ is a complex sinusoid, thus $\mathcal{E}\{|\Delta_{\mathbf{s}}(\mathbf{u})|^2\} = 1$

When performing Wiener deconvolution by calculating $G(\mathbf{u})W(\mathbf{u})$, one has:

$$G(\mathbf{u})W(\mathbf{u}) = \frac{G(\mathbf{u})F(\mathbf{u})^*}{|F(\mathbf{u})|^2 + \Psi_n(\mathbf{u})} \tag{5}$$

This operation attenuates more the frequencies that are corrupted by noise having higher variance, and it has the effect of denoising the estimate of the delta function $\delta_{\mathbf{s}}$ in space-domain. This can be easily observed by taking a pair of images related to each other by a translation (Fig. 1, top row), and among which one is corrupted by additive white Gaussian noise having $\Psi_n(\mathbf{u}) = \sigma^2$. By taking the inverse FT of (3), one obtains a noisy correlation spectrum. If we repeat the procedure by replacing (3) with (5), one obtains a visibly less noisy correlation spectrum. The denoising effect can be also observed quantitatively when comparing the amount "peakedness" measured by the sum of the kurtoses of the marginal distributions of the correlation spectrum (Fig. 1).

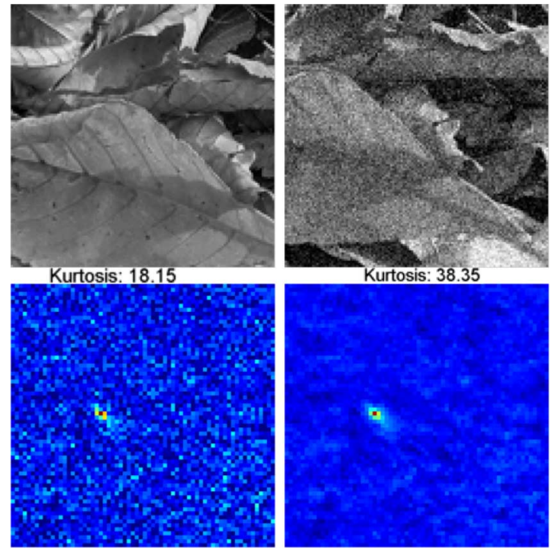

Fig. 1. Original image *(top left)*. Translated version of the original image with additive white Gaussian noise *(top right)*. Peak generated by computing the inverse FT of (3) with the kurtosis of the peak image *(bottom left)*. Peak generated by the inverse FT of (5) using $\Psi_n(\mathbf{u}) = \sigma^2$, the power spectral density of the noise; note how the kurtosis of the delta peak significantly increases. *(bottom right)*.

IV. Color Image Registration as Quaternionic Wiener Deconvolution

In this section we discuss the main result of this letter, which is the the generalization to quaternionic signals of the Wiener-Helstrom filter in Equation (4), and consequently the generalization to color images of traditional phase correlation, i.e. Equation (3). Before deriving the quaternionic Wiener filter, we must introduce the quaternion Fourier transform (QFT). Let's recall that a quaternion q can be represented as $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, where q_0, q_1, q_2, q_3 are real numbers, and the relationship $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ on the imaginary units holds. Given an image $f \in \Omega$, where $\Omega = \mathcal{L}^2_{\mathcal{E}}(\mathbb{R}^2; \mathbb{H})$ is the space of two-dimensional quaternion-valued square integrable stochastic processes, we map the color triplets (ℓ, c_1, c_2) of some luminance-chrominance color space in the following manner:

$$f(\mathbf{x}) = \ell(\mathbf{x}) + c_1(\mathbf{x})\mathbf{i} + c_2(\mathbf{x})\mathbf{j} = f_+(\mathbf{x}) + f_-(\mathbf{x})\mathbf{i} \tag{6}$$

where $f_+(\mathbf{x}) = \ell(\mathbf{x})$ and $f_-(\mathbf{x})\mathbf{i} = c_1(\mathbf{x})\mathbf{i} + c_2(\mathbf{x})\mathbf{j} = (c_1(\mathbf{x}) + c_2(\mathbf{x})\mathbf{k})\mathbf{i}$. These two terms can be represented as two complex numbers (though it must be noted that f_+ is always real), and they are respectively called *even-grade* part and *odd-grade* part of f. This representation of $f(\mathbf{x})$ as an ordered pair of complex numbers with imaginary unit $\mathbf{k}$ has several advantages: it allows one to compute the QFT with two ordinary FFT's, and it leads to a compact expression of the Fourier convolution theorem [6], which is useful in the derivation of the quaternion Wiener filter. Following [7], the QFT $F \stackrel{def}{=} \mathcal{F}\{f\}$ of (6) is defined as follows:

$$F(\mathbf{u}) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-\mathbf{k}(\mathbf{x}\cdot\mathbf{u})} d^2\mathbf{x} \tag{7}$$

and can be effectively computed by two FFT's on the two complex-valued functions f_+ and f_-, i.e. $f \stackrel{\mathcal{F}}{\mapsto} F = F_+ + F_{\mathbf{i}}$. Given $f, g \in \Omega$, the convolution theorem takes the following form [7]:

$$f{*}g \stackrel{\mathcal{F}}{\mapsto} FG_+ + \overset{\bullet}{F} G_- \mathbf{i} \tag{8}$$

where $\overset{\bullet}{F}(\mathbf{u}) \stackrel{def}{=} F(-\mathbf{u})$ and the subscripts $_+$ and $_-$ denote the even and odd-grade parts of $F(\mathbf{u})$. Note that the term on the right side denotes a sum of point-wise multiplications between the functions F and G. It is important to observe that, if g is real or complex valued, the odd-grade part g_- is zero, which happens for instance when g is a real scalar function, then Equation (8) reduces to:

$$f{*}g \stackrel{\mathcal{F}}{\mapsto} FG \tag{9}$$

A. Quaternionic Wiener-Helstrom Filter

We are now ready to introduce the quaternionic version of the Wiener-Helstrom filter. The material found in this section is presented only informally, and a more rigorous treatment is published as supplementary material in http://ieeexplore.ieee.org. Consider the space $\Omega = \mathcal{L}^2_{\mathcal{E}}(\mathbb{R}^2; \mathbb{H})$ and suppose that $g \in \Omega$ is defined as: $g = f{*}h + n$, analogously to Equation (1), where f is an image, h is a real-valued convolution kernel function, and n is additive zero-mean noise uncorrelated with f, and with known power spectral density $\Psi_n(\mathbf{u}) = \mathcal{E}\{|N(\mathbf{u})|^2\}$. By taking the QFT of g we have, according to (9) that $g \stackrel{\mathcal{F}}{\mapsto} FH + N$. If we suppose, for now, that $\Psi_n(\mathbf{u}) = 0$, it appears clear that, in order to recover F we must simply multiply $G(\mathbf{u})$ by the inverse filter $H(\mathbf{u})^{-1}$, for all $\mathbf{u} \in \mathbb{R}^2$ (note that quaternions always have an inverse). In fact, if we drop the assumption that no noise is present, it is possible to prove that the filter W that minimizes the following mean squared error:

$$\underset{W(\mathbf{u})}{\arg\min}\, \mathcal{E}\left\{ |G(\mathbf{u})W(\mathbf{u}) - F(\mathbf{u})|^2 \right\} \tag{10}$$

for all $\mathbf{u} \in \mathbb{R}^2$, where $|\cdot|^2$ denotes the squared norm of a quaternion, is given by:

$$W(\mathbf{u}) = \frac{\overline{H(\mathbf{u})}}{|H(\mathbf{u})|^2 + \frac{\mathcal{E}\{|N(\mathbf{u})|^2\}}{\mathcal{E}\{|F(\mathbf{u})|^2\}}} \tag{11}$$

Note that (11) is analogous to the formula of the complex Wiener filter, with the only exception of replacing the complex conjugate operator * with the *quaternion conjugate*, which is defined as: $\overline{H(\mathbf{u})} = H_+(\mathbf{u})^* - H_-(\mathbf{u})\mathbf{i}$. The unknown F is thus estimated by calculating:

$$F_{est}(\mathbf{u}) = G(\mathbf{u})W(\mathbf{u}) \tag{12}$$

It should be noted that, in the complex case, the functional in (10) is typically minimized by utilizing the rules for complex differentiation of Wirtinger calculus [8], but it is not completely straightforward to generalize those rules to quaternions [9], [10]. In the supplementary document we demonstrate that (11) can be easily obtained from (10) by making use of the more general concept of *multivector derivative*[11].

B. Noise Robust Registration of Color Images

Supposing $f, g \in \Omega$ are quaternionic (color) images that follow the model in Equation (2), we can treat the image f as a quaternionic convolution filter applied to the delta function $\delta_{\mathbf{s}}$ (interpreted as a real signal), as explained in Section II. Hence, we can form a quaternion Wiener filter W according to (11) in order to recover $\delta_{\mathbf{s}}$ from $g = \delta_{\mathbf{s}}{*}f + n$. The filter W will have *notationally* the same form of (4), so we can write:

$$W(\mathbf{u}) = \frac{\overline{F(\mathbf{u})}}{|F(\mathbf{u})|^2 + \Psi_n(\mathbf{u})} \tag{13}$$

where

$$|F(\mathbf{u})|^2 = |F_+(\mathbf{u})|^2 + |F_-(\mathbf{u})|^2 \tag{14}$$

It is very important to emphasize that the numerator in (13) should be interpreted as an ordered pair of complex numbers $(F_+(\mathbf{u})^*, -F_-(\mathbf{u}))$, while the denominator is always a real scalar. In analogy with (5), we calculate the Wiener filtered version of g in Fourier domain in its expanded form:

$$GW = \frac{(G_+F_+^* + G_-F_-^*) + (G_-F_+ - G_+F_-)\,\mathbf{i}}{|F|^2 + \Psi_n} \tag{15}$$

In Equation (15) for the sake of brevity we omitted the frequency variable $\mathbf{u}$ in each term, so that $F = F(\mathbf{u})$, $G = G(\mathbf{u})$. The two terms in parentheses denote the even-grade and the odd-grade part of the quaternion-valued numerator, and they are obtained expanding $G(\mathbf{u})\overline{F(\mathbf{u})}$ according to the product rule for quaternions [12, p. 177]. Again, taking the inverse QFT of (15) we obtain an estimate of $\delta_{\mathbf{s}}$. An interesting fact is that for grayscale images with no noise (i.e. $G_- = F_- = 0$, and $\Psi_n = 0$), the Wiener deconvolution in (15) coincides with the formula to calculate the ordinary phase correlation, and the term $G_+F_+^*$ at the numerator would then represent the cross-correlation of f and g in frequency domain. However it must be pointed out that for quaternionic images, the numerator of (15) does *not* coincide with the formula of quaternion cross-correlation given in [4]. Furthermore, Equation (15) represents, to the best of our knowledge, the only formulation of the phase correlation algorithm for color images that gives exactly a *delta function* corresponding to the shift.

The reader might have also noticed that since F is, strictly speaking, a stochastic process, we have slightly simplified our notation, as we should interpret F in the numerators of (4) and (13) as $\mathcal{E}\{F\}$, and similarly $|F|^2$ as $\mathcal{E}\{|F|^2\}$. In the light of this observation, we shall remark that the identity in (14) holds if the color channels of f are *uncorrelated*, hence it is convenient to represent the images in a color space such that the transformation that converts from RGB into luminance-chrominance values is *linear* and decorrelates the RGB color channels. The requirement for linearity ensures that noise remains additive after the color transformation is applied. One linear color transformation that decreases the correlation between the

RGB channels in natural images is obtained by rotating the RGB axes so that the luminance axis ℓ has RGB coordinates [0.299 0.587 0.114] (in analogy with the luminance of the YUV color space), and the two chromaticity axes c_1 and c_2 form an orthonormal base with ℓ. This is essentially the opponent color space described by Van De Weijer in [13].

V. Experiments

We compared the performance of our method in registering color images with other known approaches according to the following experiment. One hundred 4-megapixels color images were acquired with a consumer camera, and from each of these we extracted a pair of images of 255×255 pixels that have a specified amount of overlap, and such that they are related to each other by a translation in a random direction. We fixed the overlap between images to 20%. We also added a small geometric perturbation by rotating one of the two images by 2 degrees. White Gaussian noise with standard deviation σ is added to all the three color channels of one of the two images. We used $\sigma = 0, 20, 40, 60, 80, 100$ (pixel intensities range from 0 to 255). The RGB images are converted to a luminance-chrominance opponent color space (denoted by $\ell c_1 c_2$) as described in Section IV-B. Each pair of images is registered with the following methods: our registration method based on the quaternion Wiener deconvolution filter (qWF) described in Section IV; the quaternion inverse filter (qIF) obtained by setting $\sigma(\mathbf{u}) = 0$ in (13); the Wiener deconvolution filter (WF) obtained by setting $f_- = g_- = 0$ (e.g. discarding the chromaticity of both images); ordinary phase correlation (PC) [2] applied to f_+ and g_+ (e.g. the luminance channels); the quaternion cross-correlation (qXCorr) proposed in [4]; the trivial approach of performing phase correlation channel-wise and summing the three correlation spectra ($\ell + c_1 + c_2$). Results obtained by SIFT registration [14] which is a popular landmark-based method are also included for comparison. We did not include Mennesson's method [3], as there is no straightforward way to resolve its double-peak ambiguity. Since our method requires the power spectral density of the noise to be known, we present both the results of our qWF using the ground-truth variance σ^2 of the Gaussian noise, and using an estimate for σ^2 obtained by the approach described in [15]. The estimate of the shift is obtained by taking the location of the highest value in the correlation spectrum. The shift error ε is defined as the Euclidean distance between the true shift and the estimated shift. The performance is quantified according to the criterion proposed in [16], that is, by counting the number of misregistrations, where a "misregistration" occurs whenever ε exceeds a specified threshold η. We set $\eta = 3$ pixels. The results are summarized in Fig. 2.

The most important conclusions of this experiment are the following ones. The proposed method outperforms the others and its advantage becomes more apparent in case of heavy noise. The performance of our method is robust to the accuracy of the noise variance estimation; the difference between the usage of the estimated and ground-truth value is insignificant. The registration method based on SIFT features was the only one in this experiment that uses landmarks. Due to the low overlap of the images and heavy noise, it performed very poorly. This is a common drawback of most landmark-based methods, which one can partially override by introducing additional consistency check and by using global information about the landmark distribution, as proposed for instance in [17] and [18]. This would be however far beyond the scope of this letter. Low overlap lead also to a poor performance of qXCorr.

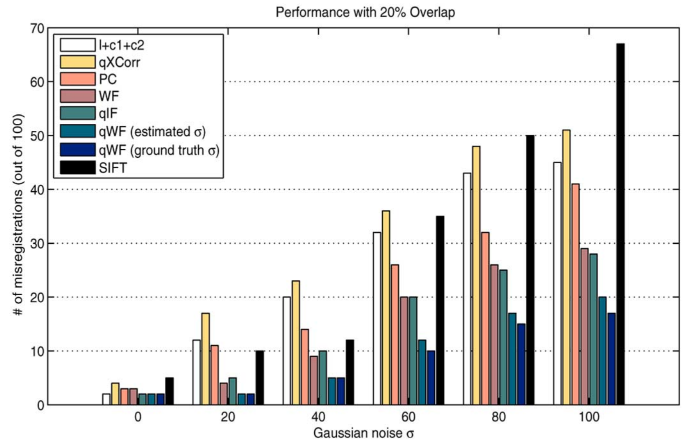

Fig. 2. Percentage of misregistrations when the images were degraded by different levels of additive white Gaussian noise. The overlap between the images was fixed to 20%.

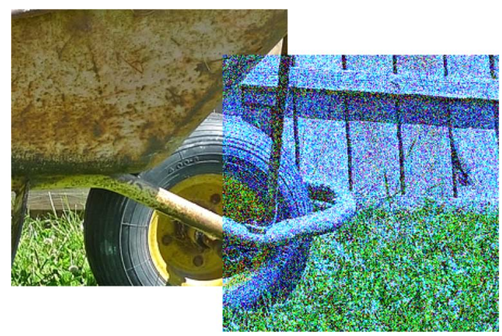

Fig. 3. A pair of registered images in the experiments with illumination changes and noise. The amount of noise in the right image corresponds approximately to $\sigma = 60$.

The same experiment was repeated including also a chromatic degradation to one of the two images, in order to simulate a change of illumination. This was obtained by multiplying each of the RGB color channels by respectively three different scalars chosen randomly in the range [0.6, 1.4]. An example of a pair of images used in this experiment is illustrated in Fig. 3, however we did not observe significant differences from the results in Fig. 2.

VI. Conclusion

We derived a quaternionic version of the Wiener deconvolution filtering technique, and we showed that it can be utilized to recover the shift between two color images (unlike typical usage in image restoration), which leads to a generalization of the traditional phase correlation method [2]. This enabled us to design a noise-robust registration method for color images. We experimentally showed that our methods outperforms the trivial channel-by-channel extensions of phase correlation, and its other generalizations found in the literature, both in ideal scenarios (noise-free images), and with heavily noisy images, including also illumination changes.

REFERENCES

[1] B. Zitova and J. Flusser, "Image registration methods: A survey," *Image Vis. Comput.*, vol. 21, no. 11, pp. 977–1000, 2003.

[2] E. de Castro and C. Morandi, "Registration of translated and rotated images using finite fourier transform," *IEEE Trans. Patt. Anal. Mach. Intell.*, vol. 9, no. 5, pp. 700–703, 1987.

[3] J. Mennesson, C. Saint-Jean, and L. Mascarilla, "A phase correlation for color images using clifford algebra," in *Proc.Conf. Applied Geometric Algebras in Computer Science and Engineering (AGACSE)*, Amsterdam, The Netherlands, 2010.

[4] C. E. Moxey, S. J. Sangwine, and T. A. Ell, "Hypercomplex correlation techniques for vector images," *IEEE Trans. Signal Process.*, vol. 51, no. 7, pp. 1941–1953, 2003.

[5] R. Frieden, *Probability, Statistical Optics, and Data Testing: A Problem Solving Approach*. Berlin, Germany: Springer, 2001, vol. 10.

[6] M. Pedone and J. Heikkila, "Local phase quantization descriptors for blur robust and illumination invariant recognition of color textures," in *21st Int. Conf. Pattern Recognition (ICPR), 2012*, 2012, pp. 2476–2479, IEEE.

[7] E. Hitzer and B. Mawardi, "Clifford Fourier transform on multivector fields and uncertainty principles for dimensions $n = 2$ (mod 4) and $n = 3$ (mod 4)," *Adv. Appl. Clifford Alg.*, vol. 18, no. 3, pp. 715–736, 2008.

[8] W. Wirtinger, "Zur formalen theorie der funktionen von mehr komplexen veränderlichen," *Math. Annalen*, vol. 97, no. 1, pp. 357–375, 1927.

[9] D. P. Mandic, C. Jahanchahi, and C. C. Took, "A quaternion gradient operator and its applications," *IEEE Signal Process. Lett.*, vol. 18, no. 1, pp. 47–50, 2011.

[10] G. M. Menanno and A. Mazzotti, "Deconvolution of multicomponent seismic data by means of quaternions: Theory and preliminary results," *Geophys. Prospect.*, vol. 60, no. 2, pp. 217–238, 2012.

[11] E. M. Hitzer, "Multivector differential calculus," *Adv. Appl. Clifford Alg.*, vol. 12, no. 2, pp. 135–182, 2002.

[12] H. E. Rose and Linear, *Algebra: A Pure Mathematical Approach*. Berlin, Germany: Springer, 2002.

[13] J. Van De Weijer, T. Gevers, and J.-M. Geusebroek, "Edge and corner detection by photometric quasi-invariants," *IEEE Trans. Patt. Anal. Mach. Intell.*, vol. 27, no. 4, pp. 625–630, 2005.

[14] D. G. Lowe, "Distinctive image features from scale-invariant keypoints," *Int. J. Comput. Vis.*, vol. 60, no. 2, pp. 91–110, 2004.

[15] A. Danielyan and A. Foi, "Noise variance estimation in nonlocal transform domain," in *2009 IEEE Int. Workshop on Local and Non-Local Approximation in Image Processing*, 2009, pp. 41–45.

[16] M. Pedone, J. Flusser, and J. Heikkila, "Blur invariant translational image registration for N-fold symmetric blurs," *IEEE Trans. Image Process.*, vol. 22, no. 9, pp. 3676–3689, 2013.

[17] G. J. Wen, J. J. Lv, and W. X. Yu, "A high-performance feature-matching method for image registration by combining spatial and similarity information," *IEEE Trans. Geosci. Remote Sens.*, vol. 46, no. 4, pp. 1266–1277, 2008.

[18] Q. Shi, G. Ma, F. Zhang, W. Chen, Q. Qin, and H. Dou, "Robust image registration using structure features," *IEEE Geosci. Remote Sens. Lett.*, vol. 11, no. 12, pp. 2045–2049, 2014.